I

Lie Groups and Symmetric Spaces

SIGURDUR HELGASON

The purpose of these lectures is to give an account of the theory of those Lie groups which have played a particular role in geometry and in physics—the so-called semisimple Lie groups. Associated with these groups are the symmetric spaces, whose theory is a kind of an intersection of Riemannian geometry and Lie group theory.

The primary prerequisites for reading these notes are some familiarity with the elements of the theory of topological groups and differentiable manifolds. The emphasis is on noncompact semisimple Lie groups and the associated (noncompact) symmetric spaces. The function theory on these spaces is treated in a relatively detailed manner; however the holomorphic function theory is omitted altogether.

Although the definitions and theorems are usually stated in full generality, complete proofs are given only if they are either very short or particularly instructive. Verification for a special case is a frequent substitute for a proof. A study of special cases is in fact very important for understanding of Lie theory. With this in mind, Chapter 1 is devoted to the special group $G = \boldsymbol{SU}(1, 1)$ and the associated symmetric space, the non-Euclidean disk. Chapters 2 and 3 deal with selected topics from the classical theory of Lie groups and symmetric spaces. The results in Chapter 4 are of more recent vintage but almost all of them have been published elsewhere. The only exceptions are the integral representation of the eigenfunctions of the Laplacian on the non-Euclidean disk (Theorem 5.1, Ch. 1) and the extension of Fatou's theorem to harmonic functions on symmetric spaces (Theorem 2.12, Ch. 4) proved by A. Korányi and the author.

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GENERAL NOTATION

We list here some standard notation which will be utilized throughout the lectures. The symbols $\boldsymbol{R}$, $\boldsymbol{C}$, and $\mathbf{Z}$ refer to the real numbers, the complex numbers, and the integers, respectively. The nonnegative reals are denoted by $\boldsymbol{R}^+$ and the nonnegative integers by $\mathbf{Z}^+$. The conjugate of a complex number c is denoted by $\bar{c}$. The empty set is denoted by $\varnothing$. If X is a set and $x \in X$ then the subset of X consisting of x alone is denoted by $\{x\}$.

If M is a manifold, the set of complex-valued indefinitely differentiable functions on M is denoted $C^\infty(M)$. The set of functions $f \in C^\infty(M)$ of compact support is denoted $C_c^\infty(M)$. If $p \in M$ the tangent space to M at p is denoted by M_p. Let M and N be manifolds and $\phi : M \to N$ a differentiable mapping. The differential of ϕ at a point $p \in M$, denoted $d\phi_p$, or just ϕ, is a mapping of M_p into $N_{\phi(p)}$ defined by $d\phi_p(X)(f) = X(f \circ \phi)$ if X is any vector in M_p and f any function in $C^\infty(N)$. If $t \to \gamma(t)$ is any curve in M with tangent vector X at the point p then $d\phi_p(X)$ is the tangent vector to the curve $t \to \phi(\gamma(t))$ at $\phi(p)$. The differentiable map $\phi : M \to N$ is called a diffeomorphism if it is a one-to-one map of M onto N and if the inverse map $\phi^{-1} : N \to M$ is differentiable.

CHAPTER 1: INTRODUCTION

1-1 Lie Groups

A *Lie group* is a group G which is also an analytic manifold such that the mapping $(g, h) \to gh^{-1}$ of the product manifold $G \times G \to G$ is analytic.

Roughly speaking, this means that, at least locally, a Lie group is parametrized by an n-tuple of real numbers such that the group operations are expressed by analytic functions in these parameters. This makes it possible to study these groups by analytical methods.

Lie group theory can be traced back to Sophus Lie's applications of group theory to geometric situations as well as to his desire to obtain a theory of differential equations which paralleled Galois' theory for algebraic equations. Since groups at that time were usually viewed as permutation groups, the geometric problems led naturally to the consideration of transformation groups with certain invariance properties. These invariance properties often give rise to a parametrization of the group, turning it into a Lie group.

Example

Let G denote the group of transformations of the plane $\boldsymbol{R}^2$ preserving distance as well as orientation. If $g \in G$ let $(x(g), y(g))$ denote the coordinates of $g \cdot 0$ (0 is the origin in $\boldsymbol{R}^2$) and $\theta(g)$ the angle from the x axis l to the line $g \cdot l$. The parametrization

$$g \to (x(g), y(g), \theta(g))$$

turns G into a Lie group.

1-2 Symmetric Spaces

Let M be a C^∞ manifold. A *Riemannian structure* on M is a positive definite inner product $\langle\ ,\ \rangle$ on the tangent space M_p at an arbitrary point $p \in M$. It is assumed that if X, Y are C^∞ vector fields on M then the function $p \to \langle X_p, Y_p \rangle$ is a C^∞ function on M. A manifold with a Riemannian structure is called a *Riemannian manifold*.

Example

The following example is of basic importance and will accompany us throughout these lectures.

