Formal group laws

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The additive formal group $G_a$ plays a fundamental role in the theory. We will see in the first section that up to isomorphism this is the only formal group over a $\mathbb{Q}$-algebra. This is far from true over other rings. The ring of endomorphisms of a formal group over a $\mathbb{Q}$-algebra $R$ is canonically isomorphic to $R$. In general things are more complicated, even for the additive formal group. For example, as we will see in Section 3, the endomorphism ring of $G_a$ over rings of characteristic $p$ is quite interesting. It is related to the algebra of Steenrod reduced powers, familiar from algebraic topology. A wish to establish an analogue of $G_a$ affording the full Steenrod algebra at an odd prime leads to the amusing project of working out the “super” analogue of formal groups, in Section 4. The final section is a misfit in this chapter: it is devoted to the relationship between the multiplicative group and gradings.

1 Formal group laws

In studying formal groups one often picks a parameter and so obtains by transport of structure a “formal group law”:

**Definition 1.1** A formal group law over $R$ is a formal group over $R$ whose underlying formal scheme is $D$.

The group structure on $D = \text{Spf} \, R[[t]]$ given by a formal group law is defined by an augmented (and hence automatically continuous) $R$-algebra homomorphism $R[[t]] \to R[[t]] \hat{\otimes} R[[t]] = R[[u, v]]$ (where $u = t \otimes 1, v = 1 \otimes t$). This homomorphism is determined by the image of $t$, which is a formal power
series \( F(u, v) \in R[[u, v]] \) such that
\[
F(0, v) = v, \quad F(u, 0) = u, \\
F(F(u, v), w) = F(u, F(v, w)), \\
F(u, v) = F(v, u).
\]

It is easy to check inductively that there is a power series \([-1]_F(u)\) such that \( F([-1]_F(u), u) = 0 = F(u, [-1]_F(u))\), so we do not need to assume the existence of an inverse. We will often write \( u +_F v \) for \( F(u, v) \) and regard it as a kind of formal addition. Giving the formal power series \( F \) is precisely equivalent to giving the formal group law, and one speaks of “the formal group law \( F \).” The formal group law associated to a parameter \( x \) on a formal group \( G \) will be denoted by \( G_x \).

For example the identity map serves as a parameter for the additive formal group, and the resulting power series is
\[
G_a(u, v) = u + v.
\]
The formal \( R \)-group functor \( G_m \) is a formal group as well. A parameter \( x : D \to G_m \) is given by on the nilpotent \( R \)-algebra \( I \) by \( x(s) = 1 - s \). The resulting power series is
\[
G_m(u, v) = u + v - uv.
\]

Formal group laws form a full subcategory \( \mathcal{F}(R) \) of the category \( \mathbf{FG}_R \) of formal groups over \( R \). A homomorphism from \( F(u, v) \) to \( G(u, v) \) is a formal power series \( \theta(x) \) such that \( \theta(0) = 0 \) and
\[
\theta(u +_F v) = \theta(u) +_G \theta(v).
\]
Since any formal group is isomorphic (by choice of parameter) to a formal group law, the category of formal groups is equivalent to its subcategory of formal group laws.

If \( G \) and \( H \) are formal group laws, the sum of two homomorphisms \( f, g : G \to H \) is given by the power series \( (f +_H g)(x) = f(x) +_H g(x) \). The canonical ring homomorphism \( \mathbb{Z} \to \text{End}_R(G) \) will be denoted by \( n \mapsto [n]_G \). The power series \([n]_G(x)\) has leading term \( nx \). For example
\[
[n]_{G_a}(x) = nx, \quad [n]_{G_m}(x) = 1 - (1 - x)^n.
\]
Another point of view is to regard a formal group law as an equivalence class of pairs \((G, x)\) where \(G\) is a formal group and \(x : D \to G\) is a parameter. \((G, x)\) is equivalent to \((H, y)\) when the composite \(y \circ x^{-1} : G \to H\) is a homomorphism (as well as an isomorphism of formal schemes).

Clearly two parameters \(x : D \to G\) and \(y : D \to G\) are related by a unique isomorphism \(f : D \to D\): \(y = x \circ f\). \(f\) is a composition-invertible power series. The formal group law, say \(F\), determined by the parameter \(x\) on \(G\) is related to the one given by \(y\) by conjugation with the formal power series \(f\):

\[
F^f(u, v) = f^{-1}(F(f(u), f(v))).
\]

Note that \(f\) is an isomorphism from \(F^f\) to \(F\). The \(n\)-series for \(F^f\) is the conjugate by \(f\) of the \([n]\)-series for \(F\):

\[
[n]_{F^f}(u) = f^{-1}([n]_F(f(u))).
\]

2 Formal groups over \(\mathbb{Q}\)-algebras

If \(G(x, y)\) is any formal group law over \(R\), we may conjugate it with an invertible formal power series \(f(t)\), to form

\[
G^f(x, y) = f^{-1}G(f(x), f(y)).
\]

