Then $u$ has boundary values given by $F$, that is,
\[
\lim_{t \to \infty} u(k \exp tH \cdot o) = F(kM)
\]  \hspace{1cm} (7)
for each $k \in K$ and each $H \in a^+$. 

**Proof.** We may assume $k = e$. We must prove that if $a_t = \exp tH$ then as $t \to \infty$
\[
\int_{K/M} F(a_t(kM)) \, dk_M \to F(eM)
\]
But by Lemma 2.5 and (6) the integral on the left equals
\[
\int_N F(a_t(k(n)M)) \psi(n) \, d\bar{n} = \int_N F(k(n^\alpha)M) \psi(n) \, d\bar{n}
\]  \hspace{1cm} (8)
Now
\[
\bar{n} = \exp \left( \sum_{x < 0} X_x \right)
\]
where $X_x \in g_x$ and by (7) and (9) in §3-1,
\[
\bar{n}^{\exp H} = \exp H \exp \left( \sum_{x} X_x \right) \exp (-H) = \exp \left( \text{Ad} \left( \exp H \right) \left( \sum_{x} X_x \right) \right)
\]
\[
= \exp \left( e^{\text{ad} H} \left( \sum_{x} X_x \right) \right) = \exp \left( \sum_{x} e^{\alpha(H) X_x} \right)
\]
But $\alpha(H) < 0$ whenever $\alpha < 0$ so we see that for each $\bar{n} \in N$, $\bar{n}^{\exp tH} \to e$. It follows (using the dominated convergence theorem) that the right-hand side of (8) has a limit
\[
\int_N F(eM) \psi(n) \, d\bar{n} = F(eM)
\]
as $t \to \infty$. This proves the theorem.

The result above is not new (cf. Karpelevič [46], Theorem 18.3.2 and also Moore [53], p. 204). Next we prove that the boundary function $\hat{u}$ in Theorem 2.4 is unique.

**Corollary 2.7.** Let $F \in L^\infty(B)$ and
\[
u(x) = \int_B p(x, b)F(b) \, db \quad (x \in X)
\]
Then if $u \equiv 0$, we have also $F \equiv 0$.

In fact, let $\phi \in L^1(G)$ be continuous and consider the function
\[
F_1(b) = \int_g \phi(g)F(g(b)) \, dg \quad b \in B
\]
The function $F_1$ is continuous (as a convolution of a continuous integrable function with a bounded function) and its Poisson integral $u_1$ is given by

$$u_1(h \cdot o) = \int_B P(h \cdot o, b)F_1(b) \, db = \int_B F_1(h(b)) \, db$$

$$= \int_B \left( \int_G \phi(g)F(gh(b)) \, dg \right) \, db = \int_G \phi(g)u(gh \cdot o) \, dg$$

Now if $u \equiv 0$ we have $u_1 \equiv 0$ so by Theorem 2.6, $F_1 \equiv 0$. But since $\phi$ is arbitrary, we conclude $F \equiv 0$.

**The Topology of $X \cup B$**

It is possible to define a topology on the union $X \cup B$ such that the limit relation (7) is convergence in this topology. A vector $Y \in p$ is called regular if its centralizer $Z_Y$ in $p$ is Abelian. A point $x = (\exp Y)K$ in $X$ is called regular if $Y$ is regular. Now a regular vector $Y \in p$ belongs to a unique Weyl chamber $b_Y$ in the maximal Abelian subspace $Z_Y$. We say that a sequence of points $x_1, x_2, \ldots$ in $X$ converges to a boundary point $b$ if

(i) Each $x_n = (\exp Y_n)K$ (where $Y_n \in p$) is regular
(ii) The Weyl chambers $b_{Y_n}$ converge to $b$ (in the topology of $B$)
(iii) The distance from $Y_n$ to the boundary of $b_{Y_n}$ in $Z_{Y_n}$ tends to $\infty$

It is not hard to verify that this convergence concept (together with the usual convergence definition on $X$ itself) defines a topology on the union $X \cup B$.

We shall now prove some measure-theoretic results due to Harish-Chandra ([25], p. 239, [27], p. 294) and give an explicit formula for the Poisson kernel $P(x, b)$ as a consequence (cf. also Schiffmann [56]).

**Lemma 2.8.** Let $dk, da$, and $dn$ be left invariant Haar measures on the groups $K$, $A$, and $N$, respectively. Then for a suitable normalization of the Haar measure $dg$ of $G$, we have

$$\int_G f(g) \, dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)} \, dk \, da \, dn$$

for all $f \in C_c^\infty(G)$. This $\rho$ is defined in §3-5 and log denotes the inverse of the mapping $\exp : a \rightarrow A$.

**Proof.** Since the mapping $(k, a, n) \rightarrow kan$ is a diffeomorphism of $K \times A \times N$ onto $G$ (§3-5) there exists a function $D(k, a, n)$ on $K \times A \times N$ such that

$$\int_G f(g) \, dg = \int_{K \times A \times N} f(kan)D(k, a, n) \, dk \, da \, dn$$

(9)
for all $f \in C_c^\infty(G)$. The groups $G, K, A, N$ are all unimodular, that is, the left invariant Haar measures are all right invariant. Thus the left-hand side of (9) does not change if we replace $f(g)$ by $f(k_1 g n_1)$, $k_1 \in K, n_1 \in N$. It follows that $D(k_1^{-1} k, a, n n_1^{-1}) \equiv D(k, a, n)$ so $D(k, a, n)$ is a function $\delta(a)$ of $a$ alone. Let $a_1 \in A$. Then

$$
\int_G f(g) \, dg = \int_G f(g a_1) \, dg = \int_{K A N} f(k a a_1) \delta(a) \, dk \, da \, dn
$$

$$
= \int_{K A N} f(k a a_1 (a_1^{-1} n a_1)) \delta(a) \, dk \, da \, dn
$$

$$
= \int_{K A N} f(k a (a_1^{-1} n a_1)) \delta(a a_1^{-1}) \, dk \, da \, dn
$$

$$
= \int_{K A N} f(k a n) \delta(a a_1^{-1}) J(a_1, n) \, dk \, da \, dn
$$

where $J(a_1, b)$ denotes the Jacobian determinant of the mapping $n \to a_1 n a_1^{-1}$ of $N$ onto $N$. The computation in the proof of Theorem 2.6 shows that

$$
J(a_1, n) = e^{2 \rho(\log a_1)}
$$

Thus

$$
\delta(a) = \delta(a a_1^{-1}) e^{2 \rho(\log a_1)}
$$

and the lemma follows.

Given $g \in G$, let $k(g) \in K, H(g) \in A, n(g) \in N$ be determined by $g = k(g) \exp H(g) n(g)$.

