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REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

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REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

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The purpose of these lectures is to give an elementary introduction to some basic topics in the theory of representations of semisimple Lie groups. Within harmonic analysis I have limited myself to a special topic which is now fairly well-developed, namely Fourier analysis of spherical functions.

The prerequisites for reading these notes are some familiarity with the theory of semisimple Lie groups; this is now easily available in many books.

The following standard notation will be used: \( \mathbb{R}, \mathbb{C}, \mathbb{Z} \) denote, respectively, the field of real numbers, the field of complex numbers, the ring of integers. Also \( \mathbb{R}^+ \) and \( \mathbb{Z}^+ \) denote the set of nonnegative elements in \( \mathbb{R} \) and \( \mathbb{Z} \), respectively. If \( G \) is a locally compact group \( dg \) denotes a left invariant Haar measure on \( G \). If \( G \) is unimodular and \( H \) a closed unimodular subgroup the \( G \)-invariant measure on the left coset space \( G/H \) derived from \( dg \) and \( dh \) is denoted \( dg_H \). Thus

\[
\int_G f(g)dg = \int_{G/H} (\int_H f(gh)dh)dg_H
\]

if \( f \) is continuous on \( G \) with compact support. We use Schwartz' notation \( \mathcal{S}(M) \) for the space of \( C^\infty \) functions of compact support on a manifold \( M \).
§1. Complex semisimple Lie algebras.

Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{R} \) or \( \mathbb{C} \); for \( X \in \mathfrak{g} \), \( \text{ad} X \) (or \( \text{ad}_{\mathfrak{g}} X \)) denotes the linear transformation \( Y \to [X,Y] \). The bilinear form

\[
\langle X, Y \rangle = \text{Tr}(\text{ad} X \text{ad} Y)
\]

(Tr = Trace)

is called the Killing form of \( \mathfrak{g} \). The Lie algebra \( \mathfrak{g} \) is called semisimple if \( \langle , \rangle \) is nondegenerate. Let \( \mathfrak{gl}(\mathfrak{g}) \) denote the Lie algebra of all linear transformations of \( \mathfrak{g} \); it is identified with the Lie algebra of the group \( \text{GL}(\mathfrak{g}) \) of all nonsingular linear transformations of \( \mathfrak{g} \). Let \( \text{Int}(\mathfrak{g}) \) denote the analytic subgroup of \( \text{GL}(\mathfrak{g}) \) with Lie algebra \( \text{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g}) \); this analytic group is called the adjoint group of \( \mathfrak{g} \). If \( G \) is a Lie group with Lie algebra \( \mathfrak{g} \) let for \( g \in G \), \( \text{Ad}_g(g) \) (or just \( \text{Ad}(g) \)) denote the automorphism of \( \mathfrak{g} \) which corresponds to the inner automorphism \( x \to g x g^{-1} \) of \( G \). Then if \( G \) is connected \( \text{Ad}_g \) is a homomorphism of \( G \) onto \( \text{Int}(\mathfrak{g}) \). A Lie algebra is called compact if \( \text{Int}(\mathfrak{g}) \) is a compact Lie group. If this is the case the topology of \( \text{Int}(\mathfrak{g}) \) is the relative topology of \( \text{GL}(\mathfrak{g}) \).

Let \( \mathfrak{u} \) be a compact semisimple Lie algebra over \( \mathbb{R} \), \( \mathfrak{t} \subset \mathfrak{u} \) a maximal abelian subalgebra. Let \( \mathfrak{g} \) be the complexification of the vector space \( \mathfrak{u} \) with the bracket operation in \( \mathfrak{g} \) given by

\[
[X+iY, Z+iT] = [X, Z] - [Y, T] + i([Y, Z] + [X, T])
\]
whereby $\mathfrak{g}$ becomes a semisimple Lie algebra over $\mathfrak{c}$. Let $\mathfrak{h}$ denote the subalgebra $\mathfrak{h} + i\mathfrak{t}$ of $\mathfrak{g}$. The Killing form $\langle , \rangle$ of $\mathfrak{g}$ gives by restriction to $\mathfrak{u} \times \mathfrak{u}$ the Killing form of $\mathfrak{u}$ and this is strictly negative definite, $\mathfrak{u}$ being compact. Each endomorphism $\text{ad}_{\mathfrak{u}}(X)$ ($X \in \mathfrak{u}$) is skew symmetric with respect to this negative definite bilinear form, hence $\text{ad}_{\mathfrak{u}}(X)$ is a completely reducible linear transformation. It follows that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, that is, it is maximal abelian and for each $H \in \mathfrak{h}$ the linear transformation $\text{ad}_H$ is completely reducible. We can therefore diagonalize these linear transformations simultaneously; for each linear function $\alpha \neq 0$ on $\mathfrak{h}$ we define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)X \text{ for } H \in \mathfrak{h}\}$$

and call $\alpha$ a root (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) if $\mathfrak{g}_\alpha \neq 0$.

Let $\Delta$ denote the set of all roots; then we have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha \quad \text{(direct sum)}.$$

For this decomposition we shall need the following properties:

(i) $\dim \mathfrak{g}_\alpha = 1$.

(ii) The Killing form is nondegenerate on $\mathfrak{h} \times \mathfrak{h}$; thus for each linear form $\lambda$ on $\mathfrak{h}$ there exists a unique vector $H_\lambda \in \mathfrak{h}$ such that $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{h}$. Put $\langle u, \lambda \rangle = \langle H_\lambda, H_u \rangle$.

(iii) For each $\alpha, \beta \in \Delta$

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$
(iv) Put $\mathfrak{h}^* = \sum_{\alpha \in \Delta} E \mathfrak{h}\mathfrak{h}_\alpha$. Then $\mathfrak{h}^* = i\mathfrak{t}$ so by (iii) each $\beta$ is real on $\mathfrak{h}^*$.

(v) $[\mathfrak{g}_\alpha^\alpha, \mathfrak{g}_\alpha^{-\alpha}] = C\mathfrak{h}_\alpha$ for $\alpha \in \Delta$.

Let $G$ be a simply connected Lie group with Lie algebra $\mathfrak{g}$, $T$ and $U$ the analytic subgroups corresponding to $\mathfrak{t}$ and $\mathfrak{u}$, respectively.

The Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ is by definition the group of linear transformations of $\mathfrak{h}$ induced by those members of $\text{Int}(\mathfrak{g})$ which leave $\mathfrak{h}$ invariant. Since each $s \in W$ permutes the vectors $H_\alpha (\alpha \in \Delta)$ which span $\mathfrak{h}$, $W$ is finite. By (v) select $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = -H_\alpha$ and put $X = \pi 2^{-\frac{1}{2}} \langle \alpha, \omega \rangle^{-\frac{1}{2}} (X_\alpha + X_{-\alpha})$. Then a simple computation shows that

$$e^{ADX} = H - 2\frac{\alpha(H)}{\langle \alpha, \omega \rangle} H_\alpha$$

so that if $x = \exp X$, $Ad(x)$ restricted to $\mathfrak{h}$ constitutes the symmetry $s_\alpha$ of $\mathfrak{h}$ with respect to the hyperplane $\alpha = 0$. Thus $s_\alpha \in W$ for each $\alpha \in \Delta$. Consider now the open set in $\mathfrak{h}^*$ where all roots are non-zero; the components of this open set are called Weyl chambers. We fix one of these, say $\mathfrak{h}^+$. It is geometrically easy to see that the subgroup $W'$ of $W$ generated by the $s_\alpha$ permutes the Weyl chambers transitively; on the other hand it can be proved that if $s \in W$, $s \mathfrak{h}^+ = \mathfrak{h}^+$ then $s = \text{identity}$. Thus $W$ is simply transitive on the set of Weyl chambers and is generated by the $s_\alpha (\alpha \in \Delta)$.
A root α is called positive if its values on \( \mathfrak{h}^+ \) are positive. Let \( \Delta^+ \) denote the set of positive elements in \( \Delta \). A root \( \alpha \in \Delta^+ \) is called simple if it cannot be written as a sum \( \alpha = \beta + \gamma \) where \( \beta, \gamma \in \Delta^+ \). Let \( \alpha_1, \ldots, \alpha_l \) be the set of all simple roots. Then \( l = \dim \mathfrak{h}^* \) and each \( \alpha \in \Delta^+ \) can be written \( \alpha = \sum_{i=1}^{l} n_i \alpha_i \) where \( n_i \in \mathbb{Z}^+ \). The hyperplanes in \( \mathfrak{h}^* \) forming the walls of \( \mathfrak{h}^+ \) are given by \( \alpha_1 = 0, \ldots, \alpha_l = 0 \).

The unit lattice \( t_e \subset \mathfrak{t} \) is defined by
\[
t_e = \{ H \in \mathfrak{t} \mid \exp H = e \}.
\]

The following theorem shows how the unit lattice is determined from the root system.

**Theorem 1.1.** The unit lattice \( t_e \) for a compact semisimple simply connected Lie group \( U \) is generated by the vectors
\[
\frac{4\pi i}{\langle \alpha, \alpha \rangle} H_{\alpha} \quad \alpha \in \Delta.
\]

The notions and results of this section apply to an arbitrary semisimple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \), because any such \( \mathfrak{g} \) has a compact real form \( \mathfrak{U} \). The customary proof of this starts by constructing a Cartan subalgebra of \( \mathfrak{g} \), appealing to theorems of Engel and Lie for nilpotent and solvable Lie algebras. A different approach was attempted by Cartan and was carried out by Richardson [30]. Cartan's idea was as follows: The Killing form is nondegenerate so there exists a basis \( e_1, \ldots, e_n \) of \( \mathfrak{g} \) such that
\[ \langle Z, Z \rangle = \sum_{i=1}^{n} z_i^2 \quad \text{if} \quad Z = \sum_{i=1}^{n} z_i e_i. \]

Let the structural constants \( c_{ijk} \) be determined by
\[ [e_i, e_j] = \sum_{k=1}^{n} c_{ijk} e_k. \]

Then
\[ \langle Z, Z \rangle = \text{Tr}(\text{ad}Z Z) = \sum_{i,j,h,k} (\sum_{i} c_{ikh} c_{jkh}) z_i z_j. \]

so by (1)
\[ \sum_{h,k} c_{ikh} c_{jkh} = \delta_{ij}. \]

But
\[ \langle [e_i, e_j], e_k \rangle + \langle e_j, [e_i, e_k] \rangle = 0 \]
so
\[ c_{ijk} + c_{ikj} = 0 \]
whence by (2)
\[ \sum_{i,h,k} c_{ikh}^2 = n. \]

The space \( \mathcal{U} = \sum_{l=1}^{n} \mathbb{R} \mathbf{e}_l \) is a real form of \( \mathcal{G} \) if and only if all the \( c_{ijk} \) are real.