\(G^f\) is another formal group law over \(R\), and \(f\) is an isomorphism from \(F^f\) to \(F\). For example the universal “strict” conjugate of the additive formal group law \(G_a(x, y) = x + y\) is given by

\[
G_a^{log}(x, y) = \exp(\log(x) + \log(y))
\]

over

\[
M = \mathbb{Z}[l_1, l_2, \ldots]
\]

where

\[
\log(t) = \sum_{i=1}^{\infty} l_{i-1} t^i, \quad l_0 = 1,
\]

and \(\exp(t)\) is its composition inverse. The notations \(\log\) and \(\exp\) are chosen because they give isomorphisms between the additive formal group law and some other formal group law.
We can attempt to compute the logarithm in terms of $G = G^{\log}_a$. Differentiating the equation
\[
\log(G(x, y)) = \log(x) + \log(y)
\]
with respect to $y$ and setting $y = 0$, we find
\[
\log'(x)G_2(x, 0) = \log'(0) = 1
\]
where $G_2$ indicates the partial derivative with respect to the second variable. Since $G(x, y) = x + y + \cdots$, $G_2(x, 0) = 1 + \cdots$ and so we can solve for $\log'(x)$:
\[
\log'(x) = \frac{1}{G_2(x, 0)}.
\]
We cannot generally integrate this power series, but over a $\mathbb{Q}$-algebra we can form
\[
\log(x) = \int_0^x \frac{dt}{G_2(t, 0)}
\]
for any formal group law $G$. This proves most of

**Lemma 2.1** Let $R$ be a $\mathbb{Q}$-algebra. Any formal group law $G$ over $R$ is isomorphic to the additive formal group law $G_a$ by a unique strict isomorphism $\log_G : G \to G_a$. Moreover, for any two formal group laws $F, G$ over $R$, the map
\[
\text{Hom}_R(F, G) \to R, \quad \theta \mapsto \theta'(0),
\]
is an isomorphism.

**Proof.** First of all the map $\text{End}_R(G_a) \to R$ is an isomorphism; the only formal power series $\theta(t)$ such that $\theta(u + v) = \theta(u) + \theta(v)$ are those of the form $\theta(t) = at$, and any $a$ gives such a series. Now if $a \in R$ we may form $\log^{-1}_G(a \log_F(t))$. It is a homomorphism from $F$ to $G$ with leading term $at$. Conversely, a homomorphism $\theta$ determines an endomorphism $\log_G \theta \log_F^{-1}$ of $G_a$ and so must be of this form. \(\blacksquare\)

For example the multiplicative formal group law $G_m(x, y) = x + y - xy$ has as logarithm
\[
\log_{G_m}(t) = -\log(1 - t) = \sum_{r=1}^{\infty} \frac{t^r}{r}
\]
and as exponential
\[
\exp_{G_m}(t) = 1 - e^{-t} = -\sum_{r=1}^{\infty} \frac{(-t)^r}{r!}.
\]
3 The endomorphism ring of $G_a$

The ring of endomorphisms of the additive formal group law over the ring $S$ consists in the formal power series $f(t) \in tS[[t]]$ such that for every pair of elements $a, b$ in a nilpotent $S$-algebra $I$, $f(a + b) = f(a) + f(b)$. This ring is hard to deal with in general but it simplifies if we put restrictions on the ring $S$. In this regard it is useful to fix a ring $R$ and look at the functor $S \mapsto \text{End}_S(G_a)$ as a functor from $R$-algebras to rings, an “$R$-ring” $\text{End}(G_a)_R$.

Suppose first that we restrict to algebras over $\mathbb{Q}$. By differentiation, an additive power series necessarily reduces to its linear term, and any power series of the form $ax$ will serve. Thus the endomorphism ring of $G_a$, as a functor on $\mathbb{Q}$-algebras, is the inclusion functor from $\mathbb{Q}$-$\text{alg}$ to $\text{Ring}$. This is an affine ring-scheme, represented by the co-ring structure on $\mathbb{Q}[t]$ in which $t$ is primitive for the additive diagonal and grouplike for the multiplicative one. In particular,

$$\text{Aut}(G_a)_{\mathbb{Q}} \cong G_{m\mathbb{Q}}.$$

By 2.1, the same statements hold for any formal group over a $\mathbb{Q}$-algebra.

Next suppose $R = \mathbb{F}_p$. Additivity is equivalent to the assertion that

$$f(t) = \sum_{i=0}^{\infty} a_i t^{p^i}$$

The ring of endomorphisms is represented by a co-ring object in the category of $\mathbb{F}_p$-algebras, with underlying $\mathbb{F}_p$-algebra

$$S = \mathbb{F}_p[\xi_0, \xi_1, \ldots].$$

The additive structure is represented by the diagonal making each $\xi_k$ primitive, with augmentation zero. The composition is represented by the diagonal given by

$$\delta \xi_k = \sum_{i+j=k} \xi_i \otimes \xi_j^{p^i},$$

and the unit is represented by the augmentation sending $\xi_0$ to 1 and $\xi_k$ to 0 for all $k > 0$.