**Corollary 2.9.** The Poisson kernel on $G/K \times K/M$ is given by

$$
P(g K, k M) = e^{-2 \rho(H(g^{-1} k))}
$$

**Proof.** The mapping $k \to k(g k)$ is a diffeomorphism of $K$ onto itself. Now fix $h \in G$. Then for $f \in C_c^\infty(G),

$$
\int f(k a n) e^{2 \rho(\log a)} \, dk \, da \, dn = \int f(g) \, dg = \int f(h g) \, dg
$$

(10)

Now if $g = k a n$, then

$$
h g = h k a n = k(h k) \exp H(h k) n(h k) a n = k(h k) \exp H(h k) a a^{-1} n(h k) a n
$$

which we write as $k_1 a_1 n_1$. Then our integral on the right-hand side of (10) equals

$$
\int f(k_1 a_1 n_1) e^{2 \rho(\log a)} \, dk \, da \, dn.
$$
But the map \( a \to \exp H(hk)a \) preserves the measure \( da \) and the map \( n \to (a^{-1}n(hk)a)n \) preserves the measure \( dn \). The integral (11) therefore equals

\[
\int f(k(hk)a_1n)e^{2\rho(\log a_1)}e^{-2\rho(H(hk))}dk \, da_1 \, dn
\]

so comparing with the left-hand side of (10), we find

\[
\int_k F(k) \, dk \equiv \int_k F(k(hk))e^{-2\rho(H(hk))} \, dk \quad (F \in C^\infty(K)) \tag{12}
\]

In particular, let us use this for \( F(k) = \phi(kM) \), \( \phi \) being an arbitrary \( C^\infty \) function on the boundary. Since

\[
\int_k F(k) \, dk = \int_{K/M} \phi(kM) \, d k_M
\]

\[
\int_k F(k(hk))e^{-2\rho(H(hk))} \, dk = \int_{K/M} \phi(k(hk)M)e^{-2\rho(H(hk))} \, d k_M
\]

and since \( k(hk)M = h(kM) \) the corollary follows from (12).

As another application let us compute the function \( \bar{n} \to \psi(\bar{n}) \) in Lemma 2.5.

**Proposition 2.10.** For a suitable Haar measure \( d\bar{n} \) on \( \bar{N} \) we have

\[
\int_{K/M} f(k_M) \, d k_M = \int_{\bar{N}} f(k(\bar{n})M)e^{-2\rho(H(\bar{n}))} \, d\bar{n} \in C^\infty(K/M).
\]

**Proof.** Fix an element \( \bar{n}_0 \in \bar{N} \) and consider the function \( f^{\bar{n}_0} : kM \to f(\bar{n}_0(kM)) \) on \( K/M \). Since \( \bar{n}_0(k(\bar{n})M) = k(\bar{n}_0 \bar{n})M \) we conclude from Lemma 2.5,

\[
\int_{K/M} f(\bar{n}_0(kM)) \, d k_M = \int_{\bar{N}} f(k(\bar{n}_0 \bar{n})M)\psi(\bar{n}) \, d\bar{n} = \int_{\bar{N}} f(k(\bar{n})M)\psi(\bar{n}^{-1} \bar{n}) \, d\bar{n},
\]

and from the definition of the Poisson kernel,

\[
\int_{K/M} f(\bar{n}_0(kM)) \, d k_M = \int_{K/M} f(kM)P(\bar{n}_0 \cdot o, kM) \, d k_M
\]

\[
= \int_{\bar{N}} f(k(\bar{n})M)P(\bar{n}_0 \cdot o, k(\bar{n})M)\psi(\bar{n}) \, d\bar{n}
\]

Comparing the formulas we conclude,

\[
\psi(\bar{n}^{-1} \bar{n}) = P(\bar{n}_0 \cdot o, k(\bar{n})M)\psi(\bar{n})
\]

so putting \( \bar{n} = e \) the proposition follows from Cor. 2.9.

To conclude this section we state two theorems without proof. Let \( \Delta \) denote the Laplace–Beltrami operator on \( X \).
**Theorem 2.11.** Let \( u \) be a bounded solution of the equation \( \Delta u = 0 \) on \( X \). Then \( u \) is harmonic.

A probabilistic proof of this theorem is given in Furstenberg [19] (cf. also Berezin [2] and Karpelevič [46]).

Using this result, A. Korányi and the author ([38]) have proved the following theorem which generalizes the classical Fatou theorem for the unit disk.

**Theorem 2.12.** Let \( u \) be a bounded solution of the equation \( \Delta u = 0 \) on \( X \). Then for almost all geodesics \( t \to \gamma(t) \) in \( X \) starting at the origin \( o \) the limit

\[
\lim_{t \to \infty} u(\gamma(t))
\]

exists.

### 4-3 Spherical Functions on Symmetric Spaces

Let \( X = G/K \) be a symmetric space of the noncompact type as in the last section. A spherical function on \( G/K \) is by definition a \( K \)-invariant eigenfunction \( \phi \) of all the operators \( D \in D(G/K) \) satisfying \( \phi(o) = 1 \). According to a theorem of Harish-Chandra the spherical functions are precisely the functions on \( G/K \) given by

\[
\phi_\lambda(gK) = \int_K e^{i(\lambda - \rho)(H(gk))} \, dk
\]

(1)

where \( \lambda \) is an arbitrary complex-valued linear function on \( a \).

In the simplest case when \( X \) is the non-Euclidean disk \( D \) from Ch. 1 the spherical functions are the Legendre functions \( P_v \) and their integral formula

\[
P_v(\cosh r) = \frac{1}{2\pi} \int_0^{2\pi} (\cosh r + \sinh r \cos \theta)^v \, d\theta
\]

is the simplest example of (1) (see, for example, [31], p. 406).

We shall now state Harish-Chandra’s result ([27], p. 612, [28], p. 48) which describes how an arbitrary \( K \)-invariant function \( f \in C_c^\infty(X) \) can be decomposed into spherical functions. In view of Theorem 7.1, Ch. 3 such a function \( f \) is completely determined by the values \( f(a \cdot o) \), \( (a \in A^+) \) and we define the transform (spherical Fourier transform) \( \tilde{f}(\lambda) \) by

\[
\tilde{f}(\lambda) = \int_{A^+} f(a \cdot o) \overline{\phi_\lambda(a)} D(a) \, da \quad (\lambda \in a^*)
\]

(2)

Here \( a^* \) is the dual of the vector space \( a \) and the function \( D(a) \) is the density for the volume element \( dx \) on \( X \) in polar coordinates (Theorem 7.1, Ch. 3). More precisely, if \( x = ka \cdot o \) then \( dx = D(a) \, dk_M \, da \).
The problem is now to invert formula (2). Motivated by the spectral
theory of singular ordinary differential operators, Harish–Chandra expands
the function $\phi_\lambda (\exp H)$ in a series of the form

$$
\phi_\lambda (\exp H) = \sum_\mu \left( \sum_{s \in W} \gamma_\mu (s \lambda) e^{i s \lambda (H)} \right) e^{-\mu (H)} \quad (H \in a^+) 
$$

(3)

Here $\mu$ runs through certain subset of $a^*$, the $\gamma_\mu$ are certain functions on $a^*$ and $W$ denotes the Weyl group (which acts on $a^*$ by duality). The domi-
ninating term in this series has the form

$$
e^{-\rho (H)} \sum_{s \in W} c(s \lambda) e^{i s \lambda (H)}
$$

(4)

where $1/c(\lambda)$ is a certain analytic function on $a^*$. From (1) above and Prop.
2.10, Harish–Chandra derives the integral formula

$$
c(\lambda) = \int_N e^{-(i \lambda \cdot \rho + H(\lambda))} d\bar{\eta}
$$

(5)

whenever the integral converges absolutely.

**Theorem 3.1.** The inverse of the spherical Fourier transform $f \rightarrow \tilde{f}$ in (2) is
given by

$$
f(a \cdot o) = \int_{a^*} \tilde{f}(\lambda) \phi_\lambda (a) |c(\lambda)|^{-2} d\lambda
$$

(6)

where $d\lambda$ is a constant multiple of the Euclidean measure on $a^*$.

