THE PALEY-WIENER SPACE FOR THE MULTITEMPORAL WAVE EQUATION

S. Helgason and H. Schlichtkrull

Let X be a Riemannian symmetric space, and let $\mathcal{E}^\sigma : \mathcal{D}(X) \times \cdots \times \mathcal{D}(X) \to L^2(\mathfrak{a}^* \times B)$ be the spectral representation related to the multitemporal wave equation on X, as defined in [H, eq. (6.7)]. In this note we characterize the image of $\mathcal{E}^\sigma$. We use the notation from [H]. Moreover, our numbering and literature labels refer to this paper, unless it is otherwise noted.

Theorem. *The range of $\mathcal{D}(X) \times \cdots \times \mathcal{D}(X)$ under $\mathcal{E}^\sigma$ is the set of functions $\varphi(\lambda, kM)$ on $\mathfrak{a}^*_C \times B$ satisfying:*

(i) $\varphi \in \mathcal{H}(\mathfrak{a}^*_C \times B)\pi$

(ii) *For each $\delta \in \hat{K}_M$*

$$\lambda \mapsto Q^{\check{\delta}}(\sigma\lambda)^{-1} \int_K \varphi(\lambda, kM)\delta(k^{-1})\, dk$$

is holomorphic on $\mathfrak{a}^*_C$.

In particular, we note that the condition (ii) for the trivial K-type δ is a consequence of (i), since the polynomial $Q^{\check{\delta}}(\lambda)$ in that case is constant.

Proof. We first prove the theorem with $\sigma = e$. Let $\mathcal{E} = \mathcal{E}^1$. In the proof we invoke the δ-spherical Fourier transform defined in §3. We introduce the following notation.

For a function φ on $\mathfrak{a}^*_C \times B$ we define a $\mathrm{Hom}(V_\delta, V_\delta^M)$-valued function φ^δ on $\mathfrak{a}^*_C$ by

$$\varphi^\delta(\lambda) = d(\delta) \int_K \varphi(\lambda, kM)\delta(k^{-1})\, dk. \tag{1}$$

Likewise, for $f \in \mathcal{D}(X)$ we define a $\mathrm{Hom}(V_\delta, V_\delta)$-valued function f^δ on X by

$$f^\delta(x) = d(\delta) \int_K f(kx)\delta(k^{-1})\, dk.$$

Then $\mathrm{Tr}(f^\delta) = d(\delta)\chi_\delta * f \in \mathcal{D}_{\check{\delta}}(X)$ is the $\check{\delta}$-typical component of f. With the δ-spherical Fourier transform on $\mathcal{D}_{\check{\delta}}(X)$ defined as in (3.10), it is then easily seen that

$$\widetilde{\mathrm{Tr}(f^\delta)}(\lambda) = \tilde{f}^\delta(\lambda), \tag{2}$$

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where the expression on the right-hand side is defined by (3.3) and (1).

Let $F = (f_1, \dots, f_w) \in \mathcal{D}(X) \times \cdots \times \mathcal{D}(X)$ and put $\varphi = \mathcal{E}(F)$. Then (i) is obvious from Theorem 6.3. Moreover, (ii) is satisfied, since as mentioned below (3.11) the function $Q^{\check{\delta}}(\lambda)^{-1}\widetilde{\mathrm{Tr}(f^{\delta})}(\lambda)$ is holomorphic, and by (2) the latter function equals $Q^{\check{\delta}}(\lambda)^{-1}\tilde{f}^{\delta}(\lambda)$.

Conversely, assume that a given function $\varphi(\lambda, kM)$ satisfies (i) and (ii). Define

$$f_j(x) = \int_{\mathfrak{a}^*_C \times B} p_j(i\lambda)\varphi(\lambda, b)e^{(i\lambda+\rho)A(x,b)} \frac{d\lambda\, db}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2} \tag{3}$$

for $x \in X$, $j = 1, \dots, w$. We claim that $f_j \in \mathcal{D}(X)$ and $\varphi = \mathcal{E}(F)$ where $F = (f_1, \dots, f_w)$.

It is well known that $\lambda \mapsto \pi(\lambda)\mathbf{c}(\lambda)$ is nonzero on $\mathfrak{a}^*$ and that its reciprocal satisfies a polynomial estimate). It then follows from the uniform rapid decrease of $\varphi(\,\cdot\,, b)$ that (3) defines a smooth function on X.

An easy computation shows that

$$f_j^{\delta}(x) = d(\delta)\int_K f_j(kx)\delta(k^{-1})\, dk = \int_{\mathfrak{a}^*} p_j(i\lambda)\Phi_{\lambda,\delta}(x)\varphi^{\delta}(\lambda)\frac{d\lambda}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2}$$

where $\Phi_{\lambda,\delta}(x)$ is the generalized spherical function $\int_K e^{(i\lambda+\rho)A(x,kM)}\delta(k)\, dk$. The function $\varphi^{\delta}(\lambda)$ is divisible by $Q^{\check{\delta}}(\lambda)$ because of our assumption (ii), and we can then write

$$f_j^{\delta}(x) = \int_{\mathfrak{a}^*} p_j(i\lambda)\Phi_{\lambda,\delta}(x)Q^{\check{\delta}}(\lambda)Q^{\check{\delta}}(\lambda)^{-1}\varphi^{\delta}(\lambda)\frac{d\lambda}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2}.$$