The group of units is represented by

$$P = \mathbb{F}_p[\xi_0^{\pm 1}, \xi_1, \ldots]$$
with the unique extension of the Hopf algebra structure on $S$; we can write

$$\text{Aut}(G_a)_{\mathbb{F}_p} \cong \text{Sp}P.$$ 

One also has the strict automorphisms, those with leading term 1, represented by

$$P_1 = P/(\xi_0 - 1) \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots].$$

This is precisely the dual of the Hopf algebra of Steenrod reduced powers. The Hopf algebra $P$ is a semi-tensor product of $P_1$ with $P_0 = \mathbb{F}_p[\xi_0^{\pm 1}]$.

There is another interesting feature of the ring of endomorphisms of $G_a$, pointed out by Goro Nishida. Fix $n$, let $q = p^n$, and look at the ring $\text{End}_{\mathbb{F}_q}(G_a)$. Let $W$ be the subset consisting of power series of the form

$$f(t) = \sum_{i=0}^{\infty} b_i t^{q^i}, \quad b_i \in \mathbb{F}_q.$$ 

Since the $q$th power map is the identity on $\mathbb{F}_q$, this is a commutative subring. In fact it is isomorphic to the power series ring $\mathbb{F}_q[[\lambda]]$, via

$$\sum_{i=0}^{\infty} b_i t^{q^i} \leftrightarrow \sum_{i=0}^{\infty} b_i \lambda^i.$$ 

The power series $\pi(x) = x^p$ is also an endomorphism, and it satisfies the identities

$$\pi^n = \lambda, \quad \pi f = \sigma(f)\pi \quad \text{for } f \in W,$$

where $\sigma$ is the “Frobenius automorphism” of $W$ given by taking the $p$th powers of the coefficients. The whole of $\text{End}_{\mathbb{F}_q}(G_a)$ is generated by $W$ and $\pi$, subject to these relations; and any endomorphism has a unique expression

$$\sum_{i=0}^{n-1} f_i(\lambda)\pi^i, \quad f_i \in W,$$

or, what is the same,

$$\sum_{i=0}^{\infty} a_i \pi^i, \quad a_i \in \mathbb{F}_q.$$ 

This is interesting because it is a “function field analogue” of the ring of endomorphisms of a formal group of height $n$ over a sufficiently large field, as studied in Chapter ??.
4 Super formal groups

Just for fun, let’s define a “super” extension of the theory of formal sets and groups. A super ring is an associative $\mathbb{Z}/2\mathbb{Z}$-graded ring $R^\bullet$ satisfying the skew-commutativity relations

$$ab = (-1)^{|a||b|} ba, \quad c^2 = 0 \text{ if } |c| = 1.$$  

An $R^\bullet$-algebra is a super ring $S^\bullet$ equipped with a super ring homomorphism from $R^\bullet$.

Giving a nilpotent $R^\bullet$-algebra is equivalent to giving an augmented $R^\bullet$-algebra with nilpotent augmentation ideal. These form a category $\text{Nil}_{R^\bullet}^s$.

An interesting case occurs when $R^1 = 0$. Write $R$ for $R^0$. Then we will call an $R^\bullet$-algebra an $R$-super algebra. There is a functor $\text{Nil}_R \to \text{Nil}_{R^\bullet}^s$, sending the augmented $R$-algebra $S$ to the augmented $R$-super algebra with $S^0 = S$, $S^1 = 0$, and the evident structure maps.

**Definition 4.1** A super formal $R^\bullet$-set is a functor from $\text{Nil}_{R^\bullet}^s$ to $\text{Set}$ sending 0 to a singleton set.

If $R^1 = 0$ and $R^0 = R$, a subformal $R^\bullet$-set has an “underlying” formal $R$-set, given by precomposing with the functor $\text{Nil}_R \to \text{Nil}_{R^\bullet}^s$.

For example we have the super formal line given by

$$D^s(S^\bullet) = S^0 \times S^1.$$  

This functor is pro-represented by the complete augmented $R^\bullet$-algebra

$$R^\bullet[[x]] \otimes_{R^\bullet} E[e], \quad |x| = 0, |e| = 1, \varepsilon(x) = 0.$$  

With natural transformations as morphisms, we have a Category $\text{FS}_{R^\bullet}^s$. An endomorphism of $D^s$ is of the form

$$(a, b) \mapsto (f_{00}(a) + f_{01}(a)b, f_{11}(a) + f_{10}(a)b)$$  

where each $f$ denotes a power series, and the second subscript denotes the degree of the coefficients. It is represented by

$$x \mapsto f_{00}(x) + f_{01}(x)e, \quad e \mapsto f_{11}(x) + f_{10}(x)e,$$  

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and the requirement that this maps commute with augmentations forces $f_{00}(0) = 0$ and $f_{11}(0) = 0$.

A subformal group over $R^\bullet$ is an abelian group $G$ in the category of subformal $R^\bullet$-sets which is isomorphic to $D^s$ as a subformal $R^\bullet$-set. An isomorphism $D^s \rightarrow G$ is called a parameter. A subformal group structure on $D^s$ itself is a subformal group law. For example, we have the additive subformal group $G^s_a$ defined by

$$G^s_a(S^\bullet) = S^0 \oplus S^1$$

with the direct sum group structure. In general we can express the group law in terms of power series (four of them now), and express the associativity as identities (eight of them) involving them, and similarly for commutativity. But the exercise seems futile.