The simplest case of this theorem is the inversion formula for the Mehler
transform stated in Ch. 1.

We shall now attempt to describe some of the main steps in the proof
of this theorem. For a restricted root $\alpha > 0$ let $m_\alpha = \dim (q_\alpha)$, where $q_\alpha$ is as
defined in §3.5. Let $(, )$ denote the inner product on $a^*$ induced by the
Killing form $B$ of $g$, restricted to $a$.

(i) The function $c(\lambda)$ is given by $c(\lambda) = I(i \lambda) / I(\rho)$, where

$$
I(v) = \prod_{\alpha > 0} B \left( \frac{1}{2} m_\alpha, \frac{1}{4} m_{\alpha/2} + \frac{v \cdot \alpha}{(\alpha, \alpha)} \right) \quad (v \in a^*)
$$

(7)

and $B$ denotes the Beta function,

$$
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}
$$

Let us first consider the case rank $(G/K) = 1$. Then $\phi_\lambda (a)$ is a function
of one real variable and is characterized by a single second-order ordinary
differential equation (which comes from $\phi_\lambda$ being an eigenfunction of $\Delta$). One finds then that $\phi_\lambda$ is given by a hypergeometric function. If one now
compares the series expansion for the hypergeometric function with the expansion (3), formula (7) follows. For the details see Harish–Chandra [27], p. 301.

Bhanu–Murthy [4, 5] extended (7) to several other special cases whereupon Gindikin and Karpelevič [21] proved (7) in general along the following lines. Let \( \alpha > 0 \) be a restricted root which is not a positive integral multiple of other restricted roots. Let \( g^\sigma \) denote the subalgebra of \( g \) generated by \( g_\alpha \) and \( g_{-\alpha} \). Then \( g^\sigma \) is semisimple and has a Cartan decomposition

\[
g^\sigma = f^\sigma + p^\sigma \quad f^\sigma = g^\sigma \cap f \quad p^\sigma = g^\sigma \cap p
\]  

(8)

Let \( G^\sigma \) and \( K^\sigma \) denote the analytic subgroups of \( G \) corresponding to \( g^\sigma \) and \( f^\sigma \), respectively. The symmetric space \( G^\sigma/K^\sigma \) (which can be identified with the orbit \( G^\sigma \cdot \circ \) and is a totally geodesic submanifold of \( G/K \)) has rank one. In fact if \( a^\sigma \) denotes the orthogonal complement in \( a \) of the hyperplane \( a(H) = 0 \) then \( a^\sigma \) is maximal Abelian in \( p^\sigma \). Now \( G^\sigma \) has an Iwasawa decomposition \( G^\sigma = K^\sigma a^\sigma N^\sigma \), and the \( c \)-function for \( G^\sigma/K^\sigma \), denoted \( c^\sigma \), is given by an integral of the form (5) over the group \( N^\sigma \). Now Gindikin and Karpelevič prove that the product of these integrals (for the various \( \alpha \)) is equal to the integral (5) over \( N \); more precisely,

\[
c(\lambda) = \prod_\alpha c^\sigma(\lambda^\sigma)
\]  

(9)

where \( \lambda^\sigma \) denotes the restriction of \( \lambda \) to \( a^\sigma \) and \( \alpha \) runs through the restricted roots specified above. Now (7) follows from the rank-one case.

Now let \( \mathcal{A}(a^\ast) \) denote the set of rapidly decreasing functions on \( a^\ast \) in the sense of Schwartz [57] and let \( \mathcal{S}(a^\ast) \) denote the set of \( W \)-invariant functions in \( \mathcal{A}(a^\ast) \). (Here \( W \) is the Weyl group.)

(ii) Let \( \mu \in a^\ast \). Then the mapping

\[
S_\mu : b \mapsto \int_{a^\ast} \overline{\varphi_\mu(a)} \left( \int_{a^\ast} b(\lambda) \phi_\lambda(a) |c(\lambda)|^{-2} d\lambda \right) D(a) \, da
\]

\([b \in \mathcal{S}(a^\ast)]\) is a tempered distribution on \( a^\ast \).

It is easy to see from (7) that the integral over \( \lambda \) is absolutely convergent. On the other hand to show that the integral with respect to \( a \) is absolutely convergent and makes \( S_\mu \) a distribution requires very detailed study of the behavior of \( \phi_\lambda(a) \) for large \( a \) (see Harish–Chandra [27], p. 588).

Eventually one wants to prove that for a suitable normalization of \( d\lambda \),

\[
S_\mu(b) = b(\mu).
\]

But first one proves

(iii) If \( p \) is a Weyl group invariant polynomial on \( a^\ast \) then

\[
pS_\mu = p(\mu)S_\mu
\]
To see this select a differential operator $D \in D(G/K)$ such that $D\phi_\lambda = p(\lambda)\phi_\lambda$ (see [27], p. 591 or [31], p. 432). Then

$$pS_\mu(b) = S_\mu(pb) = \int_X \bar{\phi}_\mu(x) \left( \int_{a^*} p(\lambda) b(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} \, d\lambda \right) \, dx$$

Here we replace $p(\lambda)\phi_\lambda(x)$ by $(D\phi_\lambda)(x)$ and carry $D$ over on $\bar{\phi}_\mu$ by replacing it with its adjoint; the result is $p(\mu)S_\mu(b)$ as desired.

As a fairly easy consequence of (iii) we obtain (cf. [27], p. 591).

(iv) There exists a function $\gamma$ on $a^*$ such that

$$S_\mu(b) = \gamma(\mu)b(\mu) \quad b \in \mathcal{F}(a^*).$$

Now we must prove that $\gamma$ is a constant. Consider for $f$ as in (2) the function $F_f$ defined by

$$F_f(a) = e^{\rho(\log a)} \int_N f(\bar{n}a \cdot o) \, d\bar{n} \quad a \in A$$

Then we have as a simple consequence of (1) and Lemma 2.8 that

$$\tilde{f}(\lambda) = \int_X f(x) \bar{\phi}_\lambda(x) \, dx = \int_A F_f(a) e^{-i\lambda(\log a)} \, da$$

If $b \in \mathcal{F}(a^*)$ consider the function

$$\phi_\lambda(x) = \int_{a^*} b(\lambda)\phi_\lambda(x) |c(\lambda)|^{-2} \, d\lambda$$

The integral for $F_{\phi_\lambda}$ can be shown to converge and by the inversion formula for the Fourier transform on $A$ and $a^*$ we obtain

$$F_{\phi_\lambda}(a) = e^{\rho(\log a)} \int_N \bar{\phi}_\lambda(\bar{n}a \cdot o) \, d\bar{n} = \int_{a^*} \bar{\phi}_\lambda(\lambda) e^{i\lambda(\log a)} \, d\lambda$$

$$= \int_{a^*} S_\lambda(b) e^{i\lambda(\log a)} \, d\lambda = \int_{a^*} \gamma(\lambda)b(\lambda) e^{i\lambda(\log a)} \, d\lambda$$

$$= \frac{1}{W} \int_{a^*} \gamma(\lambda)b(\lambda) \sum_{s \in W} e^{is\lambda(\log a)} \, d\lambda$$

where $W$ denotes the order of $W$. The relation $\gamma \equiv w$ would therefore result from the following statement.