Let \( \mathcal{J} \) denote the set of all bases \( (e_1, \ldots, e_n) \) of \( \mathcal{G} \) such that (1) holds. Consider the function \( f \) on \( \mathcal{J} \) given by
\[ f(e_1, \ldots, e_n) = \sum_{i,j,k} |c_{ijk}|^2. \]
Then
\[ \sum_{i,j,k} |c_{ijk}|^2 \geq | \sum_{i,j,k} c_{ijk} | = \sum_{i,j,k} c_{ijk}^2 = n \]
and the equality sign holds if and only if the \( c_{ijk} \) are real. Thus to prove \( \forall \) a real form one must prove

1) \( f \) has a minimum on \( J \).

2) The minimum value is attained at a point \( (e_1^0, \ldots, e_n^0) \) for which the structural constants are real.

These results are established in the quoted paper by Richardson.

§2. Finite dimensional representations.

**Definition.** Let \( \pi \) be a representation of \( \mathfrak{g} \) on a finite-dimensional vector space \( V \). A linear function \( \lambda \) on \( \mathfrak{h} \) is called a **weight** of \( \pi \) if there exists a vector \( v \neq 0 \) in \( V \) such that
\[ \pi(H)v = \lambda(H)v \quad \text{for } H \in \mathfrak{h}. \]

In this section we consider finite-dimensional representations only. Let \( G, T \) and \( U \) be as in §1. A representation \( \pi \) of \( \mathfrak{g} \) on \( V \) induces a representation of \( G \) on \( V \), also denoted \( \pi \). Since \( U \) is compact there exists a Hermitian inner product on \( V \times V \) invariant under each \( \pi(u), \ u \in U \). It follows that \( \pi(T) \) and therefore also \( \pi(\mathfrak{h}) \) form a commutative
family of completely reducible endomorphisms of \( V \) and we have a decomposition

\[
V = \sum_{\lambda} V_{\lambda} \quad \text{(direct sum)}
\]

where \( \lambda \) runs over the weights of \( \pi \) and

\[
V_{\lambda} = \{ v \in V \mid \pi(H)v = \lambda(H)v \quad \text{for} \quad H \in \mathfrak{h} \}.
\]

Since the transformations \( \pi(t) \) are skewhermitian with respect to the mentioned inner product and since \( \mathfrak{h}^* = it \) we deduce that the weights are real-valued on \( \mathfrak{h}^* \).

Theorem 2.1. Let \( \mathfrak{t}_e \) denote the unit lattice and \( \Lambda \) the set of weights of all representations of \( \mathfrak{g}_f \). Then if \( \lambda \) is a linear function on \( \mathfrak{h} \),

\[
\lambda \in \Lambda \iff \lambda(H_0) \in 2\pi i \mathbb{Z} \quad \text{for all} \quad H_0 \in \mathfrak{t}_e
\]

and if \( H \in \mathfrak{t} \),

\[
H \in \mathfrak{t}_e \iff \lambda(H) \in 2\pi i \mathbb{Z} \quad \text{for all} \quad \lambda \in \Lambda.
\]

Proof. Let \( \lambda \) be a weight for \( \mathfrak{g}_f \) and \( \pi \) a finite-dimensional representation of \( \mathfrak{g}_f \) on \( V \) such that \( \pi(H)v = \lambda(H)v \) for some \( v \neq 0 \) in \( V \). Then \( \pi(\exp H)v = e^{\lambda(H)}v \) so half of (3) follows. On the other hand if \( \lambda \) satisfies (3) there exists a homomorphism \( \gamma: T \to \mathbb{C} \) such that \( \gamma(\exp H) = e^{\lambda(H)} \).

By a standard theorem about representations of compact Lie groups (see Chevalley [4] p. 191) there exists a finite-dimensional representation \( \pi \) of \( U \) on a space \( V \) such
that the restriction of $\pi$ to $T$ contains $V$, that is $\pi(t)v = \gamma(t)v$ \( t \in T \) for some $v \neq 0$ in $V$. But then $\pi(\exp H)v = e^{\lambda(H)}v$ so $\lambda$ is a weight. This proves (3) and part $\Rightarrow$ of (4). Suppose finally $\lambda(H) \in 2\pi i \mathbb{Z}$ for each $\lambda \in \Lambda$. Using (1) we deduce that for any finite-dimensional representation $\pi$ on vector space $V$, $\pi(\exp H)v = v$ for all $v \in V$. But since $U$ has a faithful representation, $H \in \mathfrak{t}_e$.

**Corollary 2.2.** The map $\lambda \mapsto e^{\lambda}$ identifies $\Lambda$ with the character group of $T$. Moreover if $\lambda$ is a linear function on $\mathfrak{h}$,

$$\lambda \in \Lambda \iff \frac{2\langle \lambda, \alpha_\alpha \rangle}{\langle \alpha_\alpha, \alpha_\alpha \rangle} \in \mathbb{Z} \quad \text{for all} \quad \alpha \in \Delta. \tag{5}$$

This is immediate from Theorems 1.1 and 2.1. Note that since the roots are the nonzero weights of the adjoint representation we have $\Delta \subset \Lambda$ in accordance with (iii) in §1 and Cor. 2.2.

If $\alpha_1, \ldots, \alpha_k$ are the simple roots let the linear functions $\lambda_1, \ldots, \lambda_k$ on $\mathfrak{h}$ be determined by

$$2\frac{\langle \lambda_\alpha, \alpha_\alpha \rangle}{\langle \alpha_\alpha, \alpha_\alpha \rangle} = \delta_{ij}.$$

The $\lambda_\alpha$ are called the fundamental weights. The set $\Lambda$ consists of the integral linear combinations of the fundamental weights. The vectors $H_{\lambda_\alpha}$ form the edges of the Weyl chamber $\mathfrak{h}^+.

Let $\pi$ be a representation of $\mathfrak{g}$ on $V$. The dimension
m_\lambda = \dim V_\lambda \text{ in (1) is called the multiplicity of the weight } \lambda.
The function \( \chi(g) = \text{Tr}(\pi(g)) \) is called the character of \( \pi \).
Two representations with the same character are equivalent (see e.g. Chevalley [4]). We shall now give the character \( \chi \) of the irreducible representation \( \pi \). It suffices to compute \( \chi \) on \( U \) and since each \( u \in U \) is conjugate to an element in \( T \) and since \( \chi \) is invariant under conjugation it suffices to compute \( \chi \) on \( T \). Now

\[
\chi(\exp H) = \sum_\lambda m_\lambda e^{\lambda(H)},
\]

where \( \lambda \) runs over the weights of \( \pi \). By the invariance of \( \chi \) mentioned it follows that the set of weights of \( \pi \) is invariant under the Weyl group \( W \) and that \( m_{s\lambda} = m_\lambda \) (\( s \in W \)).

Let \( J \) be the set of all linear functions on \( \mathfrak{n}^\vee \), real valued on \( \mathfrak{n}^\vee \). Each \( \lambda \in J \) can be written \( \lambda = \sum_1^r c_i \alpha_i \) (\( c_i \in \mathbb{R} \)). We give \( J \) a lexicographic ordering with respect to the basis \( \alpha_1, \ldots, \alpha_r \). An element \( \lambda \in J \) is called integral (resp. dominant integral) if \( 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z} \) for \( \alpha \in \Delta \) (resp. \( 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}^+ \) for \( \alpha \in \Delta^+ \)). If \( \lambda \in \Lambda \) then exactly one of the transforms \( s\lambda \) is dominant integral.

Theorem 2.3. Let \( \lambda \) be a dominant integral function. Then there exists an irreducible finite-dimensional representation \( \pi \) of \( \mathfrak{g} \) (unique up to equivalence) such that \( \pi \) has highest weight \( \lambda \). Moreover \( m(\lambda) = 1 \).
The character $\chi$ of $\pi$ is given by

$$\chi(\exp H) = \frac{\sum_{s \in W} \varepsilon(s) e^{s(\lambda + \rho)}(H)}{\sum_{s \in W} \varepsilon(s) e^{s\rho}(H)}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in A^+} \alpha$ and $\varepsilon(s)$ denotes the determinant of $s$ as a linear transformation.

Thus $\Gamma/W$ gives an explicit parametrization of the set of finite-dimensional irreducible representations of $G$.

While the determination of the characters in principle determines all the representations it is still desirable to find explicit models for the representation spaces. Here is of course no final or unique solution because equivalent models may look quite different. One model described below is given by means of eigenspaces of differential operators. For other models see [33] and [38]. Let $N$ denote the analytic subgroup of $G$ with Lie algebra $\sum_{\alpha \in A^+} \mathfrak{g}_\alpha$. The group $N$ is closed in $G$.

**Theorem 2.4.** Let $D(G/N)$ denote the algebra of $G$-invariant differential operators on $G/N$ (as a real analytic manifold). Then

(i) $D(G/N)$ is commutative.

(ii) For each algebra homomorphism $\gamma: D(G/N) \to \mathbb{C}$
let $V_\gamma$ denote the space of holomorphic functions $f$ on $G/N$ satisfying

\begin{equation}
Df = \gamma(D)f \quad \quad D \in \mathcal{D}(G/N)
\end{equation}

and let $\pi_\gamma$ denote the natural representation of $G$ on $V_\gamma$. As $\gamma$ varies, $\pi_\gamma$ runs through all the irreducible finite-dimensional representations of $G$.

For a proof see [24], Ch. IV. One can try to generalize this by dropping the condition of holomorphy and consider the space $\mathcal{A}_\gamma$ of all $C^\infty$ solutions of the system (6), with the topology induced by the usual locally convex topology of $C^\infty(G/N)$. On the basis of results of Zhelobenko [37] and Zhelobenko-Naimark [36] one can explicitly determine all the (exceptional) $\gamma$ for which $\mathcal{A}_\gamma$ fails to be irreducible and verify that all completely irreducible representations of $G$ (with some specific exceptions) arise in this way (up to weak equivalence); cf. [24], Ch. IV.

More generally, if $Q$ is a Lie group and $P$ a closed subgroup, $\mathcal{D}(Q/P)$ the set of $Q$-invariant differential operators on the manifold $Q/P$, the joint eigenspaces of the operators in $\mathcal{D}(Q/P)$ give representation spaces for the group $Q$. For other examples of irreducible representations constructed in this way see [24], Ch. IV.
§3. Real semisimple Lie groups.

We shall now recall the part of the structure theory of semisimple Lie groups which is necessary for the sequel.