A homomorphism of subformal groups is a morphism which evaluates to a homomorphism at every nilpotent super algebra. For example, consider $\text{End}_{R^\bullet}(G^s_a)$. Requiring that $f((a, b) + (a', b')) = f(a, b) + f(a', b')$ amounts to the following eight identities.

$$f_{00}(a + a') = f_{00}(a) + f_{00}(a')$$
$$f_{01}(a) = f_{01}(a + a'), \quad f_{01}(a') = f_{01}(a) + f_{01}(a')$$
$$f_{11}(a + a') = f_{11}(a) + f_{11}(a')$$
$$f_{10}(a) = f_{10}(a + a'), \quad f_{10}(a') = f_{10}(a) + f_{10}(a')$$

Thus $f_{00}$ and $f_{11}$ are additive and $f_{01}$ and $f_{10}$ are constant, as functions on nilpotent $R^\bullet$-algebras. A power series representing such a constant function must itself reduce to its constant term. Let us relabel as follows:

$$f_0 = f_{00}, \quad f_1 = f_{11}, \quad c_1 = f_{01}(0), \quad c_0 = f_{10}(0),$$

so our endomorphism has the form

$$(a, b) \mapsto (f_0(a) + c_1b, f_1(a) + c_0b)$$

where $f_0(x)$ is a power series with coefficients in $R^0$ determining an additive function on every nilpotent $R^\bullet$-algebra, $f_1(x)$ is a power series with coefficients in $R^1$ also determining an additive function on every nilpotent
$R^\bullet$-algebra, $c_0 \in R^0$, and $c_1 \in R^1$. Write $f$ for this whole package: $f = (f_0, f_1, c_0, c_1)$. For example the identity endomorphism is $1 = (x, 0, 1, 0)$. The group structure of $\text{End}_{R^\bullet}(G^s_a)$ is given by adding each coordinate in the four-tuple.

Write $g = (g_0, g_1, d_0, d_1)$ for another endomorphism of $G^s_a$. We now compute the composition $h = g \circ f$, writing $h = (h_0, h_1, e_0, e_1)$. Using additivity several times we arrive at

\[
    h_0(x) = g_0(f_0(x)) + d_1 f_1(x)
\]

\[
    h_1(x) = g_1(f_0(x)) + d_0 f_1(x)
\]

\[
    e_0 = g'_1(0)c_1 + d_0 c_0
\]

\[
    e_1 = g'_0(0)c_1 + d_1 c_0
\]

Let us fix $R^\bullet$ and consider $S^\bullet \mapsto \text{End}_{S^\bullet}(G^s_a)$ as a functor on the category of $R^\bullet$-algebras. It is representable by a co-ring object in that category. Specialize to the case in which $pR^\bullet = 0$; in fact, without real loss we can take $R^0 = \mathbb{F}_p$ and $R^0 = 0$. A power series $f(x)$ determines an additive function on every nilpotent $\mathbb{F}_p$-algebra exactly when it is of the form

\[
    f(x) = \sum_{i=0}^{\infty} r_i x^{p^i}.
\]

Thus in this case the representing super algebra has the form

\[
    \mathbb{F}_p[\gamma_0, \xi_0, \xi_1, \ldots] \otimes E[\gamma_1, \tau_0, \tau_1, \ldots]
\]

and the universal endomorphism is given by

\[
    f_0(x) = \sum_{i=0}^{\infty} \xi_i x^{p^i} \quad f_1(x) = \sum_{i=0}^{\infty} \tau_i x^{p^i} \quad c_0 = \gamma_0 \quad c_1 = \gamma_1.
\]

Addition is of course represented by the diagonal with respect to which all the generators $\xi_k, \tau_k, \gamma_0, \gamma_1$, are primitive, and the additive unit is represented by the augmentation sending all these generators to zero.
The composition formulae translate into formulae for a diagonal map representing this product:

\[ \delta \xi_k = \sum_{i+j=k} \xi_i \otimes \xi_j^p + \gamma_1 \otimes \tau_k \]

\[ \delta \tau_k = \sum_{i+j=k} \tau_i \otimes \xi_j^p + \gamma_0 \otimes \tau_k \]

\[ \delta \gamma_0 = \tau_0 \otimes \gamma_1 + \gamma_0 \otimes \gamma_0 \]

\[ \delta \gamma_1 = \xi_0 \otimes \gamma_1 + \gamma_1 \otimes \gamma_0 \]

The augmentation representing the multiplicative unit is determined by \( \xi_0 \mapsto 1, \xi_k \mapsto 0 \) for \( k > 0 \), \( \tau_k \mapsto 0, \gamma_0 \mapsto 1, \gamma_1 \mapsto 0 \).