Averaging over W, and observing that $\lambda \mapsto \Phi_{\lambda,\delta}(x)Q^{\check{\delta}}(\lambda)$ is W-invariant (cf. [H7, p. 289, Thm. 5.15]) we obtain

$$f_j^{\delta}(x) = \frac{1}{w}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)Q^{\check{\delta}}(\lambda)\left[\sum_{s\in W} p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^{\delta}(s\lambda)\right]\frac{d\lambda}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2}. \tag{4}$$

In order to prove that $\mathrm{Tr}(f_j^{\delta})$ is compactly supported it now suffices, by [H7, p. 289, Cor. 5.14 and p. 290, Thm. 5.15], to prove that the function

$$\lambda \mapsto \left[\sum_{s\in W} p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^{\delta}(s\lambda)\right]\frac{1}{\pi(\lambda)^2} \tag{5}$$

belongs to the space $\mathcal{J}^{\delta}(\mathfrak{a}^*)$ of W-invariant functions in $\mathcal{H}(\mathfrak{a}^*_C, \mathrm{Hom}(V_\delta, V_\delta^M))$. Clearly (5) is W-invariant, and the expression in square brackets is holomorphic of exponential type, uniformly in δ. It remains to be seen that it is divisible by $\pi(\lambda)^2$. We rewrite (5) as follows

$$\left[\sum_{s\in W} \epsilon(s)p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\frac{\varphi^{\delta}(s\lambda)}{\pi(s\lambda)}\right]\frac{1}{\pi(\lambda)}.$$

Here the expression in square brackets is holomorphic by (i)-(ii), since $\pi(\lambda)$ and the determinant of $Q^{\check{\delta}}(\lambda)$ have no common zeroes, cf. [H7, p. 267, Thm. 4.2 and p. 348, Cor. 11.3]. Moreover, being skew, it is divisible by $\pi(\lambda)$. Hence $\mathrm{Tr}(f_j^\delta) \in \mathcal{D}(X)$ with support uniform in δ, and so $f_j = \sum_\delta \mathrm{Tr}(f_j^\delta) \in \mathcal{D}(X)$.

It remains to be seen that $\mathcal{E}(F) = \varphi$. It follows from the above that the δ-spherical Fourier transform $\widetilde{\mathrm{Tr}(f_j^\delta)}$ of $\mathrm{Tr}(f_j^\delta)$ is given by $Q^{\check{\delta}}(\lambda)$ times (5). By (2) $\widetilde{\mathrm{Tr}(f_j^\delta)} = \widetilde{f_j}^\delta$ and hence

$$\pi(\lambda)^2 \sum_j q^j(i\lambda)\widetilde{f_j}^\delta(\lambda) = \sum_j q^j(i\lambda)Q^{\check{\delta}}(\lambda) \sum_{s\in W} p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^\delta(s\lambda)$$

$$= \sum_{s\in W} \left(\sum_j q^j(i\lambda)p_j(is\lambda) \right) Q^{\check{\delta}}(\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^\delta(s\lambda).$$

Now $\sum_j q^j(i\lambda)p_j(is\lambda) = 1$ for $s = e$, and otherwise it vanishes (use the remark below Lemma 4.3), hence we obtain

$$\pi(\lambda)^2 \sum_j q^j(i\lambda)\widetilde{f_j}^\delta(\lambda) = \varphi^\delta(\lambda).$$

Since this expression holds for all δ we conclude that

$$\pi(\lambda)^2 \sum_j q^j(i\lambda)\widetilde{f_j}(\lambda, b) = \varphi(\lambda, b)$$

for all $b \in B$, as claimed.

We now prove the theorem with arbitrary σ. We use the notation g^σ, φ^σ for $\lambda \mapsto g(\sigma^{-1}\lambda)$, $(\lambda, b) \mapsto \varphi(\sigma^{-1}\lambda, b)$. The proof will be based on the observation that the conditions (i) and (ii) are independent of the basis $p = (p_1, \dots, p_w)$ of $H(\mathfrak{a})$ used.

Let $\mathcal{E}_p^\sigma$ denote the mapping $\mathcal{E}^\sigma$ with the basis p. Thus

$$\mathcal{E}_p^e(\mathcal{D} \times \cdots \times \mathcal{D}) = \mathcal{E}_{p'}^e(\mathcal{D} \times \cdots \times \mathcal{D})$$

if $p' = (p_1', \dots, p_w')$ is any homogeneous basis of $H(\mathfrak{a})$ with $p_1' = 1$. In particular

$$\mathcal{E}_p^e(\mathcal{D} \times \cdots \times \mathcal{D}) = \mathcal{E}_{p^\sigma}^e(\mathcal{D} \times \cdots \times \mathcal{D}). \tag{6}$$

On the other hand,

$$(\mathcal{E}_p^\sigma F)(\lambda, b) = \sum_k (q^k)^\sigma(i\sigma\lambda)\widetilde{f_k}(\sigma\lambda, b) = (\mathcal{E}_{p^\sigma}^e)(\sigma\lambda, b)$$

so

$$\mathcal{E}_p^\sigma F = (\mathcal{E}_{p^\sigma}^e F)^{\sigma^{-1}}. \tag{7}$$

Using (6) and (7) the proof of (ii) is easily completed. □

References

[H] S. Helgason, *Integral geometry and multitemporal wave equations.*