Let D be the open unit disk $|z| < 1$ in $\boldsymbol{R}^2$ with the usual manifold structure but given the following Riemannian structure: If u, v are tangent vectors at the point $z \in D$, put

$$\langle u, v\rangle = \frac{(u, v)}{[1 - |z|^2]^2} \tag{1}$$

$(,)$ denoting the usual inner product on $\boldsymbol{R}^2$. Since

$$\frac{\langle u, v\rangle^2}{\langle u, u\rangle\langle v, v\rangle} = \frac{(u, v)^2}{(u, u)(v, v)}$$

the angle between u and v in the new Riemannian structure coincides with the Euclidean angle.

The length of a curve $\gamma(t)$ $(\alpha \leqslant t \leqslant \beta)$ on a Riemannian manifold is defined by

$$L(\gamma) = \int_\alpha^\beta \langle\gamma'(t), \gamma'(t)\rangle^{1/2}\, dt$$

and the distance between two points $p, q \in M$ is defined by

$$d(p, q) = \inf_\gamma L(\gamma)$$

the infimum taken over all curves joining p and q. In our case if $\gamma(t) = (x(t), y(t))$ and $s(\tau)$ is the arc-length of the segment $\gamma(t)$ $(0 \leqslant t \leqslant \tau)$, we get

$$\left(\frac{ds}{d\tau}\right)^2 = \frac{1}{\{1 - [\alpha(\tau)^2 + y(\tau)^2]\}^2}\left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2\right]$$

In classical terminology this is written

$$ds^2 = \frac{dx^2 + dy^2}{[1 - (x^2 + y^2)]^2} \tag{2}$$

In particular, if $\gamma(\alpha) = 0$, $\gamma(\beta) = x$ (point on the x axis) and we denote by γ_0 the line segment from 0 to x, we get from

$$\frac{x'(\tau)^2}{[1 - x(\tau)^2]^2} \leqslant \frac{x'(\tau)^2 + y'(\tau)^2}{\{1 - [x(\tau)^2 + y(\tau)^2]\}^2}$$

the inequality

$$L(\gamma_0) \leqslant L(\gamma)$$

Thus

$$d(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} \tag{3}$$

and the straight lines through the origin are geodesics.

Let us now determine the group $I(D)$ of all isometrics on D. If $a, b \in \mathbf{C}$ then the transformation

$$g : z \to \frac{az + b}{\bar{b}z + \bar{a}} \qquad |a|^2 - |b|^2 = 1 \tag{4}$$

maps D onto itself. Let us verify that g preserves the Riemannian structure (1): Let $z(t)$ be a curve with $z(0) = z$, $z'(0) = u$. Then

$$g \cdot u = \left\{\frac{d}{dt} g[z(t)]\right\}_{t=0} = \text{the vector } \frac{z'(0)}{(\bar{b}z + \bar{a})^2} \text{ at } g \cdot z$$

and the relation

$$\langle g \cdot u, g \cdot u \rangle = \langle u, u \rangle$$

follows immediately. Now if $h \in I(D)$ is arbitrary, there exists a g as in (4) such that gh^{-1} leaves the x axis pointwise fixed. But then gh^{-1} is either the identity or the conjugation $z \to \bar{z}$. Thus $I(D)$ is generated by the transformation (4) and the conjugation $c : z \to \bar{z}$. Denoting as usual

$$\boldsymbol{SU}(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| \; |a|^2 - |b|^2 = 1 \right\}$$

and by I the identity matrix, we have

$$I(D) = (\boldsymbol{SU}(1, 1)/\pm I) \cup c(\boldsymbol{SU}(1, 1)/\pm I)$$

In particular, $I(D)$ is a Lie group (a fact which was proved for all Riemann manifolds in Myers and Steenrod [55]).

Since the group of transformations (4) is transitive on D we deduce that the geodesics in D are the circular arcs perpendicular to the boundary $|z| = 1$. Since the expression for $d(0, z)$ can be written by means of the cross ratio

$$d(0, z) = \frac{1}{2} \log \left(\frac{0 - z/|z|}{0 + z/|z|} : \frac{z - z/|z|}{z + z/|z|} \right)$$

and since the cross ratio is invariant under fractional linear transformations we obtain

$$d(z_1, z_2) = \frac{1}{2} \log \left(\frac{z_1 - b_2}{z_1 - b_1} : \frac{z_2 - b_2}{z_2 - b_1} \right) \qquad z_1, z_2 \in D \tag{5}$$

(b_1, b_2 being shown in Fig. 1). But the space D with this distance d is of course the classical Poincaré model of non-Euclidean geometry.

Definition. A Riemannian manifold M is called *symmetric* (or *globally symmetric*) in the sense of É. Cartan if for each $p \in M$ there is an isometry s_p of M onto itself which reverses the geodesics through p (s_p is called the *geodesic symmetry* with respect to p).