The general question of picking out the invertible morphisms seems complicated, but we certainly have the “special units,” those morphisms \( f \) for which

\[ f'_0(0) = 1, \quad c_0 = 1, \quad c_1 = 0. \]

The super Hopf algebra representing this functor is precisely the Milnor dual \( A \) of the Steenrod algebra, for \( p \) odd, with its \( \mathbb{Z} \) grading reduced to a \( \mathbb{Z}/2\mathbb{Z} \) grading. Somewhat more generally we may require

\[ f'_0(0) \in (S^0)^\times, \quad c_0 = 1, \quad c_1 = 0, \]

so that the inverse is the four-tuple \( g \) with

\[ g_0(x) = f_0^{-1}(x), \quad g_1(x) = -f_1(f_0^{-1})(x), \quad d_0 = 1, \quad d_1 = 0. \]

This is represented by the semi-tensor product of \( \mathbb{F}_p[\xi_0^\pm] \) with \( A \), and it carries grading information; a comodule for it is precisely equivalent to a graded comodule for the usual (graded) Milnor dual of the Steenrod algebra.

## 5 Comodules and gradings

Gradings play a fundamental role in topology and algebra. A grading is often accompanied with a sign rule for commuting elements. In this section we will explain how gradings over a set \( J \) may be handled in terms of comodules over a coalgebra, and describe a theory of exchange maps.

We begin with a definition.
**Definition 5.1** Let $A$ be a bialgebra over $R$. An $A$-comodule is an $R$-module $M$ together with an $R$-module map $\psi : M \rightarrow M \otimes_R A$ which is unital and associative:

$$
\begin{align*}
M & \xrightarrow{\psi} M \otimes_R A \\
M \otimes_R A & \xrightarrow{1 \otimes \epsilon} M \otimes_R A
\end{align*}
$$

There are two immediate examples: $R$ is an $A$-comodule with coaction $\psi : R = R \otimes_R R \xrightarrow{1 \otimes \eta} R \otimes_R A$, and $A$ is an $A$-comodule with coaction $\delta : A \rightarrow A \otimes_R A$.

**Definition 5.2** Let $J$ be a set and $R$ a commutative ring. A $J$-graded $R$-module is choice of $R$-module $M_j$ for each $j \in J$.

The $R$-module $RJ$ is a commutative coalgebra, with structure maps given by

$$
\Delta[j] = [j] \otimes [j], \quad \epsilon[j] = 1.
$$

**Lemma 5.3** There is an equivalence of categories between the category of $J$-graded $R$-modules and the category of $RJ$-comodules.

**Proof.** Let $M$ be an $RJ$-comodule, with coaction $\psi : M \rightarrow M \otimes_R RJ$. The equation

$$
\psi(x) = \sum_{j \in J} \pi_j x \otimes [j]
$$

defines $R$-linear operators $\pi_j : M \rightarrow M$. Associativity forces

$$
\pi_i \pi_j = 0 \text{ if } i \neq j, \quad \pi_j^2 = \pi_j.
$$
The finiteness of the sum in the expression for $\psi(x)$ implies that for any $x \in M$, $\pi_j x$ is nonzero for only finitely many $j$, and the unital condition then implies that
\[ \sum_{j \in J} \pi_j = 1. \]

Conversely, given the collection $M_j, j \in J$, form
\[ M = \bigoplus_{j \in J} M_j \]
and define $\psi$ by the extending $\psi(x) = x \otimes [j]$ for $x \in M_j$ to a map on the direct sum.

An abelian group structure on $J$ naturally determines a symmetric monoidal structure on the category of $J$-graded $R$-modules:
\[ (M \otimes N)_k = \sum_{i+j=k} M_i \otimes N_j. \]
The unit for this structure is $R$, defined by $R_0 = R, R_j = 0$ for $j \neq 0$.

The abelian group structure on $J$ determines an abelian Hopf algebra structure on $R_J$, and the given symmetric monoidal structure on $J$-graded $R$-modules corresponds to the standard symmetric monoidal structure on the category of comodules over $R_J$: $\psi : M \otimes N \to M \otimes N \otimes A$ by the composite $(1 \otimes 1 \otimes \mu) \circ 1(\otimes T \otimes 1) \circ (\psi \otimes \psi)$, where $T(x \otimes y) = y \otimes x$.

We wish to compare $M \otimes N$ with $N \otimes M$. The canonical switch map $T : M \otimes N \to N \otimes M$ is a comodule isomorphism. Even in this case, however, this may not be the appropriate way to exchange factors. Typically we want a unit to intervene. To express this, suppose we are given a bicharacter: a map
\[ \varphi : J \times J \to R^\times, \]
which is bimultiplicative:
\[ \varphi(i+j, k) = \varphi(i, k) \varphi(j, k), \quad \varphi(i, j+k) = \varphi(i, j) \varphi(i, k), \quad \varphi(0, j) = 1 = \varphi(i, 0). \]
The product of two bicharacters is again a bicharacter, and, because we have assumed that bicharacters take values in $R^\times$, rather than merely in the multiplicative monoid of $R$, the set of bicharacters forms a group with identity element given by the trivial bicharacter $\varphi_0(i, j) = 1$. Precomposing
with the switch map $T$ gives an involution on bicharacters, and a bicharacter $\varphi$ is symmetric if

$$\varphi^{-1} = \varphi T.$$  

A motivating example is given by the symmetric bicharacter

$$\varphi(i,j) = (-1)^{ij}$$
on $J = \mathbb{Z}$. In this case the Hopf algebra $RJ$ is none other than the Hopf algebra representing the multiplicative group scheme $\mathbb{G}$, and, without trying to make this precise, we may regard a $\mathbb{Z}$-graded abelian group as a representation of the multiplicative group scheme.