Let \( g \) be a semisimple Lie algebra over \( \mathbb{R} \), \( \langle , \rangle \) the Killing form; \( g \) has a Cartan involution, that is an involutive automorphism \( \theta \) such that the symmetric bilinear form \( B_\theta: (X,Y) \to -\langle X, \theta Y \rangle \) is positive definite. Let \( g = k + p \) be the (Cartan) decomposition into eigenspaces of \( \theta \), \( (k \) the fixed point set). Each \( \text{ad}X (X \in p) \) is symmetric with respect to \( B_\theta \) so if \( \alpha \subset p \) is a maximal abelian subspace the endomorphisms \( \text{ad}H (H \in \alpha) \) of \( g \) admit a simultaneous diagonalization. Thus if \( \alpha \) is a real-valued linear function on \( \alpha \) put

\[
g_\alpha = \{ X \in g \mid [H,X] = \alpha(H)X \text{ for } H \in \alpha \}.
\]

If \( \alpha \neq 0 \) and \( g_\alpha \neq \{ 0 \} \) then \( \alpha \) is called a restricted root and \( m_\alpha = \dim( g_\alpha ) \) is called its multiplicity. If \( \Sigma \) is the set of restricted roots,

\[
(1) \quad g = \bigoplus_{\alpha \in \Sigma} g_\alpha + g_0 \quad \text{(direct sum)}
\]

and \( g_0 = \mathcal{N} + \mathcal{Z} \) where \( \mathcal{N} \) is the centralizer of \( \alpha \) in \( k \). Select a component \( \alpha^+ \) of the open set in \( \alpha \) where all restricted roots are \( \neq 0 \), and call a restricted root positive if it has positive values on the Weyl chamber \( \alpha^+ \).

Let \( \Sigma^+ \) denote the set of positive restricted roots and put
\[ \gamma = \sum_{\alpha > 0} \alpha' \alpha, \quad \rho = \frac{1}{2} \sum_{\alpha > 0} (\dim \alpha') \alpha. \]

Let \( G \) be any connected Lie group with finite center with Lie algebra \( \mathfrak{g} \), \( K, A, N \) the analytic subgroups corresponding to \( \mathfrak{k}, \mathfrak{a} \) and \( \gamma \), respectively. Let \( M \) (resp. \( M' \)) denote the centralizer (resp. normalizer) of \( A \) in \( K \) and \( W \) the (finite) factor group \( M'/M \) (the Weyl group). Let \( w \) denote the order of \( \mathfrak{w} \) and \( m_1', \ldots, m_W' \) a complete set of representatives in \( M' \) (mod \( M \)). Let \( A^+ = \exp \mathfrak{a}^+ \), \( C\mathfrak{z} = \text{closure} \), \( P \) the group \( MAN \). Then we have the decompositions

(2) \[ G = K C\mathfrak{z}(A^+)K \quad \text{(Cartan decomposition)} \]

(3) \[ G = KAN \quad \text{(Iwasawa decomposition)} \]

(4) \[ G = \bigcup_{i=1}^{W} P m_i' P \quad \text{(Bruhat decomposition).} \]

Here (2) means that each \( g \in G \) can be written \( g = k_1 a(g) k_2 \) where \( k_1, k_2 \in K \) and \( a(g) \) is unique in \( C\mathfrak{z}(A^+) \). In (3) each \( g \in G \) can be uniquely written

\[ g = k(g) \exp H(g) n(g), \quad k(g) \in K, \quad H(g) \in \mathfrak{a}, \quad n(g) \in N. \]

In (4) the union is a disjoint union.

These decompositions enter constantly in analysis on \( G \). While (2) goes back to É. Cartan, (3) was proved by Iwasawa [27] where the extension from complex \( G \) to real \( G \) is attributed to Chevalley; (4) was proved at least for \( G = \text{SL}(n, \mathbb{C}) \) by Gelfand-Naimark[10], by F. Bruhat for the remaining complex classical groups and by Harish-Chandra [16] for all real \( G \).
The system $\Sigma^+$ of positive restricted roots has properties very similar to the system $\Delta^+$ in §1. In particular property (iii) in §1 holds for $\gamma^+$, so if two members of $\gamma^+$ are proportional the proportionality factor is $\frac{1}{2}$, 1 or 2.

If we consider the subsystem $\gamma_0^+ = \{ \alpha \in \Sigma^+ \mid \frac{1}{2} \alpha \notin \Sigma^+ \}$ it is known ([31],[1]) that $\gamma_0^+$ is a root system for a complex semisimple Lie algebra. In particular, if $\ell = \dim A$ there exist elements $\alpha_1, \ldots, \alpha_\ell \in \Sigma^+$ (called simple restricted roots) such that each $\alpha \in \Sigma^+$ is a positive integral linear combination of the $\alpha_i$. The walls of the Weyl chamber $\Delta^+$ lie on the hyperplanes $\alpha_i = 0$. The integer $\ell$ is called the real rank of $G$.

Example. Let $G$ be the group $SU(1,1)$, that is the group of matrices of determinant 1 leaving invariant the Hermitian form $-z_1 \overline{z}_1 + z_2 \overline{z}_2$. This group appears also as the group of conformal mappings of the unit disk $|z| < 1$; explicitly

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mid |a|^2 - |b|^2 = 1 \right\}$$

It is easy to see that the corresponding Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{su}(1,1) = \begin{Bmatrix} \begin{pmatrix} i \alpha & \beta \\ -\beta & -i \alpha \end{pmatrix} \mid \alpha \in \mathbb{R}, \beta \in \mathbb{C} \end{Bmatrix}.$$ 

The Killing form is here found to be

$$\langle X, Y \rangle = 4 \text{Tr}(XY)$$

and the decomposition
\[
\begin{pmatrix}
i \alpha & \beta \\
-\beta & -i \alpha
\end{pmatrix} = \begin{pmatrix}
i \alpha & 0 \\
0 & -i \alpha
\end{pmatrix} + \begin{pmatrix}
0 & \beta \\
\bar{\beta} & 0
\end{pmatrix}
\]

is easily seen to give a Cartan decomposition \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \).

For \( \mathfrak{h} \) we take the space

\[
\mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and since

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \begin{pmatrix}
i \alpha & \beta \\
\bar{\beta} & -i \alpha
\end{pmatrix} = \begin{pmatrix}
\bar{\beta} - \beta , -2i \alpha \\
2i \alpha , \beta - \bar{\beta}
\end{pmatrix}
\]

it is readily verified that the root space decomposition (1) is

\[
\mathfrak{g} = \mathbb{R} \begin{pmatrix} i & -i \\ i & -i \end{pmatrix} + \mathbb{R} \begin{pmatrix} i & 1 \\ -i & -i \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

For a Weyl chamber \( \mathfrak{h}^+ \) we select the open subset

\[
\mathbb{R}^+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

so

\[
\forall \mathcal{W} = \mathbb{R} \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}
\]

and

\[
N = \left\{ \begin{pmatrix} 1 + \text{i}n & -\text{i}n \\ \text{i}n & 1 - \text{i}n \end{pmatrix} \mid n \in \mathbb{R} \right\}.
\]

The groups \( K \) and \( A \) are given by

\[
K = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid |t| = 1 \right\}, \quad A = \left\{ \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \mid s \in \mathbb{R} \right\}.
\]
§4. Infinite-dimensional representations.

Let the notations be as in §3. Let $\pi$ be a representation of $G$ on a Banach space $\mathcal{V}$ and let $\delta$ be an irreducible (hence finite-dimensional) representation of $K$. We would like to assign meaning to the statement that $\delta$ is contained a finite number of times in $\pi$. If $\pi$ is finite-dimensional one simply restricts $\pi$ to $K$, decomposes into irreducible components and counts how many are equivalent to $\delta$. For a general $\pi$ a vector $v \in \mathcal{V}$ is called $K$-finite if $\pi(k)v$ spans a finite-dimensional subspace and $K$-finite of type $\delta$ if in addition the representation $k \mapsto \pi(k)$ of $K$ on this subspace decomposes into finitely many copies of $\delta$. Let $\mathcal{V}(\delta)$ denote the set of vectors of type $\delta$. If $\dim \mathcal{V}(\delta)$ is finite the representation $k \mapsto \pi(k)$ of $K$ on this subspace decomposes into $p$ copies of $\delta$. In this case $\pi$ is said to contain $\pi$

Let $\chi_\delta$ be the character of $\delta$, multiplied by its dimension $d(\delta)$. Then the operator

$$E(\delta) = \int_K \pi(k)\text{conj } \chi_\delta(k)dk$$

is a continuous projection of $\mathcal{V}$ onto $\mathcal{V}(\delta)$. Hence $\mathcal{V}(\delta)$ is closed.

If $v \in \mathcal{V}$, the vector $E(\delta)v$ is called the $\delta$-Fourier component of $v$. This notion is appropriate because the following lemma holds [15,III]. A vector $v \in \mathcal{V}$ is called
differentiable (resp. analytic) if the mapping \( g \rightarrow \pi(g)v \)
of \( G \) into \( \mathcal{V} \) is differentiable (resp. analytic).

**Lemma 4.1.** The space of differentiable vector is dense in \( \mathcal{V} \).

If \( v \in \mathcal{V} \) is differentiable the Fourier series

\[
\sum_{\delta \in \hat{K}} E(\delta)v
\]

converges absolutely to \( v \), \( \hat{K} \) denoting the set of equivalence classes of irreducible representations of \( K \).

**Theorem 4.2.** Suppose \( \pi \) is a unitary irreducible representation of \( G \) on a Hilbert space \( \mathcal{V} \). Then there exists an integer \( N \) such that each irreducible representation \( \delta \) of \( K \) is contained in \( \pi \) at most \( Nd(\delta) \) times.

**Remark.** This was proved by Harish-Chandra for \( \pi \) a "quasisimple" irreducible representation on a Banach space.

Godement gave in [12] a new and simpler proof in case \( G \) has a faithful representation and \( \pi \) is a completely irreducible representation of \( G \) on a Banach space.

With the assumptions in Theorem 4.2, \( \pi(z) \) is a scalar operator for each \( z \in Z \), the center of \( G \). Also each vector in \( \mathcal{V}(\delta) \) is an analytic vector. Let \( \mathcal{V}^\infty \) denote the algebraic direct sum \( \sum_{\delta} \mathcal{V}(\delta) \); this is the set of all \( K \)-finite vectors in \( \mathcal{V} \). By Lemma 4.1, \( \mathcal{V}^\infty \) is a dense subspace of \( \mathcal{V} \). The representation \( \pi \) of \( G \) on \( \mathcal{V} \) induces a representation of \( \mathcal{V}^\infty \), and hence of the universal enveloping algebra \( U(\mathcal{V}) \), on \( \mathcal{V}^\infty \).
Theorem 4.3. Let $\pi$ be a representation of $G$ on a Banach space $\mathcal{V}$. Assume

(i) $\pi(z)$ is a scalar for $z \in Z$.

(ii) For each $\delta \in \hat{K}$, $\dim \mathcal{V}(\delta) < \infty$.

Then each vector in $\mathcal{V}^\infty = \sum_\delta \mathcal{V}(\delta)$ is analytic; if $d\pi$ denotes the representation of $U(G)$ on $\mathcal{V}^\infty$ induced by $\pi$ then $\pi$ is irreducible if and only if $d\pi$ is (algebraically) irreducible.