(v) The relation

$$|c(\lambda)|^{-2} e^{\rho(\log a)} \int_N \bar{\phi}_\lambda(\bar{n}a \cdot o) \, d\bar{n} = \sum_{s \in W} e^{is\lambda(\log a)}$$

holds in the weak sense in $\lambda$, that is, it gives the right result when integrated against any $b \in \mathcal{F}(a^*)$. 
This is carried out by means of a beautiful analysis in §15, p. 597, of Harish–Chandra [27]. Here we have to settle for a vague plausibility argument. Writing \( \bar{n}a = k_1 a'k_2 \) \( (k_1, k_2 \in K, a' \in A^+) \) we have (loc. cit. p. 604)

\[
\log a' \sim \log a + H(\bar{n})
\]
as \( a \to \infty \) in \( A^+ \). Since (4) is the dominating term in the expansion for \( \phi_\lambda (\exp H) \) let us replace \( \phi_\lambda (\bar{n}a \cdot o) = \phi_\lambda (a' \cdot o) \) by

\[
e^{-\rho(\log a + H(\bar{n}))} \sum_{s \in \mathcal{W}} c(s\lambda) e^{is\lambda(\log a + H(\bar{n}))}
\]

When this expression is integrated over \( \mathcal{N} \) we obtain from (5) the expression

\[
e^{-\rho(\log a)} \sum_{s \in \mathcal{W}} c(s\lambda) c(-s\lambda) e^{is\lambda(\log a)}
\]

which equals \( e^{-\rho(\log a)} |c(\lambda)|^2 \sum_{s \in \mathcal{W}} e^{is\lambda(\log a)} \) in accordance with (11).

In order to deduce Theorem 3.1 from the relation \( S_\mu (b) = (\text{const})b(\mu) \) \( (b \in \mathcal{I}(a^*)) \) we still have to prove the following statement.

(vi) Each \( K \)-invariant function \( f \in C_c^\infty(X) \) can be written in the form

\[
f(x) = \int_{a^*} b(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} \, d\lambda \quad b \in \mathcal{I}(a^*)
\]

This was stated as a conjecture in Harish–Chandra [27], p. 612, and was finally proved by him in [28], p. 48. Since this proof involves so much work on the general Plancherel formula for \( G \) (in particular, the discrete series) it would not be feasible to describe it here. Instead let me outline a different effort [37] at proving (vi).

Let \( F \) be a \( \mathcal{W} \)-invariant function in \( C_c^\infty(A) \) and \( F^* \) its Fourier transform

\[
F^*(\lambda) = \int_A F(a) e^{-i\lambda(\log a)} \, da
\]

Writing the expansion (3) as

\[
\phi_\lambda(\exp H) = \sum_\mu \psi_\mu(\lambda, H) \quad (H \in a^+)
\]

we assume that the term-by-term integration

\[
\int_{a^*} F^*(\lambda) \phi_\lambda(\exp H) |c(\lambda)|^{-2} \, d\lambda = \sum_\mu \int_{a^*} F^*(\lambda) \psi_\mu(\lambda, H) |c(\lambda)|^{-2} \, d\lambda
\]

is permissible. Then we have (loc. cit. p. 302).

(vii) For \( H \in a \) let \( |H| = B(H, H)^{1/2} \). Suppose \( R > 0 \) such that \( F(\exp H) = 0 \) for \( |H| > R \). Then

\[
\int_{a^*} F^*(\lambda) \psi_\mu(\lambda, H) |c(\lambda)|^{-2} \, d\lambda = 0 \quad \text{for } |H| > R
\]
This is proved by translating the integration into the complexification \( a^* + i\alpha^* \) by use of Cauchy's theorem. Because of the formula (7) the function \( c(\lambda) \) can be extended to a function on \( a^* + i\alpha^* \) with singularities, whose location can be determined. The functions \( \psi_\mu(\lambda, H) \) are determined by certain recursion formulas which result from \( \phi_\lambda \) being an eigenfunction of each \( D \in D(G/K) \). It is therefore possible to describe the sets of singularities of the functions \( \psi_\mu(\lambda, H) \) and the integration in \( a^* \) can by Cauchy's theorem be translated away from these sets. This leads to estimates of the integral, which prove (14).

In order to prove (vi) let \( f \in C_c^\infty(X) \) be \( K \)-invariant and let us use (14) on the function \( F(a) = F_F(a), (a \in A) \). We put

\[
g(x) = \int_{a^*} F^*(\lambda)\phi_\lambda(x) |c(\lambda)|^{-1} d\lambda
\]

and by (13) and (14) we have \( g \in C_c^\infty(X) \) and \( K \)-invariant. On the other hand, we have by (10) and the result \( S_\mu(b) = b(\mu) \) (with \( d\lambda \) suitably normalized),

\[
\tilde{g}(\lambda) = Fg^*(\lambda) = F^*(\lambda)
\]

The Euclidean Fourier transform \( F \to F^* \) is one-to-one so the last relation implies

\[
Fg(a) = F(a) = F_F(a)
\]

Thus, in view of (10), the function \( h = f - g \) is a \( K \)-invariant function in \( C_c^\infty(X) \) satisfying

\[
\int_X h(x)\phi_\lambda(x) dx = 0
\]

for all complex-valued linear forms \( \lambda \) on \( a^* \). It is well-known (see, for example, [31], p. 409, 453) that this implies \( h = 0 \), so

\[
f(x) = \int_{a^*} F^*(\lambda)\phi_\lambda(x) |c(\lambda)|^{-1} d\lambda
\]

which gives (vi).

What is lacking in this proof of (vi) is a justification of the term-by-term integration (13). In the quoted paper this justification is given for the case rank \( (G/K) = 1 \); in this case the proof also gives a Paley-Wiener type of theorem for the transform \( f \to \tilde{f} \), that is, an intrinsic characterization of the functions \( \tilde{f}(\lambda) \) as \( f \) runs through the \( K \)-invariant functions in \( C_c^\infty(X) \).

We conclude this section with a simple remark on the formulas

\[
\tilde{f}(\lambda) = \int_{A^*} f(a \cdot o)\phi_\lambda(a)D(a) da
\]

\[
f(a \cdot o) = \int_{a^*} \tilde{f}(\lambda)\phi_\lambda(a)\delta(\lambda) d\lambda \quad \delta(\lambda) = |c(\lambda)|^{-2}
\]
In analogy with the product formula (9)

$$\delta(\lambda) = \prod_{a} \delta_{a}(\lambda^{a})$$

(17)

one can prove (and this is an elementary result) that

$$D(\exp H) = \prod_{a} D_{a}(\exp H^{a})$$

(18)

where $D_{a}$ is the $D$ function for the space $G^{a}/K^{a}$, and $H^{a}$ is the projection of $H$ on $a^{a}$. It seems conceivable that a fuller understanding of the reason for the product formulas (17) and (18) might lead to a reduction of Theorem 3.1 to the rank-one case.