If $J$ is commutative we may define a new switch map on $RJ$-comodules,

$$c : M \otimes N \to N \otimes M,$$

by

$$c(x \otimes y) = \varphi(j,i)y \otimes x, \quad x \in M_i, \quad y \in N_j.$$  

The map $c$ is a natural isomorphism since we have assumed that $\varphi$ takes values in $R^\times$.

## 6 Cobordism comodules

The functor assigning to a ring $R$ the groupoid $\mathcal{F}(R)$ of formal group laws over $R$ is representable. In this section we will study the representing object and representations of it. These play a fundamental role in the topological applications of formal groups, and these applications confront us with issues of gradings and signs which, from the present perspective, are best handled using the most general framework available for such things, the theory of braided Hopf algebras. The previous section offers a guide to the final definitions.

Fix a ring $R$ and let $\mathcal{F}(R)$ denote the category of formal group laws over $R$ and their isomorphisms. This category is a groupoid, i.e. a small category in which every morphism is an isomorphism. A ring homomorphism $R \to S$ determines a functor $\mathcal{F}(R) \to \mathcal{F}(S)$: $\mathcal{F}$ is a functor from rings to groupoids.

**Lemma 6.1** The functor $\mathcal{F}$ is representable.
Proof. The universal formal group is easily constructed: form the symmetric algebra \( \mathbb{Z}[a_{i,j} : i, j \geq 1] \) and divide it out by the ideal generated by the relations implied by requiring

\[
G_u(s, t) = x + y + \sum_{i,j \geq 1} a_{i,j} s^i t^j
\]
to be a formal group law. The quotient ring is the Lazard ring \( L \). We will discover its structure in Lecture ??, but this information is not essential to us now. Given a formal group \( G/R \), the set of morphisms in \( \mathcal{F} \) into it is precisely given by the set of invertible power series over \( R \): an invertible power series \( \varphi \in R[[t]] \) is an isomorphism to \( G \) from a unique formal group law, namely \( G^\varphi \) given by

\[
G^\varphi(s, t) = \varphi^{-1}(G(\varphi(s), \varphi(t))).
\]

Thus the set of all morphisms in \( \mathcal{F} \) is representable by the ring

\[
W = L[b_0^{\pm 1}, b_1, \ldots]
\]
with universal morphism given by the invertible power series

\[
\varphi_u(t) = \sum_{i=1}^{\infty} b_{i-1} t^i
\]
regarded as an isomorphism to \( G_u \).  

A functor into groupoids is represented by a cogroupoid object, which we describe in the following definition.

Definition 6.2 A Hopf algebroid is a cogroupoid object in the category of commutative rings.

Explicitly, we have a pair of rings \( (A, H) \) together with structural ring homomorphisms

- source: \( \eta_R : A \to H \)
- target: \( \eta_L : A \to H \)
- identity: \( \epsilon : H \to A \)
- composition: \( \delta : H \to H \otimes_A H \)
- inverse: \( \chi : H \to H \)
satisfying the axioms below. There, and in the expression for the target of \( \delta \), we will systematically regard \( H \) as a left \( A \)-module via \( \eta_L \) and a right \( A \)-module via \( \eta_R \). When we right \( M \otimes_A N \) it is to be understood that \( M \) is a right \( A \)-module and \( N \) is a left \( A \)-module. If we wish to form the left \( A \)-module tensor product of two left \( A \)-modules, we will write \( M^A \otimes N \). If we wish to form the right \( A \)-module tensor product of two right \( A \)-modules, we will write \( M \otimes^A N \). The following diagrams are required to commute.

\[
\begin{align*}
A & \xrightarrow{\eta_L} H & H & \xleftarrow{\eta_R} A \\
\downarrow \text{id} & & \downarrow \text{id} & \\
A & & A & \xrightarrow{\epsilon}
\end{align*}
\]

\[
\begin{align*}
H & \xrightarrow{\delta} H \otimes_A H \\
\downarrow \delta & & \downarrow \delta \otimes_A 1 \\
H^A \otimes H & \xrightarrow{1 \otimes_A \delta} H \otimes_A H \otimes_A H
\end{align*}
\]

Definition 6.3 Suppose a group \( G \) acts from the right on a set \( X \), with action map \( X \times G \rightarrow X \). The translation groupoid associated to this group action is defined as follows. The set of objects is \( X \); the set of morphisms is \( X \times G \); and we have structure maps

- source: \((x, g) \mapsto xg\)
- target: \((x, g) \mapsto x\)
- identity: \(x \mapsto (x, 1)\)
- composition: \((x, g), (xg, h) \mapsto (x, gh)\)
- inverse: \((x, g) \mapsto (xg, g^{-1})\).
A groupoid is split if it is isomorphic to the translation groupoid of a group action.