This is Harish-Chandra's fundamental theorem in [15,I], p. 228, stated in a somewhat simplified form in order to emphasize its principal quality of relating (topological) irreducibility for $G$ to algebraic irreducibility for $U(G)$. A simple proof of the "only if" part was given by Godement [12], p. 545, even for $G$ not necessarily semisimple.

Since the definition of characters of finite-dimensional representations breaks down for infinite-dimensional representations an indirect method is adopted. Let $\pi$ be a representation of $G$ on a Hilbert space $\mathcal{V}$. If $f \in \mathcal{L}(G)$ the integral

$$\pi_f = \int_G f(g) \pi(g) dg$$

is a well-defined operator on $\mathcal{V}$. An operator $A$ on $\mathcal{V}$ is said to be of trace class if for each orthonormal basis $(e_\alpha)$ of $\mathcal{V}$ the series $\sum_\alpha (e_\alpha, A e_\alpha)$ converges absolutely. If this is so the sum is independent of the choice of basis and is denoted $\text{Tr}(A)$. 
Theorem 4.4. Let \( \pi \) be an irreducible unitary representation of \( G \) on a Hilbert space. Then for each \( f \in \mathcal{A}(G) \) the operator \( \pi_f \) is of trace class. The linear form \( f \to \text{Tr}(\pi_f) \) is a distribution on \( G \).

This theorem was proved by Harish-Chandra [15, III] (again with the unitary condition replaced by quasi-simple).

The proof is a not too difficult consequence of Lemma 4.1 and Theorem 4.2 so we sketch it. If we select an orthonormal basis \( e_{\delta,i} \) in each of the finite-dimensional spaces \( \mathcal{H}(\delta) \) we obtain an orthonormal basis of \( \mathcal{H} \). We shall prove

\[
\sum_{\delta_1, \delta_2 \in \mathcal{K}, i,j} |(e_{\delta_1,i}, \pi_f e_{\delta_2,j})| < \infty
\]

because then in particular \( \pi_f \) is of trace class. Since

\[
|(e_{\delta_1,i}, \pi_f e_{\delta_2,j})| \leq \|E(\delta_1)\pi_f E(\delta_2)\|
\]

the norm on the right hand side being operator norm, and since by Theorem 4.2,

\[
\dim \mathcal{H}(\delta) \leq \text{Nd}(\delta)^2
\]

relation (1) would follow if we prove

\[
\sum_{\delta_1, \delta_2 \in \mathcal{K}} d(\delta_1)^2 d(\delta_2)^2 \|E(\delta_1)\pi_f E(\delta_2)\| < \infty.
\]

But (2) can be deduced from the absolute convergence in Lemma 4.1. For this let \( \mathcal{M}(\mathcal{H}) \) denote the space of all bounded operators on \( \mathcal{H} \) and let \( \phi \) denote the representation
of $G \times G$ on $M(\mathfrak{k})$ given by
\[ \phi(x, y)A = \pi(x)A\pi(y^{-1}). \]

Now we have in a canonical fashion $(K \times K)^{\wedge} = \hat{K} \times \hat{K}$ by associating to any pair $(\delta_1, \delta_2) \in \hat{K} \times \hat{K}$ the irreducible representation
\[ (k_1, k_2) \rightarrow \delta_1(k_1) \otimes \delta_2(k_2) \]
of $K \times K$. Hereby
\[ \chi(\delta_1, \delta_2)(k_1, k_2) = \chi_{\delta_1}(k_1)\chi_{\delta_2}(k_2) \]
whence by the formula for $E(\delta)$,
\[ E(\delta_1, \delta_2)A = E(\delta_1)AE(\delta_2^{-1}) \quad A \in M(\mathfrak{k}) \]
the prime denoting contragredient representation. Now if $f \in \mathcal{S}(G)$
\[ \phi(x, y)\pi_f = \int_G f(x^{-1}gy)\pi(g)dg \]
so $\pi_f$ is a differentiable vector for the representation $\phi$ of $G \times G$ on $M(\mathfrak{k})$. Thus by (3) and Lemma 4.1,
\[ \sum_{\delta_1, \delta_2 \in \hat{K}} \|E(\delta_1)\pi_fE(\delta_2)\| < \infty. \]

Let $w$ be an element in the center $Z(\mathfrak{k})$ of the universal enveloping algebra of $\mathfrak{k}$. Assume $w$ invariant under the antiautomorphism $X \rightarrow -X$ of the universal enveloping algebra. By irreducibility of $\delta$, $\delta(w)$ is a scalar $c(\delta)$ and since
π restricted to K induces on \( \kappa(\delta) \) finitely many copies of \( \delta \) we see that \( \pi(w) = c(\delta) \) on \( \kappa(\delta) \). From (3) and the definition of \( \phi \) we deduce

\[
\phi(w, w) = c(\delta_1)c(\delta_2) \text{ on } M(\kappa)(\delta_1, \delta_2).
\]

But then if \( F = \ell(w)r(w)F \) (\( \ell \) and \( r \) being the left and right regular representations of \( G \), respectively),

\[
c(\delta_1)c(\delta_2)E(\delta_1)E(\delta_2) = \phi(w, w)E(\delta_1)\pi_F E(\delta_2)
\]

so using (4) with \( F \) replacing \( f \) we get

\[
\sum_{\delta_1, \delta_2} c(\delta_1)c(\delta_2) \| E(\delta_1)\pi_F E(\delta_2) \| < \infty
\]

Now \( w \) can be chosen such that \( c(\delta) \geq d(\delta)^2 \) for all \( \delta \in \hat{K} \) (Harish-Chandra [19], p.9) so (2) follows. Thus \( \text{Tr}(\pi_F) \) exists; a closer look at the calculation shows that \( f \to \text{Tr}(\pi_F) \) is continuous for the Schwartz topology on \( \mathcal{D}(G) \) and thus defines a distribution on \( G \), as stated.

The distribution \( T_\pi : f \to \text{Tr}(\pi_F) \) in Theorem 4.4 is called the character of \( \pi \). It is rather easy to see that if \( \Omega \) is any differential operator on \( G \) which is invariant under left and right translations then \( T_\pi \) is an eigeendistribution of \( \Omega \), \( \Omega T_\pi = \chi(\Omega)T_\pi \) where \( \chi(\Omega) \in \mathcal{C} \). Harish-Chandra now transfers these differential equations to any Cartan subgroup \( H \). Let \( H' \) denote the set of regular elements in \( H \),
V the open subset $\bigcup_{g \in G} gH'g^{-1}$ of $G$ and $G^*$ the factor
space $G/H$ which has a $G$-invariant measure $dg_H$. Then
there exists a distribution $\tau_{\pi}$ on $H'$ with the following
property: If $f \in \mathcal{D}(V)$ then $T_{\pi}(f) = \tau_{\pi}(F)$ where $F \in \mathcal{D}(H')$
is given by

$$F(h) = D(h) \int_{G/H} f(ghg^{-1})dg_H$$

where $D(x)$ $(x \in G)$ is the indicated coefficient in the
expansion

$$P(\lambda) = \det(\lambda - \text{Ad}(x)) = (\lambda - 1)^N + ... + D(x)(\lambda - 1)^4.$$

The differential equations for $T_{\pi}$ now give a system of
differential equations for $\tau_{\pi}$

$$(6) \quad \gamma(\Omega)(|D|^{\frac{1}{2}} \tau_{\pi}) = \chi(\Omega)(|D|^{\frac{1}{2}} \tau_{\pi})$$

where $\gamma(\Omega)$ is a differential operator on $H$ with constant
coefficients. From this follows that $|D|^{\frac{1}{2}} \tau_{\pi}$ satisfies an
elliptic differential equation on $H'$ so is an analytic
function; this shows (H being arbitrary) that $T_{\pi}$ actually
is an analytic function on the set $G'$ of regular elements
in $G$.

A deeper study of the differential equations (6) shows
that $T_{\pi}$ actually is a locally integrable function on $G$
(Harish-Chandra [18]). The first indication of the possibility
of such a result is that the function $|D|^{-\frac{1}{2}}$ is locally
summable on $G$. 
Another basic result on characters is that two irreducible unitary representations with the same character are equivalent [15, III].

Example. The spherical principal series.

Let $\mathcal{A}^*$ denote the dual of the vector space $\mathcal{A}$ (in §3) and $\mathcal{A}_c^*$ the set of all complex-valued linear forms on $\mathcal{A}$.

For $\lambda \in \mathcal{A}_c^*$ let $\mathcal{V}_\lambda$ denote the space of measurable functions on $G$ satisfying

(i) $F(g\text{man}) = F(g)e^{(i\lambda - \rho)\log a}$

(ii) $\int_K |F(k)|^2 dk < \infty$

and let $|F|_{\lambda}$ denote the square-root of the integral in (ii).

Then $\mathcal{V}_\lambda$ is a Hilbert space under the norm $F \mapsto |F|_{\lambda}$. We define a representation $\pi_\lambda$ of $G$ on $\mathcal{V}_\lambda$ by

$$(\pi_\lambda(g)F)(x) = F(g^{-1}x).$$

Let us compute the character of $\pi_\lambda$ (cf. [10], §20 and [14]).

As usual we define for $f \in \mathcal{B}(G)$

$$\pi_\lambda(f) = \int_G \pi_\lambda(g)f(g)dg.$$ 

Extend $\mathcal{A}$ to a maximal abelian subalgebra $\mathcal{H}$ of $\mathcal{g}$, consider the corresponding complexifications $\mathcal{H}^c \subset \mathcal{g}^c$. Then $\mathcal{H}^c$ is a Cartan subalgebra of $\mathcal{g}^c$; let $\Delta$ denote the corresponding set of roots. Then $\Sigma$ consists of the nonzero restrictions of the elements of $\Lambda$ to $\mathcal{A}$. The sets $\Sigma$ and $\Delta$ are given compatible
orderings such that $\alpha \in \Delta^+$ implies that the restriction of $\alpha$ to $\mathfrak{a}$ if nonzero lies in $\Sigma^+$. Let $P_+$ denote the set of $\alpha \in \Delta^+$ which do not vanish identically on $\mathfrak{a}$, $P_- = \Delta^+ - P_+$.

Let $\mathfrak{h}'$ denote the set of regular elements in $\mathfrak{h}$ and put $H' = \exp \mathfrak{h}'$, $G' = \bigcup_{g \in G} gH'g^{-1}$. The algebra $\mathfrak{h}$ is invariant under the Cartan involution $\theta$ so $\mathfrak{h} = \mathfrak{h}_K + \mathfrak{a}$ where $\mathfrak{h}_K = \mathfrak{h} \cap \mathfrak{k}$. Let $H = \exp \mathfrak{h}$, $H_K = \exp \mathfrak{h}_K$, $A = \exp \mathfrak{a}$.

Let $N_H$ denote the normalizer of $H$ in $G$ and let $W_H$ denote the factor group $N_H/H$. It is not hard to prove that $N_H = A(N_H \cap K)$ and therefore $W_H$ is a finite group.