4-4 Fourier Transform on Symmetric Spaces

As before let $X$ denote the symmetric space $G/K$. Now we would like to define a Fourier transform for arbitrary functions $f \in C_{c}^{\infty}(X)$, not just for the $K$-invariant ones. We motivate this by means of the definition given in §1-3 for the non-Euclidean disk $D$. In this case the group $G$ equals $SU(1, 1)$ and as calculated in §3-5 the group $N$ consists of the group of matrices

$$\begin{pmatrix}
1 + in & -in \\
in & 1 - in
\end{pmatrix} \quad n \in \mathbb{R}
$$

The orbit $N \cdot O$ consists of the points $in/(in - 1)$, which clearly form a horocycle and it is a simple matter to verify that the horocycles in $D$ are the orbits in $D$ of all groups of the form $gNg^{-1}$.

Hence, we define for the general symmetric space $X = G/K$ a horocycle to be an orbit in $X$ of a subgroup of $G$ of the form $gNg^{-1}$, $g$ being an arbitrary element in $G$.

**Lemma 4.1.** The group $G$ permutes the horocycles transitively.

**Proof.** The most general horocycle $\xi$ is of the form $\xi = gNg^{-1}h \cdot o$, $g$ and $h$ being fixed elements in $G$. By the Iwasawa decomposition we can write $h^{-1}g = kan$ and deduce (since $aN\alpha^{-1} \subset N$) that $gNg^{-1}h \cdot o = hkN \cdot o$. In other words, the element $ hk \in G$ maps the horocycle $\xi_{o} = N \cdot o$ onto $\xi$, so the lemma is proved.

In particular, all the horocycles are submanifolds of $X$ of the same dimension and since $N \cap K = \{ e \}$ the mapping $n \rightarrow n \cdot o$ is a diffeomorphism of $N$ onto $\xi_{o}$.

**Lemma 4.2.** Each horocycle $\xi$ can be written

$$\xi = ka \cdot \xi_{o}$$

(1)

where $a \in A$ is unique and the coset $kM \in K/M$ is unique.
Although the proof of this lemma is not difficult we shall not stop to prove it here. For the case \( X = D \) the lemma is quite obvious.

**Definition.** The Weyl chamber \( kM \) in (1) is called the **normal** to the horocycle \( \xi \); the element \( a \in A \) in (1) is called the **complex distance** from \( o \) to \( \xi \).

Considering the example \( X = D \) the term "normal" is quite reasonable; so is the term "complex distance" because the point \( ka \cdot o \) is the unique point in \( \xi \) at minimum distance from \( o \). (If \( a = \exp H, H \in a \), the distance is \( B(H, H)^{1/2} \), cf. [37], p. 306.)

We recall now that given the maximal Abelian subspace \( a \subset p \), the group \( N \) is determined following a choice of a Weyl chamber \( n^+ \subset a \).

**Lemma 4.3.** Let \( a_1, \ldots, a_w \) denote the various Weyl chambers in \( a \) and \( N_1, \ldots, N_w \) the corresponding Iwasawa groups. Then the horocycles \( N_1 \cdot o, \ldots, N_w \cdot o \) all have the same tangent space at the point \( o \).

**Proof.** The projection \( \pi : G \to G/K \) given by \( \pi(g) = g \cdot o \) maps \( N \) onto \( \xi_o \) and the differential \( d\pi : g \to (G/K)_o \) maps \( n \) onto \( (\xi_o)_o \). But the map \( d\pi : p \to (G/K)_o \) is an isomorphism so let \( q \subset p \) be the subspace which \( d\pi \) maps onto \( (\xi_o)_o \). We shall prove that the manifolds \( N \cdot o \) and \( A \cdot o \) are orthogonal at \( o \) and since

\[
(\xi_o)_o = d\pi(q) \quad (A \circ o)_o = d\pi(a)
\]

it suffices, because of the choice of metric on \( G/K (§3-3) \), to prove \( B(q, a) = 0 \), that is, \( q \) and \( a \) are orthogonal with respect to \( B \). But if \( H \in a \), \( X \in q \) then there exists an \( X_1 \in n \) such that \( d\pi(X) = d\pi(X_1) \). Thus \( X - X_1 \in l \) so since \( B(a, l) = 0 \) and \( B(a, n) = 0 \), we obtain

\[
B(X, H) = B(X_1, H) = 0
\]

Thus each of the tangent spaces \( (N_1 \circ o)_o \) is perpendicular to the tangent space \( (A \circ o)_o \) and since \( \dim N \circ o + \dim A \circ o = \dim G/K \), the lemma follows.

**Lemma 4.4.** Given \( x \in X, b \in B \), there exists exactly one horocycle passing through \( x \) with normal \( b \).

**Proof.** Let \( b = kM \). We must find a unique \( a \in A \) such that \( x \) lies on the horocycle \( \xi = ka \cdot \xi_o \). But \( x \in \xi \) means \( x = kan \circ o \) for some \( n \in N \) so \( an \circ o = k^{-1} \circ x \). Thus, by the Iwasawa decomposition, \( a \) is uniquely determined by \( k \) and \( x \).

We denote the horocycle determined by this lemma by \( \xi(x, b) \) and write \( \exp A(x, b) (A(x, b) \in a) \) for the complex distance from \( o \) to \( \xi(x, b) \). We can now write down the analogs of the functions \( e^{\alpha \circ x, b} \) in §1-3.
For \( b \in B \) and \( \lambda \) a complex-valued linear function on \( a \), define the function \( e_{\lambda, b} \) by

\[
e_{\lambda, b} : x \mapsto e^{i\lambda(A(x, b))} \quad x \in X
\]

We state without proof two properties of \( e_{\lambda, b} \), the second of which is trivial.

(i) \( e_{\lambda, b} \) is an eigenfunction of each operator \( D \in D(G/K) \).

(ii) \( e_{\lambda, b} \) is constant on each horocycle with normal \( b \). A function on \( X \) with this property will be called a plane wave with normal \( b \).

One can also prove that these two properties characterize the functions \( e_{\lambda, b} \) (if certain singular eigenvalue systems are excluded). In accordance with the definition in §1-3 we define Fourier analysis on the symmetric space \( X \) to be a decomposition of "arbitrary" functions on \( X \) into functions of the form \( e_{\lambda, b} \).

As before let \( dx \) denote the volume element on \( X \) and

\[
\rho = \frac{1}{2} \sum_{\alpha > 0} \dim (g_{\alpha}) \alpha
\]

Let \( a^* \) denote the dual of \( a \), that is, the set of real linear functions on \( a \). Then the following theorem holds (cf. [35]).

**Theorem 4.5.** For \( f \in C_c^\infty(X) \) define the Fourier transform \( \tilde{f} \) on \( a^* \times B \) by

\[
\tilde{f}(\lambda, b) = \int_X f(x)e^{i\lambda + \rho(A(x, b))} \, dx \quad \lambda \in a^*, \ b \in B
\]

Then

\[
f(x) = \int_{a^*} \int_B \tilde{f}(\lambda, b)e^{i\lambda + \rho(A(x, b))}|c(\lambda)|^{-2} \, d\lambda \, db
\]

if the Euclidean measure \( d\lambda \) on \( a^* \) is suitably normalized.

This theorem is proved by reducing it to Theorem 3.1 in a way which is similar to the reduction of Theorem 3.1, Ch. 1, to the inversion formula for the Mehler transform. That reduction made use of the geometric identity

\[
\langle \tau \cdot z, \tau \cdot b \rangle = \langle z, b \rangle + \langle \tau \cdot o, \tau \cdot b \rangle \quad (2)
\]

and the formula

\[
\left| \frac{d(\tau \cdot b)}{db} \right| = e^{2\langle \tau^{-1} \cdot o, b \rangle} \quad (3)
\]

valid for an arbitrary isometry \( \tau \) of the non-Euclidean disk \( D \).