Let $\mathcal{F}_0(R)$ be the set of formal group laws over $R$. It is acted on by the group of coordinate changes $\Gamma(R)$. This is the group (under composition) of composition-invertible power series (without constant term) over $R$. $f$ is invertible if and only if $f'(0)$ is a unit in $R$. If $f \in \Gamma(R)$ and $F \in \mathcal{F}_0(R)$, $F$ acted on by $f$ is given by

$$F^f(x, y) = f^{-1}(F(f(x), f(y))).$$

It is easy to check that $F^f$ is again a formal group law over $R$, and we will call it $F$ conjugated by $f$. The power series $f$ is an isomorphism of formal group laws

$$f : F^f \to F.$$

**Lemma 6.4** The category $\mathcal{F}(R)$ is naturally isomorphic to the translation category of this action. \[\square\]

The functor $\mathcal{F}_0 : \text{Ring} \to \text{Set}$ is corepresentable by the Lazard formal group law $G_u/L$. As usual, any two representing pairs are canonically isomorphic, so one speaks of the representing object. This reflects an unspoken habit of mathematics: one tends to think of the objects in a unicursal category—that is, one with exactly one morphism between any two objects—as identical. This habit will be given some exercise below.

The entire groupoid-valued functor $\mathcal{F}$ is representable as well. The $\mathbb{Z}$-group $\Gamma$ is represented by a Hopf algebra with underlying algebra

$$B = \mathbb{Z}[b_0^{\pm 1}, b_1, b_2, \ldots]$$

together with the universal invertible power series

$$\theta(t) = \sum_{i=1}^{\infty} b_{i-1} t^i \in B[[t]].$$

In $B$, $\epsilon b_0 = 1$, $\epsilon b_i = 0$ for $i > 0$, and the diagonal represents the composition of power series, and so is determined by the equation

$$\sum_k (\Delta b_{k-1}) t^k = \sum_i (b_{i-1} \otimes 1)(\sum_j 1 \otimes b_{j-1} t^j)^i.$$
The action $\mathcal{F}_0 \times \Gamma \to \mathcal{F}_0$ is represented by a coaction $\psi : L \to B \otimes L$, making $L$ into a comodule algebra for the Hopf algebra $B$: the structure maps for the algebra $L$ are $B$-comodule morphisms. This coaction determines a cogroupoid object in $\text{Ring}$, a Hopf algebroid, representing the groupoid-valued functor $\mathcal{F}$. Morphisms are represented by $W = B \otimes L$, and there are structure maps

- source: $\eta_R = \eta_B \otimes 1 : L \to W$
- target: $\eta_L = \psi : L \to W$
- identity: $\epsilon = \epsilon_B \otimes 1 : W \to L$
- composition: $\Delta = \Delta_B \otimes 1 : W \to W \otimes_W W.$

This cogroupoid represents $\mathcal{F}$. Thus $W$ supports the universal isomorphism between formal groups, namely,

$$\theta : \eta_R G_u \to \eta_L G_u.$$

This perspective is due to Peter Landweber [?].

We now turn to “representations” of this structure.

**Definition 6.5** An even cobordism comodule (or comodule, for short) is a right $L$-module $M$ together with an $L$-module map $\psi : M \to M \otimes_L W = M[b_0^{\pm 1}, b_1, \ldots]$ such that

$$M \xrightarrow{\psi} M \otimes_L W \xrightarrow{1 \otimes_L \psi} M \otimes_L W \otimes_L W$$

commute.

Here and below we systematically regard $W$ as an $L$-bimodule, using $\eta_L$ to define the left $L$-module structure and $\eta_R$ to define the right $L$-module structure.

Equivalently, an even cobordism comodule is a $B$-comodule $M$ with an $L$-module structure $\psi : L \otimes M \to M$ which is a $B$-comodule map. This may be called an $L$-module over $B$. $L$ is a monoid object in the monoidal category of $B$-comodules, and $M$ is an object with a right action of this monoid.
The category of even cobordism comodules forms an abelian category. The existence of kernels depends upon the fact that \( W \) is flat over \( L \).

We will digress to describe some alternative perspectives on this category, and a mild extension.

The \( \mathbb{Z} \)-group \( \Gamma \) sits in an extension

\[
\Gamma_1 \xrightarrow{i} \Gamma \xrightarrow{\pi} \mathbb{G}_m
\]

where \( \Gamma_1(R) \) is the group of power series over \( R \) (under composition) with leading term \( t \) and \( \mathbb{G}_m \) is the multiplicative group scheme. \( i \) is the obvious inclusion and \( \pi(f(t)) = f'(0) \). This extension is split by \( \sigma(r) = rt \). By conjugating with the splitting we obtain an action of \( \mathbb{G}_m \) on \( \Gamma_1 \), namely

\[
(f \cdot r)(t) = r^{-1}f(r t).
\]

An action of \( \Gamma \) on a \( \mathbb{Z} \)-set \( X \) is the same thing as an action of \( \mathbb{G}_m \) on \( X \) together with an equivariant action of \( \Gamma_1 \) on \( X \).