Proposition 4.5. Suppose $G$ has center $\{e\}$ and that $M$ is connected. For each $\lambda \in \mathfrak{a}^*$ the operator $\pi_\lambda(f)$ has a trace for $f \in \mathcal{A}(G)$; moreover there exists a function $\Theta_\lambda$ on $G$ such that

$$\text{Tr}(\pi_\lambda(f)) = \int_{G} \Theta_\lambda(g) f(g) \, dg$$

($f \in \mathcal{A}(G)$)

and $\Theta_\lambda$ satisfies

(i) $\Theta_\lambda(g) = 0$ for $g \in G - G'$

(ii) $\Theta_\lambda(gxg^{-1}) = \Theta_\lambda(x)$ for $x, g \in G$

(iii) If $H$ is the $\mathfrak{a}$-component of $H \in \mathfrak{h}$,

$$\Theta_\lambda(\exp H) = c \frac{\sum_{s \in W_H} e^{-is\lambda(H)}}{\prod_{\alpha \in P_+} |e^{\frac{i}{2}\alpha(H)} - e^{-\frac{i}{2}\alpha(H)}|}$$

($H \in \mathfrak{h}$).

where $c$ is a constant.
Proof. Since we are not computing the value of \( c \) we can be somewhat casual about the normalization of the measures which enter below. It is convenient to transfer the representation space for \( \pi_\lambda \) from \( \mathcal{V}_\lambda \) to \( L^2(K/M) \) by associating with each \( F \in \mathcal{V}_\lambda \) the function \( \varpi \in L^2(K/M) \) given by
\[
\varpi(kM) = F(k).
\]
The group \( G \) acts on \( K/M \) via the Iwasawa decomposition, \( g \cdot kM = k(gk)M \). Then if \( \tau_\lambda \) is the representation \( \pi_\lambda \) carried over to \( L^2(K/M) \)
\[
(\tau_\lambda(g)\varpi)(kM) = (\pi_\lambda(g)F)(k) = F(k(g^{-1}k)\exp H(g^{-1}k)n(g^{-1}k)) = F(k(g^{-1}k))e^{i\lambda - \omega}(H(g^{-1}k))
\]
so
\[
(7) \quad (\tau_\lambda(g)\varpi)(kM) = e^{i\lambda - \omega}(H(g^{-1}k))\varpi(g^{-1}kM).
\]
As usual we define for \( f \in \mathcal{D}(G) \),
\[
\tau_\lambda(f) = \int_G f(g)\tau_\lambda(g)dg
\]
and we shall now express this as an integral operator on \( K/M \).
If \( \varpi \in L^2(K/M) \) we have
\[
(\tau_\lambda(f)\varpi)(kM) = \int_G f(g)(\tau_\lambda(g)\varpi)(kM)dg = \int_G f(kg^{-1})(\tau_\lambda(kg^{-1})\varpi)(kM)dg.
\]
The measure \( dg \) splits as follows relative to the Iwasawa decomposition
\[
(2) \quad \int_G f(g)dg = \int_{KAN} f(kan)e^{2\omega(\log a)}dkd\alpha d\nu.
\]
Using this and (7) in the previous integral we obtain
\[(\tau_\lambda(f)_M)(kM) = \int_{KAN} f(k(uan)^{-1})\varphi(uM)d^{1+\rho}(\log a)\,dudan.\]

Define now

\[F_f(k,u) = \int_{AN} f(kn^{-1}a^{-1}u^{-1})e(i\lambda+\rho)(\log a)\,dadn\]

\[= \int_{AN} f(kan^{-1})e^{-i\lambda}(\log a)\,dadn\]

\[= \int_{AN} f(kan^{-1})e^{-i\lambda}(\log a)\,dadn.\]

Then

\[(\tau_\lambda(f_\varphi)(kM) = \int_K F_f(k,u)\varphi(uM)du\]

Since \(F_f(km,u) = F_f(k,um^{-1})\) we can define \(\eta \in C^\infty(K/M \times K/M)\) by

\[\eta(kM,uM) = \int_M F_f(km,u)\,dm\]

and then we obtain the integral operator form of \(\tau_\lambda(f)\),

\[(\tau_\lambda(f_\varphi)(kM) = \int_{K/M} \eta(kM,uM_\varphi(uM)du_M.\]

Then \(\tau_\lambda(f)\) is of trace class and

\[\text{Tr}(\tau_\lambda(f)) = \int_{K/M} \eta(kM,kM)dk_M = \int_M (\int_K F_f(km,k)dk)\,dm.\]

The integral inside the parenthesis is invariant under a conjugation \(m \rightarrow m_mm^{-1}\) by any element \(m_1\) of \(M\), so since \(H_K\) is a maximal torus in the connected group \(M\) the last integral can by Weyl's integral formula be written
\[ \frac{1}{w} \int_{H_K} \int_{K} F_k(kh, k) dk \ j_-(h)^2 dh \]

where \( w \) is the order of the Weyl group of \( M \) and

\[ j_-(\exp H) = \prod_{\alpha \in \Delta_-} \left( e^{\frac{i}{2} \alpha(H)} - e^{-\frac{i}{2} \alpha(H)} \right) \quad H \in \mathfrak{g}_K. \]

Thus we have proved

\[ (9) \quad \text{Tr}(\tau_\lambda(f)) = (w)^{-1} \int_{K} \int_{H_K} j_-(h)^2 dh \int_{AN} f(kh^{-1}) e^{-i\lambda + \rho}(\log a)_{\text{d}a} \]

Now we need a lemma which is quite analogous to Weyl's integral formula just used.

**Lemma 4.6.** The mapping \( \varphi: G/H \times H' \to G' \) given by

\[ \varphi(gH, h) = ghg^{-1} \]

has Jacobian given by

\[ \det(d\varphi)(gH, h) = j(h)^2 \]

where

\[ j(\exp H) = \prod_{\alpha \in \Delta^+} |e^{\frac{i}{2} \alpha(H)} - e^{-\frac{i}{2} \alpha(H)}| \quad H \in \mathfrak{g}. \]

Since the Jacobian is nonsingular, \( G' \) is an open subset of \( G \). Moreover each point in \( G' \) has exactly \( |W_H| \) pre-images. Thus we have for a suitable normalization of the measures

\[ (10) \quad \int_{G'} f(g) dg = \int_{G/H \times H} f(ghg^{-1}) |j(h)|^2 dh dg_H \]

for each \( f \in \mathcal{D}(G') \).

Consider now the function \( j_+ \) on \( H \) given by
\[ J_+ (\exp H) = \prod_{\alpha \in \mathcal{P}_+} |e^{\frac{1}{2} \alpha(H)} - e^{-\frac{1}{2} \alpha(H)}|, \quad H \in \mathfrak{h}. \]

This function is indeed well defined because on \( H_K \) it equals \( j(j_\perp)^{-1} \). Next we define the function \( \Theta_\lambda \) by (i), (ii) and (iii). In order to see that the invariance in (ii) is consistent with (iii) one must remark that \( \mathcal{W}_H \) actually leaves \( \mathcal{A} \) invariant (because in fact \( N_H = A(N_H \cap K) \)) and permutes the set \( P_+ \cup (-P_+) \). If \( f \in \mathcal{J}(G) \) we have (ignoring the convergence question for the time being) by (10)

\[ \int_{G/H} \Theta_\lambda(g) f(g) dg = \int_{(G/H) \times H'} f(ghg^{-1}) \Theta_\lambda(h) j(h)^2 dg_H dh \]

and since each factor in this integral is \( \mathcal{W}_H \)-invariant this can be written

\[ \int_{(G/H) \times H} f(ghg^{-1})(j_\perp)^{-1}(h)e^{-i\lambda(\log h_\perp)} j(h)^2 dg_H dh \]

where \((h_\perp)^2 = h_\theta(h)^{-1} \). But by the formula

\[ \int_{G/A} F(gA) dg_A = \int_{G/H} \left( \int_{H/A} F(ghA) dh_A \right) dg_H \]

we have (with appropriate normalization)

\[ \int_{G/H} f(ghg^{-1}) dg_H = \int_{G/A} f(ghg^{-1}) dg_A. \]

But under the identification \( G/A = KN \) via \( G = KNA \) the measure \( dg_A \) corresponds to \( dkdn, \) ([21], p.374). Thus

\[ \int_{G/H} \Theta_\lambda(g) f(g) dg = \int_{H_+} (j_\perp)^{-1} j(h)^2 e^{-i\lambda(\log h_\perp)} \int_{KN} f(khn^{-1}k^{-1}) dkdn dh. \]
Now we use the formula

$$\int_N F(n)dn = \int_N F(h^{-1}nhn^{-1})dn \prod_{\alpha \in \mathbb{P}_+} |e^{-\frac{1}{2}a(h)}| j_+(h)$$

if \( h = \exp H \) ([14], p. 509, [21], p. 375) and since each \( \alpha \in \mathbb{P}_+ \) is purely imaginary on \( \mathfrak{h}_k \), we obtain since \( H = H_K A \)

$$\int_G \Theta_\lambda (g)f(g)dg = \int_H j(h)^2 \mathcal{J}_+(h)^{-2}e^{-i\lambda+n}(\log h_+)^{-1}f(\kappa h h^{-1}_K)dkdh.$$  

Since each \( \alpha \in \mathbb{P}_- \) vanishes identically on \( \mathfrak{a} \) this expression becomes

(11)  $$\int_K dk \int_{H_K} j_-(t)^2 dt \int_{H_K} f(kt h^{-1}_K) e^{-i\lambda+n}(\log a)\, \mathrm{d}a \, \mathrm{d}n.$$  

In view of (9) this proves the proposition except for the checking of the convergence of the integral

(12)  $$\int_G |\Theta_\lambda (g)f(g)|\, \mathrm{d}g.$$  

But this can be done by repeating the reduction above; the integral (11) being absolutely convergent the integral (12) is finite.
§5. Fourier analysis of spherical functions.

Let the notation be as in §3. A complex-valued function $f$ on $G$ is called spherical if $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$, $g \in G$. A continuous function $f \neq 0$ on $G$ satisfying the identity

$$\text{(1)} \quad \int_K f(xk) dk = f(x) f(y)$$

is necessarily spherical; the solutions of (1) are called zonal spherical functions. We now quote a few well-known results about spherical functions, the proofs of which can for example be found in [21], where however the terminology is slightly different.

(i) Let $C^\mathbb{H}(G)$ denote the algebra of continuous spherical functions of compact support, the multiplication being the convolution product

$$(f_1 \ast f_2)(g) = \int_G f_1(g) f_2(g^{-1} x) dg.$$ 

Then $C^\mathbb{H}(G)$ is commutative ([7]).

(ii) Let $\varphi$ be a continuous spherical function on $G$. Then $\varphi$ is zonal if and only if the mapping

$$f \mapsto \int_G f(g) \varphi(g) dg$$

is a homomorphism of $C^\mathbb{H}(G)$ into $C$, [7].