The generalization of the formula (2) to the symmetric space \( X \) is

\[
A(g \cdot x, g(b)) = A(x, b) + A(g \cdot o, g(b)) \quad (4)
\]

for \( g \in G, x \in X \) and \( b \in B \). (Here the action of \( G \) on \( X \) and on \( B \) is denoted as in §2.) In order to prove (4) let \( x = hK, \ b = kM \). Then

\[
h \cdot o \in k \exp A(x, b)N \cdot o
\]
so for some \( n_1 \in N, k_1 \in K \)

\[
gh = gk \exp A(x, b)n_1k_1
\]

which by the Iwasawa decomposition can be written

\[
gh = k(gk) \exp H(gk)n(gk) \exp A(x, b)n_1k_1
\]

Since \( aNa^{-1} \subset N (a \in A) \), this relation implies

\[
g \cdot x \in k(gk) \exp (H(gk) + A(x, b))N \cdot o
\]

and since \( k(gk)M = g(kM) \), we conclude

\[
A(g \cdot x, g(b)) = H(gk) + A(x, b)
\]  \( (5) \)

On the other hand, we have by the definition of \( A(g \cdot o, kM) \) that for some \( n_2 \in N, k_2 \in K \),

\[
g = k \exp A(g \cdot o, kM)n_2k_2
\]

so

\[
H(g^{-1}k) = -A(g \cdot o, kM)
\]  \( (6) \)

Hence, (5) becomes

\[
A(g \cdot x, g(b)) = -A(g^{-1} \cdot o, b) + A(x, b)
\]

In particular, putting \( x = o \), we get \( A(g \cdot o, g(b)) = -A(g^{-1} \cdot o, b) \), so the desired formula (4) follows. The generalization of (3) to the space \( X \) is given by

\[
\left| \frac{d(g(b))}{db} \right| = e^{2\rho(A(g^{-1} \cdot o, b))}
\]  \( (7) \)

and this of course is a direct consequence of Cor. 2.9 and (6). Now the proof of Theorem 4.5 proceeds essentially as the proof of Theorem 3.1 in Ch. 1.

Finally we observe that the Poisson integral representation of bounded harmonic functions on \( X \) (cf. (5) in §2) can be written

\[
u(x) = \int_B e^{2\rho_A(x, b)} \hat{u}(b) \, db
\]

and is, therefore, according to our definition, to be regarded as a formula in Fourier analysis on \( X \).

4-5 Interpretation by Representation Theory; Eigenfunctions of the Invariant Differential Operators

Since the group \( G \) leaves the volume element \( dx \) on \( X \) invariant we get a unitary representation \( T_x \) of \( G \) on \( L^2(X) \) by associating to each \( g \in G \) the operator \( f \mapsto f^{\tau(g)} \) on \( L^2(X) \). (Here \( f^{\tau(g)} \) denotes the function \( x \mapsto f(g^{-1} \cdot x) \).) We shall now indicate how Theorem 4.5 gives a decomposition of this representation into irreducible ones.
For \( \lambda \in a^* \) let \( \mathcal{S}_\lambda \) denote the vector space

\[
\mathcal{S}_\lambda = \left\{ h_\lambda(x) = \int_B e^{i\lambda \cdot \rho(A(x, b))} h(b) \, db \mid h \in L^2(B) \right\}
\]
of functions on \( X \). If \( \lambda \) is regular, that is, \( s\lambda \neq \lambda \) for all \( s \neq e \) in the Weyl group \( W \), one can use an irreducibility criterion of Bruhat [7], p. 193, to prove that the function \( h \in L^2(B) \) above is uniquely determined by \( h_\lambda \). If we define a Hilbert space norm on \( \mathcal{S}_\lambda \) by

\[
\| h_\lambda \| = \left( \int_B |h(b)|^2 \, db \right)^{1/2}
\]
then the mapping which assigns the operator \( h_\lambda(x) \to h_\lambda(g^{-1} \cdot x) \) to each \( g \in G \) is by (4) and (7) seen to be a unitary representation \( T_\lambda \) of \( G \) on \( \mathcal{S}_\lambda \). Using the irreducibility criterion cited, one can show this representation to be irreducible. Now with the notation of Theorem 4.5 there is a Plancherel formula, namely,

\[
\int_X |f(x)|^2 \, dx = \int_{a^*} \int_B |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} \, d\lambda \, db
\]

In terms of direct integrals of representations (see, for example, Dixmier [15]), Theorem 4.5 can therefore be written:

\[
L^2(X) = \int_{\mathcal{S}_\lambda} |c(\lambda)|^{-2} \, d\lambda \quad T_x = \int T_\lambda |c(\lambda)|^{-2} \, d\lambda
\]

\( \lambda \) running through \( a^* (\text{mod } W) \).

The functions in \( \mathcal{S}_\lambda \) are eigenfunctions of each \( D \in D(G/K) \). More generally, if \( T \) is an analytic functional on \( B \) and \( \mu \in C \) the function

\[
f(x) = \int_B e^{\mu(A(x, b))} \, dT(b)
\]
is an eigenfunction of each \( D \in D(G/K) \); it appears likely that for sufficiently general functionals \( T \) these functions constitute all the simultaneous eigenfunctions of the operators \( D(G/K) \) (cf. Theorem 5.1, Ch. 1).

## 4-6 Invariant Differential Equations on Symmetric Spaces

We shall now discuss general existence theorems for invariant differential equations on the symmetric space \( G/K \). In order to motivate the method followed we first describe a well-known geometric method for solving differential equations in \( \mathbb{R}^n \) with constant coefficients (Courant–Lax [14], Gelfand–Shapiro [20], John [44]). The basis of the method is a formula of Radon–John which in an explicit manner describes a function on \( \mathbb{R}^n \) by means of its integrals over the various hyperplanes in \( \mathbb{R}^n \).
For \( f \in C_c^\infty(\mathbb{R}^n) \) let \( \hat{f}(\omega, p) \) denote the integral of \( f \) over the hyperplane \((x, \omega) = p\) (here \( \omega \) is a unit vector and \( p \in \mathbb{R} \) and \((,\) the scalar product). The function \( \hat{f} \) is called the Radon transform of \( f \).

**Theorem 6.1.** For the Radon transform \( f \rightarrow \hat{f} \) the following inversion formula holds:

\[
\hat{f}(x) = c(\Delta)^{(n-1)/2}\left( \int_{S^{n-1}} \hat{f}(\omega, (x, \omega)) \, d\omega \right)
\]

(1)

for \( f \in C_c^\infty(\mathbb{R}^n) \). Here \( \Delta \) denotes the Laplacian, \( d\omega \) is the surface element on the unit sphere \( S^{n-1} \), and \( c \) is a constant.

For the proof see [44]. There the cases \( n = \text{odd} \) and \( n = \text{even} \) are presented in different forms; the unified version can be found in [34], p. 163.