Equivalently, the Hopf algebra \( B \) sits in a split coextension

\[
B_0 \xrightarrow{i} B \xrightarrow{p} B_1
\]

of Hopf algebras. Here, as in Chapter ??, \( B_0 \) is the Hopf algebra representing \( \mathbb{G}_m, B_0 = \mathbb{Z}[b_0^\pm 1] \). \( B_1 = B/(b_0 - 1) \) is the “Landweber-Novikov algebra,” corepresenting \( \Gamma_1 \). In saying that this is a split coextension we mean that the composite \( pi \) factors as \( B_0 \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\eta} B_1 \), \( p \) is surjective, \( B_0 \) maps isomorphically to the cotensor product (or Hopf kernel)

\[
\mathbb{Z} \square_{B_1} B = \{ x \in B : (p \otimes 1)\Delta x = 1 \otimes x \},
\]

and there is a Hopf algebra map \( s : B \rightarrow B_0 \) such that \( si \) is the identity map of \( B_0 \). The maps are given by \( i(b_0) = b_0, p(b_0) = 1, p(b_k) = b_k \) for \( k > 0 \), and \( s(b_0) = b_0, s(b_k) = 0 \) for \( k > 0 \).

This extension imposes a coaction of \( B_0 \) on the ring \( B_1 \) given by

\[
\psi b_n = b_n \otimes b_0^n.
\]

As we saw in Section 5, this coaction amounts to a grading on the Hopf algebra \( B_1 \), with \( |b_n| = 2n \), and a \( B \)-comodule is simply an evenly graded \( B_1 \)-comodule. Consequently a \( W \)-comodule is precisely what traditionally would be called an evenly graded \( MU, MU \)-comodule.
Not every $MU_\ast MU$-comodule is evenly graded, and we now describe the full category of $MU_\ast MU$-comodules using the apparatus of signed Hopf algebras developed in Section 5. To begin with consider comodules over the Hopf algebra $B$. We equip it with the trivial bicharacter.

To extend to general gradings we extend the Hopf algebra $B$ to

$$\tilde{B} = \mathbb{Z}[e^{\pm 1}, b_1, \ldots] = \tilde{B}_0 \otimes_{B_0} B.$$ 

It represents the functor $\tilde{\Gamma}$ sending a ring $R$ to the group of pairs $(f, u)$ consisting of an invertible power series $f$ over $R$ and an element $u \in R$ such that $u^2 = f'(0)$. Equip $\tilde{B}$ with the bicharacter determined by

$$\langle e, e \rangle = -1, \quad \langle e, b_i \rangle = \langle b_i, e \rangle = \langle b_i, b_j \rangle = 1.$$ 

There is a splitting map $\tilde{B} \to \tilde{B}_0$, compatible with the bicharacters. Then the category of $\tilde{B}$-comodules is equivalent with its symmetric monoidal structure to the category of $\mathbb{Z}$-graded $B_1$-comodules.

Like any other $B$-comodule, $L$ may be considered as a $\tilde{B}$-comodule. It is a commutative monoid in this category, and we make the following

**Definition 6.6** A cobordism comodule is an $L$-module over $\tilde{B}$.

This again forms an abelian category. The symmetric monoidal structure on the category of $\tilde{B}$-comodules yields one on the cobordism comodules by the usual method: given to cobordism comodules, $M$ and $N$, form the coequalizer diagram

$$M \otimes N \otimes L \Rightarrow M \otimes N \to M \otimes^L N$$

in the category of $\tilde{B}$-comodules. The maps are given by $x \otimes y \otimes a \mapsto xa \otimes y$ and $x \otimes y \otimes a \mapsto x \otimes ya$, and the coequalizer is naturally an $L$-module over $\tilde{B}$.

**Remark 6.7** The interest of this concept in Topology arises from a construction due to Frank Adams [?, ?]. He considered a commutative and associative ring spectrum $E$ with the property that $E_\ast(E)$ is flat as an $E_\ast$-module. Adopt the convention that $E_\ast X = \pi_\ast(X \wedge E)$, so $E_\ast X$ is naturally a right $E_\ast$-module. Then “inner” Künneth map

$$E_\ast(X) \otimes_{E_\ast} E_\ast(E) \to E_\ast(X \wedge E)$$

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is a morphism between homology theories which is an isomorphism on the sphere, and hence is an isomorphism for any spectrum $X$. If we let $\eta : S \to E$ denote the unit map for this ring spectrum, we may thus form the map $\psi$ in

$$
\begin{array}{c}
E_*(X) \\
\downarrow_{E_*(1 \wedge \eta)} \\
E_*(X \wedge E) \\
\overset{\cong}{\leftarrow} \\
E_*(X) \otimes_{E_*} E_*(E)
\end{array}
$$

In the particular case of $X = E$ we obtain a diagonal map which renders $(E_*, E_*(E))$ a Hopf algebroid, and in general an $(E_*, E_*(E))$-comodule structure on $E_*(X)$. The case we have focused on is $E = MU$. There is a “right” Künneth map of cobordism comodules

$$
MU_*(X) \otimes^L MU_*(Y) \to MU_*(X \wedge Y).
$$