(iii) Let $I^\mathbb{H}(G)$ be the closure of $C^\mathbb{H}(G)$ in $L^1(G)$. Then $I^\mathbb{H}(G)$ is a commutative, semisimple Banach algebra.
Its continuous multiplicative functionals are

\[ f \rightarrow \int \frac{f(g)\varphi(g)}{G} dg \]

where \( \varphi \) is a bounded zonal spherical function on \( G \).

([9]; [21], p. 410).

(iv) Let \( D_o(G) \) denote the set of differential operators on \( G \) which are invariant under all left translations by elements of \( G \) and all right translations by elements of \( K \). Then a spherical function \( f \) is zonal if and only if \( f(e) = 1 \) and if \( f \) is an eigenfunction of each operator in \( D_o(G) \).

([12]).

(v) A representation \( \pi \) of \( G \) on a vector space \( \mathcal{V} \) is called spherical if there exists a nonzero vector \( e \in \mathcal{V} \) fixed under \( \pi(K) \). If \( \pi \) is an irreducible spherical unitary representation then the function

\[ (2) \quad \varphi(g) = \langle e, \pi(g)e \rangle \]

is a positive definite zonal spherical function. Conversely, every positive definite zonal spherical function can be written in this way, ([13]).

(For an extension of (2) to non-unitary representations, dropping the positive definiteness see [24] Ch. III § 5.)

(vi) The zonal spherical functions are precisely the functions

\[ (3) \quad m_\lambda(g) = \int_K e^{(i\lambda - \rho)(H(gk))} dk, \quad g \in G, \]
\( \lambda \) being arbitrary. Moreover if \( \lambda, \mu \in \mathcal{A}_c^* \) then \( \varphi_{\lambda} = \varphi_{\mu} \) if and only if \( \lambda = s\mu \) for some \( s \in W \), ([15,II]).

We shall now consider the problem of decomposing an "arbitrary" spherical function into zonal spherical functions. If \( f \) is a spherical function on \( G \) we define the spherical Fourier transform \( \tilde{f} \) by

\[
(4) \quad \tilde{f}(\lambda) = \int_G f(g) \varphi_{-\lambda}(g) dg
\]

for all \( \lambda \in \mathcal{A}_c^* \) for which the integral exists. By general functional analysis methods one can prove that there exists a measure \( \mu \) on \( \mathcal{A}_c^* \) such that

\[
(5) \quad \int_G |f(g)|^2 dg = \int_{\mathcal{A}_c^*} |\tilde{f}(\lambda)|^2 d\mu(\lambda)
\]

for all \( f \in C^b(G) \) (cf. Godement [13]; see also Mautner [29] and Harish-Chandra [17]). We shall outline (in steps A-F below) a proof of Harish-Chandra's formula of this type which explicitly relates the measure \( \mu \) to the structure of \( G \).

This outline follows basically [17] but takes into account contributions from [11], [22], [23], [6] and [24]. In this way we obtain at the same time characterizations of the spherical Fourier transforms of the spaces \( I_c(G) \) and \( I^2(G) \) below.

In addition to the spaces \( C^b(G) \) and \( L^2(G) \) considered above we shall also use the space

\[
I_c(G) = C^b(G) \cap C^\infty(G)
\]
of $C^\infty$ spherical functions of compact support. For each $p \geq 1$ we consider the space $L^p(G)$ of rapidly decreasing spherical functions in $L^p(G)$. Roughly speaking, a function $f \in C^\infty(G)$ lies in $L^p(G)$ if and only if for each left invariant differential operator $D$ on $G$ the product of $(Dr)(g)$ with any power of the distance $B_{\theta}(\log a(g), \log a(g))$ tends to 0 at $\infty$ faster than the reciprocal of the $p^{th}$ root of the volume element on $G$. For a more precise definition see [24], Ch. I, §2. The space $L^2(G)$ coincides with the space $L(G)$ in [17], p.585.

**A. The expansion for $\varphi_\lambda$.**

Let $\lambda \in \mathcal{A}_C^*$. Since $\varphi_\lambda$ is spherical and since $G = K C \mathfrak{a}(A^+) K$ it is clear that $\varphi_\lambda$ is completely determined by its restriction to $A^+$. If $D \in D_0(G)$ (cf. (iv) above) and $f$ a $C^\infty$ spherical function the map $\bar{f} \to (Dr)^{-}$ (bar denoting restriction to $A^+$) can be realized by a differential operator $\Delta(D)$ on $A^+$, the "radial part" of $D$. Thus by (iv) $\varphi_\lambda$ satisfies a system of independent differential equations

$$
\Delta(D_1)\varphi_\lambda = a_1 \varphi_\lambda \ldots \quad \Delta(D_k)\varphi_\lambda = a_k \varphi_\lambda \quad \text{on } A^+,
$$

there $\lambda = \dim A$, the $a_i$ being constants. Now it can be proved that (6) has at most $w = |W|$ linearly independent solutions. Thus if one could somehow produce $w$ such solutions then $\varphi_\lambda$ would be a linear combination of those.
These special solutions of (6) are constructed by means of the Casimir operator \( \omega \). This is the Laplacian with respect to the indefinite Riemannian metric given by the Killing form of \( G \). Then \( \omega \in \mathcal{D}_0(G) \) (in fact \( \omega \) is invariant under both left and right translations) and its radial part can be computed to be

\[
\Lambda(\omega) = \Lambda_A + \sum_{\alpha \in \Pi^+} m_\alpha (\coth \alpha) H_\alpha.
\]

Here \( \Lambda_A \) denotes the Laplacian on \( A \) and for each \( \lambda \in \mathfrak{c}^\ast \) the vector \( H_\lambda \in \mathfrak{a} + i \mathfrak{a} \) is determined by \( \langle H_\lambda, H \rangle = \lambda(H) \) \( (H \in \mathfrak{a}) \). In (7) \( H_\alpha \) is viewed as a first order differential operator on \( A^+ \). Let \( \alpha_1, \ldots, \alpha_k \) denote the set of simple restricted roots and \( L \) the set of all linear combinations \( n_1 \alpha_1 \) \( (n_1 \in \mathbb{Z}^+) \). Since ([17] p. 271)

\[
\Lambda(\omega)^\alpha_\lambda = -\langle \lambda, \lambda \rangle + \langle \rho, \alpha_\lambda \rangle \sigma_\lambda
\]

we look for a solution \( \xi_\lambda \) of the equation

\[
\Lambda(\omega) \xi_\lambda = -\langle \lambda, \lambda \rangle + \langle \rho, \sigma_\lambda \rangle \xi_\lambda
\]

of the form

\[
\xi_\lambda(\exp H) = \sum_{u \in L} r_u(\lambda) e^{(i\lambda-\sigma-u)(H)} \quad H \in \mathfrak{a}^+;
\]

expression (7) then gives rise to a recursion formula for \( r_u(\lambda) \), namely

\[
\langle u, iu \rangle - 2i\langle u, iu \rangle \sum_{\alpha \in \Pi^+} m_\alpha \sum_{k \geq 1} \frac{r_{u-2ka}}{u-2ka} \langle u+\sigma-2ka, \sigma-i\alpha \rangle,
\]

expression (7) then gives rise to a recursion formula for \( r_u(\lambda) \), namely

\[
\langle u, iu \rangle - 2i\langle u, iu \rangle \sum_{\alpha \in \Pi^+} m_\alpha \sum_{k \geq 1} \frac{r_{u-2ka}}{u-2ka} \langle u+\sigma-2ka, \sigma-i\alpha \rangle,
\]
where \( k \) runs over all integers \( \geq 1 \) for which \( \nu - 2k\alpha \in L \). Putting \( \Gamma_0 = 1 \), relation (11) defines \( \Gamma_u \) recursively as a rational function on \( \mathfrak{A}_C^* \). Concerning the convergence of (10) one proves directly from (11) by induction ([23], p.300) that for each \( H \in \mathfrak{A}^+ \) and each \( \lambda \in \mathfrak{A}_C^* \) for which the denominators of \( \Gamma_u \) are all \( \neq 0 \) there exists a constant \( K_{\lambda,H} \) such that

\[
|\Gamma_u(\lambda)| \leq K_{\lambda,H} e^{u(H)}.
\]

Then convergence of (10) is assured.

Now it turns out remarkably enough that \( \phi_\lambda \) satisfies not only (9) but the entire system (6). Since \( \varphi_{s\lambda} = \varphi_\lambda \) (\( s \in W \)) the eigenvalues \( a_\lambda \) are invariant under \( W \) in their dependence on \( \lambda \) so for each \( s \in W \), \( \phi_{s\lambda} \) also satisfies (6). It is clear that for a generic \( \lambda \) these functions are linearly independent so by the statement above about the solution space for (6), \( \varphi_\lambda \) is a linear combination,

\[
\varphi_\lambda(a) = \sum_{s \in W} c_s(\lambda) \phi_{s\lambda}(a), \quad a \in A^+
\]

for all such \( \lambda \). Replacing here \( \lambda \) by \( \sigma \lambda \) (\( \sigma \in W \)) we see that \( c_s(\lambda) = c(s\lambda) \) where \( c = c_e \). This gives the desired expansion

\[
\varphi_\lambda(a) = \sum_{s \in W} c(s\lambda) \phi_{s\lambda}(a) \quad a \in A^+
\]

valid for \( \lambda \in \mathfrak{A}_C^* \) with the exception of a countable number of hyperplanes.
B. The formula for $c(\lambda)$.

The spectral theory of ordinary second order differential operators (see particularly Weyl [35], p. 266 or [5], Exercise 4, p. 255) suggests that if the rank $\ell$ equals 1, the measure $d\mu$ in (5), which is just the "spectral function" for the ordinary differential equation (8) should be given by $|c(\lambda)|^{-2}d\lambda$. The remarkable fact is that this turns out to be true for $\ell > 1$ as well and the following fundamental theorem holds.