Formula (1) states that when for \( x \in \mathbb{R}^n \) we form the integral of \( f \) over each hyperplane through \( x \), then take the average of these integrals, and finally apply the operator \( \Delta^{(n-1)/2} \), we recover the function \( f \). However, for the applications indicated, the important feature of (1) is an explicit decomposition of \( f \) into plane waves. (A plane wave is a function which is constant on each hyperplane with a given normal vector; this normal vector is then called the normal to the plane wave.) In fact, for any fixed \( \omega \in S^{n-1} \) the function \( f_\omega : x \rightarrow \hat{f}(\omega, (x, \omega)) \) is a plane wave with normal \( \omega \).

We shall now apply formula (1) to differential equations. Let \( D \) be a differential operator on \( \mathbb{R}^n \) with constant coefficients and consider a differential equation

\[
Du = f
\]

(2)

where \( f \in C_c^\infty(\mathbb{R}^n) \) is a given function. We begin by considering the differential equation

\[
Dv = f_\omega
\]

(3)

where \( f_\omega \) is as above and we look for a solution \( v \) which is a plane wave with normal \( \omega \). But a plane wave with normal \( \omega \) is just a function of one variable; furthermore if \( v \) is a plane wave with normal \( \omega \) then so is the function \( Dv \). Our problem of finding \( v \) of the specified type satisfying (3) is therefore just an ordinary differential equation with constant coefficients. Pick a solution \( u_\omega \) and assume that this choice can be made smoothly in \( \omega \). Then the function

\[
u = c \Delta^{(n-1)/2} \int_{S^{n-1}} u_\omega \, d\omega
\]

(4)

is a solution of the equation (1). In fact, since differential operators with constant coefficients commute we have (at least for \( n \) odd)

\[
Du = c \Delta^{(n-1)/2} \int_{S^{n-1}} Du_\omega \, d\omega = c \Delta^{(n-1)/2} \int_{S^{n-1}} f_\omega \, d\omega = f
\]
This proof actually works also for $n$ even. The weakness of the method lies in the assumption that $u_\omega$ can be chosen so as to vary smoothly in $\omega$. In fact the example $D = \delta^2/\delta x_1 \delta x_2$, $\omega = (1, 0)$ shows that $u_\omega$ may not exist for all $\omega$.

For a symmetric space $X = G/K$ the inversion formula for the Fourier transform (Theorem 4.5) does give a decomposition of an arbitrary function $f \in C_c^\infty(X)$ into plane waves. In fact let as before

$$\hat{f}(\lambda, b) = \int_X f(x)e^{-i\lambda \cdot x + \rho(A(x, b))} \, dx \quad \lambda \in a^*, \ b \in B$$

and put

$$f_b(x) = \int_{a^*} \hat{f}(\lambda, b) e^{i\lambda \cdot x + \rho(A(x, b))} |c(\lambda)|^{-2} \, d\lambda$$  \hspace{1cm} (5)

Then $f_b(x)$ is a plane wave with normal $b$ so the formula

$$f(x) = \int_B f_b(x) \, db$$  \hspace{1cm} (6)

does indeed give a decomposition of $f$ into plane waves. We shall now apply this formula to the problem of solving a differential equation

$$Du = f$$  \hspace{1cm} (7)

where $D$ is a given differential operator in $D(G/K)$ and $f \in C_c^\infty(X)$ is a given function. First we need a simple lemma concerning the action of invariant differential operators on plane waves (cf. [27], p. 247, or [45]).

**Lemma 6.2.** Let $D \in D(G/K)$. Then there exists a unique differential operator $\delta(D)$ on the submanifold $A \cdot o \subset X$ such that if bar denotes restriction to $A \cdot o$,

$$\bar{D}F = \delta(D)\bar{F}$$

for every $F \in C^\infty(X)$ which is $N$-invariant (that is, a plane wave with normal $a^+$). This differential operator $\delta(D)$ is invariant under $A$.

**Proof.** Since the mapping $(n, a \cdot o) \to na \cdot o$ is a diffeomorphism of $N \times (A \cdot o)$ onto $X$ the existence and uniqueness of $\delta(D)$ is obvious. Hence we just have to prove its invariance under $A$. Let $a \in A$ and, as before, if $F \in C^\infty(X)$ let $F^{(a)}$ denote the function $x \to F(a^{-1} \cdot x)$ on $X$. If $F$ is invariant under $N$ then the function $F^{(a)}$ is too; in fact,

$$F^{(a)}(n \cdot x) = F(a^{-1}n \cdot x) = F(n_1a^{-1} \cdot x)$$

for some $n_1 \in N$. Thus $F^{(a)}(n \cdot x) = F^{(a)}(x)$, and of course $F^{(a)} = (\bar{F})^{(a)}$. 

Thus,
\[(\delta(D)^{\tau(a)F} = (\delta(D)(\bar{F})^{\tau(a^{-1})})^{\tau(a)} = (\delta(D)\bar{F}^{\tau(a^{-1})})^{\tau(a)} = ((DF)^{\tau(a^{-1})})^{\tau(a)} = (DF)^{\tau(a^{-1})})^{\tau(a)} = DF = \delta(D)\bar{F})\]
This proves the lemma because each function in $C^\infty(A \cdot o)$ can be extended to an $N$-invariant function in $C^\infty(X)$.

In order to solve the differential equation (7) we begin by considering the differential equation
\[Du = f_b\]
for an arbitrary $b \in B$. We look for a solution $v = v^b$ which like the function $f_b$ [cf. (5)] is a plane wave with normal $b$. For example, consider the case $b = a^\tau$. Then the function $f_b$ is invariant under $N$ and so is the required function $v^b$. According to Lemma 6.2, the differential equation $Du^b = f_b$ on $X$ amounts to the differential equation
\[\delta(D)v^b = f_b\]
which is by the $A$-invariance of $\delta(D)$ a differential equation with constant coefficients on the Euclidean space $A \cdot o$. But by a result of Ehrenpreis [16] and Malgrange [52], a differential operator on $R^n$ with constant coefficients maps the space $C^\infty(R^n)$ onto itself. Hence a solution $v = v^b$ exists. Now we assume that $v^b$ can be chosen so that it depends smoothly on $b$. Then we put
\[u(x) = \int_B v^b(x)\, db \quad x \in X\]
and have
\[Du = \int_B Du^b\, db = \int_B f_b\, db = f\]
This is not an existence proof for the differential equation (7) because of the smoothness assumption about $v^b$ (see, however, Trèves [59], p. 131). Nevertheless, we have the following general theorem (Helgason [33], p. 577–578).

**Theorem 6.3.** Let $D \neq 0$ be an arbitrary $G$-invariant differential operator on the symmetric space $G/K$. For each $f \in C_c^\infty(G/K)$ the differential equation $Du = f$ has a solution $u \in C^\infty(G/K)$.