**Theorem 5.1.** (The inversion and the Plancherel formula).

**For a suitable normalization of** $d\lambda$,

$$f(g) = \int_{\mathfrak{t}^*} \tilde{f}(\lambda) \varphi_{\lambda}(g) |c(\lambda)|^{-2}d\lambda$$

$$\int_G |f(g)|^2 dg = \int_{\mathfrak{t}^*} |\tilde{f}(\lambda)|^2 |c(\lambda)|^{-2}d\lambda$$

**for all** $f \in I_c(G)$.

In order to outline the proof of this theorem we must investigate the function $c(\lambda)$. We go back to the integral representation (3) for $\varphi_{\lambda}$ which we can also write in the form

$$\varphi_{\lambda}(g) = \int_{K/M} e^{(i\lambda \cdot \sigma)(H(gk))} dk_M.$$  

Now it is a simple consequence of the Bruhat decomposition that if $\bar{N} = \Theta N$ then the map $\bar{n} \to \kappa(\bar{n})M$ is a diffeomorphism of $\bar{N}$ onto an open subset of $K/M$ whose complement has lower dimension. By evaluating the Jacobian of this map one finds
\[ (13) \quad \int_{K/M} F(kM) dM = \int_{N} F(k(N)M) e^{-2\rho(H(N))} dN, \quad F \in C^\infty(K/M) \]

and as a consequence

\[ (14) \quad \phi_\lambda(a) = e^{(i \lambda - \rho) \log a} = e^{(i \lambda - \rho) (H(a \tilde{a} - 1)} e^{-(i \rho + \rho) H(N)} dN, \quad a \in A. \]

Letting \( a \to \infty \) in \( A^+ \) one can deduce from (14) and (12) that \( c(\lambda) = I(i \lambda)/I(\rho) \) where

\[ (15) \quad I(\nu) = \int_{N} e^{-(\nu + \rho) (H(N))} dN, \]

whenever \( \text{Re} \langle \nu, \omega \rangle > 0 \) for \( \alpha \in \Sigma^+ \). Consider for the moment the case \( \lambda = 1 \). Then \( \phi_\lambda \) is a function of one real variable and can by means of (8) be expressed by means of the hypergeometric function. If one compares the series expansion for the hypergeometric function with the expansion (12) one obtains ([17], p.301)

\[ (16) \quad I(\nu) = \prod_{\alpha \in \Sigma^+} B \left( \frac{1}{2} + \frac{m}{m} \frac{\lambda}{\alpha} + \frac{\langle \nu, \omega \rangle}{\langle \alpha, \omega \rangle} \right) \quad \nu \in \mathcal{A}^*_c \]

B being the Beta-function. Thus \( c \) extends to a meromorphic function on \( \mathcal{A}^*_c \). A somewhat more direct proof of (16) is given in [24], Ch. III, Cor. 1.17.

Bhanu-Murthy [2,3] extended (16) to several other cases by explicit computation of \( H(N) \) and then Gindikin-Karpelevič [11] obtained (16) for \( G/K \) of any rank by means of the following method. Let \( \alpha \in \Sigma^+_O \) and consider the subalgebra \( \mathfrak{g}_\alpha \) generated by the eigenspaces \( \mathfrak{g}_\alpha \) and \( \mathfrak{g}_{-\alpha} \). Then \( \mathfrak{g}_\alpha \) is semisimple and has a Cartan decomposition \( \mathfrak{g}_\alpha = \mathfrak{k}_\alpha + \mathfrak{p}_\alpha \) compatible with that
of $\mathcal{C}_f$. The corresponding group $G^\alpha$ has real rank one and
has an Iwasawa decomposition $G^\alpha = K^\alpha A^\alpha N^\alpha$ compatible with
that of $G$. The $c$-function for $G^\alpha$ is given by an integral
(15) over $N^\alpha$ and it turns out that the product of these
integrals (for the various $\alpha$) is equal to the integral (15)
over $N$; thus (16) holds for $G$ of any real rank.

C. The asymptotic behavior of $\gamma_\lambda$.

The expansion of $\gamma_\lambda$ given by (12) suggests that if
$\lambda \in \mathcal{C}_f^*$ then $\gamma_\lambda(a)$ should behave like $e^{-\rho(\log a)}$ for
large $a \in A^+$. In fact the following more precise result
holds:

Let $D$ be any left invariant differential operator
on $G$. Then there exists a $k \in \mathbb{Z}^+$ and a constant $C$ such that

$$ (17) \quad (1 + \langle \lambda, \lambda \rangle)^{-k} \pi(\lambda) D^\alpha \gamma_\lambda(a) e^{\rho(\log a)} \leq C $$

for $\lambda \in \mathcal{C}^*$, $a \in A^+$. Here the function $\pi$ is given by

$$ \pi(\lambda) = \prod_{\alpha \in \mathcal{R}_+^0} \langle \alpha, \lambda \rangle. $$

(In [17], p.583 this result is given with a factor $\pi_0$ of $\pi$
replacing $\pi$ but in [22] it is noted that $\pi$ and $\pi_0$ actually
coincide.)

Estimate (17) is proved by induction on the dimension of
$G$. For this fix a point $a \in C_f(A^+) - A^+$ and let $G_1$ be the
connected semisimple part of the centralizer of $a$ in $G$.
Then $G_1$ has a Cartan decomposition $G_1 = K_1C(S(A_1^+))K_1$ compatible
with that of G. Harish-Chandra now obtains a precise
relationship between certain limit values of \( \varphi_\lambda \) and its
derivatives on the one hand and the zonal spherical functions
\( \theta_u \) on \( G_1 \) on the other hand. This leads to estimates of
essentially the differences between the two zonal spherical
functions which then can be used in order to prove (17) by
induction.

D. The inverse transform.

Let \( \mathcal{S}(\alpha^*) \) denote the space of rapidly decreasing
function on \( \alpha^* \) and \( \mathcal{J}(\alpha^*) \) the set of \( \mathcal{W} \)-invariant
functions in \( \mathcal{S}(\alpha^*) \). Let \( u \in \alpha^* \). Then the mapping

\[
S_u : b \to \int_{G} \left( \int_{\alpha^*} b(\lambda) \varphi_\lambda (g) |c(\lambda)|^{-2} \, d\lambda \right) \, dg
\]

for \( b \in \mathcal{J}(\alpha^*) \) is a \( \mathcal{W} \)-invariant tempered distribution on \( \alpha^* \).

The existence of the integral over \( \alpha^* \) follows easily
from (16) and then (17) can be used to show that this integral
gives a function in the space \( \mathcal{L}^2(G) \). For the integral over
\( G \) a refinement of (17) is needed so as to cover also the
case \( \lambda = 0 \). Here one has ([17], p. 279) for a constant \( C_0 \),

\[
|\varphi_\lambda (a)| \leq C_0 (1 + \log a, \log a) \) d e^0(\log a), \quad a \in A^+,
\]

where d is the number of roots in \( \tau_0^+ \).

Eventually we wish to prove that

\[
S_u (b) = b(u)
\]
because this would mean that the inverse transform

\[ b \rightarrow \int_{\mathfrak{N}^*} b(\lambda) \varphi_{\lambda}(g) |c(\lambda)|^{-2} d\lambda \]

followed by the spherical Fourier transform is the identity mapping.

**E. Determination of the distribution \( S_{\mu} \).**

Before proving that \( S_{\mu}(b) = b(u) \) (for a suitable normalization of \( d\lambda \)) we first prove

\[ (18) \quad pS_{\mu} = p(u)S_{\mu} \]

for every \( W \)-invariant polynomial \( p \) on \( \mathfrak{N}^* \).

For this we make use of the fact ([17], p. 260 or [21], p. 432) that there exists a \( D \in \mathfrak{D}_{\mathfrak{O}}(G) \) such that \( D\varphi_{\lambda} = p(\lambda)\varphi_{\lambda} \). Then

\[ pS_{\mu}(b) = S_{\mu}(pb) = \int_{G} \varphi_{-u}(g) \left( \int_{\mathfrak{N}^*} b(\lambda)p(\lambda)\varphi_{\lambda}(g) |c(\lambda)|^{-2} d\lambda \right) dg. \]

Here we replace \( p(\lambda)\varphi_{\lambda}(g) \) by \( (D\varphi_{\lambda})(g) \) and carry \( D \) over on \( \varphi_{-u} = \text{conj} \varphi_{u} \) by replacing it with its adjoint. The result is found to be \( p(u)S_{\mu}(b) \) proving (18).

From (18) we shall now derive as a consequence

\[ (19) \quad S_{\mu}(b) = \gamma(u)b(u) \quad u \in \mathfrak{N}^*, \ b \in \mathcal{F}(\mathfrak{N}^*). \]

where \( \gamma \) is a function on \( \mathfrak{N}^* \) (cf. [17] p. 591). For this fix \( u \in \mathfrak{N}^* \) and suppose \( \nu \in \mathfrak{N}^* \) is not on the orbit \( W^u \).
Select a $W$-invariant polynomial $p_o$ such that $p_o(v) \neq p_o(u)$ and put $p = p_o - p_o(u)$. Then $p \mathcal{S}_u = 0$ so since $p(v) \neq 0$, $v$ is not in the support of $\mathcal{S}_u$. Thus the support of $\mathcal{S}_u$ is contained in the orbit $W \cdot u$ so by general distribution theory, $\mathcal{S}_u$ is a sum of derivatives of the $\delta$-function at the points $s u$ ($s \in W$). But then (18) implies easily that only zero order derivatives can occur. This proves (19).

Now we must prove that the function $\gamma$ is a constant. For $f \in \mathcal{I}^2(G)$ consider the function (the integral converges absolutely)

$$F_f(a) = e^0(\log a) \int_N f(na) d\overline{n}, \quad a \in A.$$ 

Then formula (2) in §4 shows that

$$(20) \quad \tilde{f}(\lambda) = \int_G f(g)m^{-\lambda}(g) d\lambda = \int_A F_f(a)e^{-i\lambda(\log a)} da,$$ 

in other words, the spherical Fourier transform of $f$ equals the ordinary Fourier transform of $F_f$. If $b \in \mathcal{I}(\mathcal{A}^*)$ consider the inverse transform

$$m_b(g) = \int_{\mathcal{A}^*} b(\lambda)m^{-\lambda}(g) |c(\lambda)|^{-2} d\lambda.$$ 

Then (17) can be used to show $\phi_b \in \mathcal{I}^2(G)$ so $F_{\phi_b}$ is defined and by the inversion formula for the Fourier transform on $A$ and $\mathcal{A}^*$ we obtain
\[ F_{\phi_b}(a) = e^{\varphi(\log a)} \int_{\overline{N}} \widetilde{c}(\overline{n}a) d\overline{n} = \int_{\mathcal{C}^*} \varphi(\lambda) e^{i\lambda(\log a)} d\lambda \]

\[ = \int_{\mathcal{C}^*} S_{\lambda}(b)e^{i\lambda(\log a)} d\lambda = \int_{\mathcal{C}^*} \gamma(\lambda)b(\lambda)e^{i\lambda(\log a)} d\lambda \]

\[ = \frac{1}{W} \int_{\mathcal{C}^*} \gamma(\lambda)b(\lambda) \sum_{s \in W} e^{is\lambda(\log a)} d\lambda. \]

The constancy of \( \gamma \) would therefore follow from the following statement.

The relation

\[ (21) \quad |c(\lambda)|^{-2} e^{\varphi(\log a)} \int_{\overline{N}} \gamma(\overline{n}a) d\overline{n} = \sum_{s \in W} e^{is\lambda(\log a)} \]

holds in the weak sense in \( \lambda \), that is it gives the right result when integrated against any \( b \in \mathcal{J}(\mathcal{C}^*) \).

This is proved in Harish-Chandra's paper [17,II], §15 by means of very delicate analysis, which is too complicated to summarize here. Instead let us give a vague heuristic argument, assuming that the integral in (21) converges (which actually is not the case). The function

\[ g \rightarrow \int_{\overline{N}} \gamma(\overline{n}g) d\overline{n} \]

is an eigenfunction of each \( D \in D_0(G) \) left invariant under \( \overline{N} \), right invariant under \( K \). Viewing it as a function on \( A \) (since \( G = \overline{N}A\overline{K} \)) the operators \( D \) give certain differential operators on \( A \) which can be shown to be \( A \)-invariant. The resulting differential equation on \( A \) can be solved explicitly and one obtains if \( s \lambda \neq \lambda \) for all \( s \neq e \) in \( W \),

\[ (22) \quad e^{\varphi(\log a)} \int_{\overline{N}} \psi(\overline{n}a) d\overline{n} = \sum_{s \in W} a_s(\lambda)e^{is\lambda(\log a)}, \]
where $a_s(\lambda) \in \mathcal{C}$. Since the left hand side is $W$-invariant in $\lambda$, $a_s(\lambda)$ is independent of $s$. To see that it equals $|c(\lambda)|^2$ we use formula (15) giving

\begin{equation}
(23) \quad c(\lambda) = \int_{\overline{N}} \epsilon^{(i\lambda + \phi)}(H(n)) d\overline{n}
\end{equation}

(with an appropriate normalization of $d\overline{n}$) and the expansion (12) for $\varphi_{\lambda}$ which implies that

\begin{equation}
(24) \quad e^{\Theta}(log a) \varphi_{\lambda}(a) - \sum_{s \in \mathcal{W}} c(s \lambda)e^{is\lambda}(log a)
\end{equation}

tends to $0$ as $a \to +\infty$ in $A^+$. Now writing in (22)

$$\overline{na} = k_1a'k_2 \quad (k_1, k_2 \in K, \quad a' \in Cl(A^+))$$

one has ([17], p.604),

$$log a' \sim log a + H(\overline{n})$$

as $a \to +\infty$ in $A^+$. Combining this with (24) we see that the left hand side of (22) behaves for large $a$ in $A^+$ as

\begin{align*}
& e^{\Theta}(log a) \int_{\overline{N}} \epsilon^{\rho}(log a + H(\overline{n})) \sum_s c(s \lambda)e^{is\lambda}(log a + H(\overline{n})) d\overline{n} \\
= & \sum_s c(s \lambda)c(-s \lambda)e^{is\lambda}(log a) = |c(\lambda)|^2 \sum_s e^{is\lambda}(log a),
\end{align*}

since $c(s \lambda)c(-s \lambda) = c(\lambda)c(-\lambda) = |c(\lambda)|^2$ for $\lambda \in \mathcal{U}^*$. This completes the heuristic justification of (21).

The expression obtained for $S_u$ now shows that

\begin{equation}
(25) \quad b(u) = \int_{G^{-u}} (g) \left( \int_{\mathcal{U}^*} b(\lambda)e_{\lambda}(g)|c(\lambda)|^{-2} d\lambda \right) dg
\end{equation}

for all $b \in \mathcal{J}(\mathcal{U}^*)$. In order for (25) to give the inversion formula in Theorem 5.1 it remains to show that each $f \in I_c(G)$
can be written in the form
\[ f(\xi) = \int_{\mathcal{A}^*} b(\lambda) \psi_\lambda(\xi) |c(\lambda)|^{-2} \, d\lambda \]

with \( b \in \mathcal{J}(\mathcal{A}^*) \). This was proved by Harish-Chandra [17], [19]; we shall now outline a simpler proof of a more precise result, characterizing the spherical Fourier transforms of the members of \( I_c(G) \).

We shall call a holomorphic function \( F \) on \( \mathcal{A}_c^* \) a rapidly decreasing holomorphic function of exponential type if it satisfies the following condition:

There exists a constant \( A > 0 \) such that for each \( N \in \mathbb{Z}^+ \) there exists a constant \( C_N \) such that
\[
|F(\xi + in)| \leq C_N (1 + |\xi + in|)^{-N} e^{A|n|}
\]
for all \( \xi, n \in \mathcal{A}^* \). Here \( |\xi + in| = (\langle \xi, \xi \rangle + \langle n, n \rangle)^{1/2} \).

Theorem 5.2. (The Paley-Wiener theorem). The spherical Fourier transform \( f \rightarrow \tilde{f} \) is a bijection of \( I_c(G) \) onto the space of \( W \)-invariant rapidly decreasing holomorphic functions on \( \mathcal{A}_c^* \) of exponential type.

If \( f \in I_c(G) \) then the function \( F_f \), which can also be written
\[
F_f(a) = e^{(\log a) \gamma} \int f(an) \, dn,
\]
has also compact support so by (20), \( \tilde{f} \) extends to a \( W \)-invariant rapidly decreasing holomorphic function on \( \mathcal{A}_c^* \).
of exponential type.

On the other hand let $F$ be $W$-invariant holomorphic function on $\mathfrak{a}_c^*$ satisfying (26). We define $f$ on $G$ by the inverse transform

$$
(27) \quad f(g) = \int_{\mathfrak{a}_c^*} F(\lambda) \varphi_\lambda(g) |c(\lambda)|^{-2} d\lambda.
$$

Then $f$ is spherical and it suffices to prove

$$
(28) \quad f(\exp H) = 0 \quad \text{if} \quad H \in \mathfrak{a}^+ \quad \text{and} \quad |H| > A.
$$

In fact then (25) implies $\tilde{f}(\lambda) = F(\lambda)$. In order to prove (28) we use the expansion (12) which we write in the form

$$
(29) \quad \varphi_\lambda(\exp H) = \sum_{u \in L_u} \varphi_u(\lambda, H), \quad H \in \mathfrak{a}^+,
$$

where

$$
(30) \quad \varphi_u(\lambda, H) = \sum_{s \in W} c(s\lambda) r_u (s\lambda) e^{(is\lambda - s^2 u)(H)}.
$$

The following lemma gives the analog of (28) for each term in the expansion (29).

**Lemma 5.3.** Let $u \in L$. Then

$$
(\mathfrak{a}_c^*) \quad \int \varphi_u(\lambda, H) |c(\lambda)|^{-2} d\lambda = 0
$$

for $H \in \mathfrak{a}_c^+$ and $|H| > A$.

We sketch the proof from [23] of this lemma. For $\lambda$ real we have $|c(\lambda)|^2 = c(\lambda)c(-\lambda) = c(s\lambda)c(-s\lambda)$ for $s \in W$. Since $F$ is also $W$-invariant it suffices by (30) to prove
\[ (31) \quad \int_{A^*} F(-\xi)c(\xi)^{-1}e^{-i\xi(H)}\Gamma_{\mu}(-\xi)d\xi = 0 \]

provided \( H \in A^+ \), \( |H| > A \). To do this we shift the integration in (31) into the complexification \( A^*_C \) in such a way that the singularities of the functions \( \lambda \rightarrow c(\lambda)^{-1} \) and \( \lambda \rightarrow \Gamma_{\mu}(-\lambda) \) are simultaneously avoided, Cauchy's theorem applicable, and such that effective estimates can be obtained for the new integrand.

The singularities of \( c(\lambda)^{-1} \) can be read off from the Gindikin-Karpelevic formula (16), the poles of the \( \Gamma \) function being at the nonpositive integers. The singularities of \( \Gamma_{\mu}(-\lambda) \) can be read off from the recursion formula (11). It is therefore easy to see that the singularities of both functions are contained in the set

\[ \left\{ \lambda \in A^*_C \mid i\langle \nu, \lambda \rangle < 0 \text{ for some } \nu \in L \right\}. \]

Thus the function

\[ (32) \quad F(-\lambda)c(\lambda)^{-1}e^{-i\lambda(H)}\Gamma_{\mu}(-\lambda) \]

of the variable \( \lambda = \xi + i\eta \) \((\xi, \eta \in A^*)\) is holomorphic in the tube

\[ \left\{ \xi + i\eta \mid \xi, \eta \in A^*, \ -H\eta \in A^+ \right\}. \]

Using well known estimates for the Gamma function the behavior of (32) at \( \infty \) is found good enough to conclude via Cauchy's theorem that
\[ \int_{\mathcal{A}^*} F(-\xi)c(\xi)\gamma^*_{\xi}(H)\Gamma_u(-\xi)d\xi \]

\[ = \int_{\mathcal{A}^*} F(-\xi\cdot i\eta)c(\xi+i\eta)\gamma_{\xi+i\eta}(H)\gamma_{\xi-i\eta}(H)d\xi \]

provided \(-H, \eta \in \mathcal{A}^+\). But this last integral, say \(Q(H)\), is easily estimated by

\[ |Q(H)| \leq C e^{A|\eta|} e^{\eta(H)} \quad \text{for all } H, -H, \eta \in \mathcal{A}^+, \]

\(C\) being a constant. Put \(H, \eta = -tH\) \((t > 0)\). Then

\[ |Q(H)| \leq e^{t|H|(A-|H|)} C \]

so letting \(t \to +\infty\), (31) follows.

In order to conclude (28) we substitute the series (29) into (27). Integrating term-by-term we would conclude from the lemma,

\[ f(\exp H) = \int_{\mathcal{A}^*} F(\lambda)c(\exp H)|c(\lambda)|^{-2}d\lambda \]

\[ = \sum_{u \in \mathbb{L}} \int_{\mathcal{A}^*} F(\lambda)c_u(\lambda,H)|c(\lambda)|^{-2}d\lambda = 0 \]

if \(H \in \mathcal{A}^+, |H| > A\). The justification of the interchange of integration and summation was done by Gangolli [6] (see also [24] Ch. II, Theorem 2.4).

The outline we gave of the proof of Theorem 5.1 differs a bit from that of Harish-Chandra's proof in that the \(c\)-function was involved in the definition of the distribution \(S_u\). This leads to the following result (the "into" statement is in [17], p. 595, 596, the "onto" statement in [22], §3).
Theorem 5.4. The spherical Fourier transform maps $\mathcal{I}^2(G)$ onto $\mathcal{J}(\mathfrak{a}^*)$.

In view of (iii) in the beginning of this section the following theorem ([25]) gives a parametrization of the maximal ideal space of the Banach algebra $L^k(G)$.

**Theorem 5.5.** Let $\lambda \in \mathfrak{a}^*_c$ and write $\lambda = \text{Re}\lambda + i \text{Im}\lambda$ ($\text{Re}\lambda, \text{Im}\lambda \in \mathfrak{a}^*$). Then $\varphi_\lambda$ is bounded if and only if $\text{Im}\lambda$ lies in the convex hull of the points $s \circ s$ ($s \in W$).

Thus the maximal ideal space is a certain tube over a convex set. This result is used in [24], Ch. II, §2 on the problem of characterizing intrinsically the Fourier transforms of the functions in $\mathcal{I}^1(G)$. 
References


8. __________, and D. A. Raikov, Irreducible unitary representations of locally compact groups, Mat. Sbornik 13 (1943), 301-316.