It suffices to find a distribution $T$ on $X$ satisfying the differential equation $DT = \delta$, where $\delta$ is the delta-distribution at the origin $o$. In fact, the desired solution is then $u = f \times T$, where $\times$ is the operation on distributions on $X$ which is induced by the convolution product of distributions on $G$. Since $D$ and $\delta$ are $K$-invariant we look for a $K$-invariant $T$. For this we use the transform $f \mapsto F_f$ discussed in §3. As proved in Harish-Chandra [28],
p. 46, this transform is one-to-one on the space $I(X)$ of $K$-invariant, square-integrable functions on $X$ which are rapidly decreasing on $X$ in a certain technical sense, and the transform maps $I(X)$ into the space $I(A)$ of Weyl group invariant functions on $A$ which are rapidly decreasing on $A$ (considered as a Euclidean space). On the other hand, it is proved in Helgason [33] that the range of the mapping $f \mapsto F_f \ (f \in I(X))$ is precisely $I(A)$ and furthermore, $F_{Df} = \gamma(D)F_f$, where $\gamma(D)$ is a certain constant-coefficient differential operator on $A$. The isomorphism $f \mapsto F_f$ of $I(X)$ onto $I(A)$ has a transpose, mapping the dual $I'(A)$ of $I(A)$ onto the dual $I'(X)$ of $I(X)$. Under this isomorphism the differential equation $DT = \delta$ on $X$ is transformed into a differential equation for tempered distribution on $A$, and this last differential equation has constant coefficients since $\gamma(D)$ does. But by a theorem of Hörmander [40] and Łojasiewicz [49] any differential operator on $\mathbb{R}^n$ with constant coefficients maps the space of tempered distributions on $\mathbb{R}^n$ onto itself. This leads to the desired distribution $T$ on $X$, proving the theorem.

4.7 The Wave Equation on Symmetric Spaces

We shall now discuss a different method for solving differential equations on the symmetric space $X$. It uses the Radon transform on $X$ which we now define. Let $\Xi$ denote the set of all horocycles in $X$. For $f \in C_c^\infty(X)$ we define the function $\hat{f}$ on $\Xi$ by

$$\hat{f}(\xi) = \int_X f(x) \, d\sigma(x) \quad (\xi \in \Xi)$$

(1)

where $d\sigma$ is the volume element on $\xi$. (The Riemannian structure on $X$ induces in an obvious way a Riemannian structure on the submanifold $\xi$.) The function $\hat{f}$ is called the Radon transform of $f$.

If $x \in X$ the (compact) subgroup $K_x$ of $G$ which keeps $x$ fixed permutes the horocycles through $x$ transitively. For $x = o$ this is obvious from Lemma 4.2 and in general it follows by the homogeneity of $X$. The set of horocycles passing through $x$ has a unique normalized measure, say $v$, invariant under $K_x$.

If $\phi$ is a function on $\Xi$ the function $\check{\phi}$ on $X$ is defined by

$$\check{\phi}(x) = \int_{x \in \xi} \phi(\xi) \, dv(\xi)$$

(2)

**Theorem 7.1.** Suppose all Cartan subgroups of $G$ are conjugate. Then for a certain fixed differential operator $\Box \in D(G/K)$

$$f = \Box((\hat{f})^*) \quad \in C_c^\infty(G/K)$$

(3)
This formula is analogous to the inversion formula of Radon–John (Theorem 6.1) for the case of an odd-dimensional Euclidean space. The even-dimensional Euclidean case corresponds here to the existence of non-conjugate Cartan subgroups and in this case (3) still holds in a slightly modified form (cf. [35], p. 759). We emphasize that the differential operator \( \Box \) can be written down quite explicitly.

By means of (3) one can write down a solution of the wave equation on \( X \),

\[
\frac{\partial^2 u}{\partial t^2} = \Delta u
\]

with initial data

\[
\begin{align*}
\frac{\partial u}{\partial t} \bigg|_{t=0} &= f(x) \\
u(x, 0) &= 0
\end{align*}
\]

Here \( \Delta \) denotes the Laplace–Beltrami operator on \( X \) and \( f \) is an arbitrary given function in \( C_c^\infty(X) \).

In the notation of §1, let \( \Box \in D_K(G) \) be an operator satisfying \( (\Box f) \sim = \Box f \) for all \( f \in C^\infty(G/K) \). Let \( |\rho| \) denote the norm of the linear form \( \rho \), and let \( dn \) be a Haar measure on \( N \) which corresponds to the volume element \( d\sigma \) on \( \xi_o = N \cdot o \) under the diffeomorphism \( n \to n \cdot o \). Let \( \Delta_A \) denote the Laplacian on the Euclidean space \( A \).

**Theorem 7.2.** The solution to the wave equation (4) with initial data (5) is given by

\[
u(g \cdot o, t) = \Box_g \left( \int_K V_{k,g}(e, t) \, dk \right)
\]

where \( V_{k,g} \) is the solution to the equation for damped waves on \( A \times R \),

\[
(\Delta_A - |\rho|^2)V_{k,g} = \frac{\partial^2}{\partial t^2} V_{k,g}
\]

\[
V_{k,g}(a, 0) = 0, \quad \left\{ \frac{\partial}{\partial t} V_{k,g}(a, t) \right\}_t=0 = e^{\rho(\log a)}F_{k,g}(a)
\]

where

\[
F_{k,g}(a) = \int_N f(gk an \cdot o) \, dn
\]

Although the verification of this theorem is not long (cf. [32], p. 688) we omit it here because it requires some further preparation. The function \( V_{k,g} \) is given as a convolution of a certain Bessel function with \( F_{k,g} \) so the solution (6) is explicitly given in terms of the initial data \( f(x) \).
Huygens’ Principle

Let $M$ be an analytic pseudo-Riemannian manifold with Lorentzian signature, in short, a Lorentzian manifold. Since our considerations will be local we assume that $M$ is convex, that is, any two points in $M$ can be joined by a unique geodesic. The geodesics of zero length through a point $p \in M$ generate the light cone $C_p$ in $M$ with vertex $p$. A submanifold $S \subset M$ is called spacelike if each tangent vector to $S$ is spacelike. Let $\Delta$ denote the (hyperbolic) Laplace–Beltrami operator on $M$, and suppose now that a Cauchy problem is posed for the wave equation $\Delta u = 0$ with initial data on a spacelike hypersurface $S \subset M$. Hadamard proved that the value $u(p)$ of the solution at a point $p \in M$ only depends on the initial data on the piece $S^* \subset S$ which lies inside the light cone $C_p$. Huygens’ principle (in the strong sense) is said to hold for $\Delta u = 0$ if the value $u(p)$ only depends on the initial data in an arbitrary small neighborhood of the edge $s$ of $S^*$, $s = C_p \cap S$. It is known that this is a property of the space $M$ and does not depend on the particular choice of $S \subset M$. The wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2} = 0$$

for an odd-dimensional $R^{n-1}$ satisfies Huygens’ principle. A conjecture, attributed to Hadamard, was that these were essentially the only second-order hyperbolic equations satisfying Huygens’ principle. A counter-example of the form $\Delta u + cu = 0$ ($n = 6$) was given by Stellmacher [58] in 1953, and in 1965, P. Günther [23] gave a whole series of counter-examples for the pure equation $\Delta u = 0$ ($n = 4$). These are based on Hadamard’s criterion that Huygens’ principle holds if and only if $n$ is even and $\geq 4$ and the logarithmic part of the fundamental solution (in Hadamard’s sense) vanishes.

If $M$ is symmetric the evidence available seems to indicate that “Hadamard’s conjecture” might hold for the pure equation $\Delta u = 0$. For $M$ of constant curvature (a “de Sitter space” or an “anti de Sitter space”) this is indeed so (cf. [29], p. 296; see also [13].) The answer is also affirmative if $M$ has the form $M = M_o \times R$, where $M_o$ has dimension 3 and constant curvature (Hölder [39]). Finally the answer is affirmative if $M = X \times R$, where $X$ is a symmetric space whose group of isometries is a complex semisimple Lie group (Helgason [33], p. 582).

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