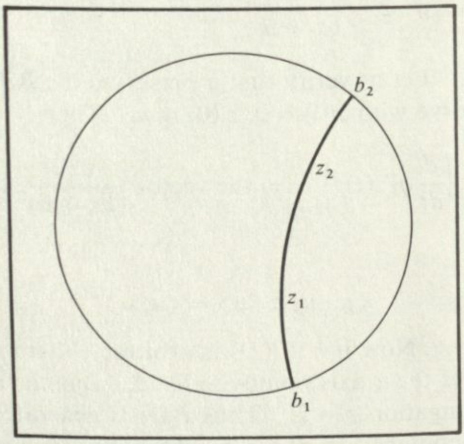

FIGURE 1

Since the symmetry $s_0 : z \to -z$ is of the form (4) it is an isometry of D. If $g \in I(D)$, then the isometry gs_0g^{-1} reverses the geodesics through $g \cdot 0$; $I(D)$ being transitive, D is therefore symmetric.

Let $\gamma(t)$ $(-\infty < t < \infty)$ be a geodesic in a symmetric space M, let $s_t = s_{\gamma(t)}$, and let τ_t denote the Levi–Civita parallel transport along γ from 0 to t. If L is a tangent vector to M at $\gamma(t)$, then since s_0 preserves parallelism and $s_0(\tau_{-t} L) = -\tau_{-t} L$ we see that $s_0(L) = -\tau_{-2t} L$. Consequently, the isometry $T_t = s_{t/2} s_0$ realizes the parallelism from 0 to t along γ. The isometries T_t actually form a one-parameter group—the group of *transvections* along the geodesic γ.

Let M be a Riemannian manifold, (U, ϕ) a local coordinate system and $\phi(q) = (x_1, \ldots, x_n)$ for $q \in U$. We put

$$g_{ij}(q) = \left\langle \left(\frac{\partial}{\partial x_i}\right)_q, \left(\frac{\partial}{\partial x_j}\right)_q \right\rangle,$$

$$g = \det (g_{ij}),$$

$$g^{ij} = (g_{ij})^{-1}$$

Then we can define a measure μ on U by

$$\mu(C) = \int_{\phi(C)} \sqrt{g}\, dx_1 \cdots dx_n \tag{6}$$

(where we have written $\sqrt{g}$ for $\sqrt{g} \circ \phi^{-1}$). This definition is invariant under coordinate changes and defines a measure on M, the *Riemannian measure*. Somewhat imprecisely (M is not necessarily orientable) one refers to $\sqrt{g}\, dx_1 \ldots dx_n$ as the *volume element* on M.

We also recall the *Laplace–Beltrami operator* defined for $f \in C^\infty(U)$ by

$$\Delta : f \to \frac{1}{\sqrt{g}} \sum_k \frac{\partial}{\partial x_k} \left(\sum_i g^{ik} \sqrt{g} \frac{\partial}{\partial x_i} \right)(f) \tag{7}$$

Again, the expression on the right can be shown to be invariant under coordinate changes and so defines a differential operator on M.

In the case of D we find at once from (1),

$$g_{ij} = [1 - |z|^2]^{-2} \delta_{ij} \qquad (\delta_{ij} = \text{Kronecker delta})$$

$$g^{ij} = [1 - |z|^2]^2 \delta_{ij} \qquad g = (1 - |z|^2)^{-4}$$

The volume element is therefore given by

$$[1 - (x^2 + y^2)]^{-2}\, dx\, dy \tag{8}$$

and the Laplace–Beltrami operator is

$$\Delta = [1 - (x^2 + y^2)]^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{9}$$

1-3 Non-Euclidean Fourier Analysis

We shall now define a Fourier transform on the non-Euclidean disk D. First we recall the Fourier inversion formula on $\boldsymbol{R}^n$. For $f \in L^1(\boldsymbol{R}^n)$ put

$$\tilde{f}(u) = \int_{\boldsymbol{R}^n} f(x) e^{-i(x,u)}\, dx \tag{1}$$

$(\ ,\ )$ denoting the usual inner product on $\boldsymbol{R}^n$. Then if $f \in C_c^\infty(\boldsymbol{R}^n)$,

$$f(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} \tilde{f}(u) e^{i(x,u)}\, du \tag{2}$$

Let us introduce polar coordinates $u = \lambda w$, $\lambda \geqslant 0$, and w is a unit vector. Then (1) and (2) become

$$\tilde{f}(\lambda w) = \int_{\boldsymbol{R}^n} f(x) e^{-i\lambda(x,w)}\, dx \tag{3}$$

$$f(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^+} \int_{S^{n-1}} \tilde{f}(\lambda w) e^{i\lambda(x,w)} \lambda^{n-1}\, d\lambda\, dw \tag{4}$$

where $\boldsymbol{R}^+ = \{\lambda \in \boldsymbol{R} \mid \lambda \geqslant 0\}$ and dw is the volume element on the unit sphere S^{n-1}.

Because the functions $e_u : x \to e^{i(x,\,u)}$ are characters of the group $\boldsymbol{R}^n$, the Fourier transform (1) can be generalized to locally compact Abelian groups. Since D is not a group this viewpoint is not directly applicable here. However, the functions e_u have the following properties:

(i) e_u is an eigenfunction of the Laplace operator on $\boldsymbol{R}^n$;

(ii) e_u is constant on each hyperplane perpendicular to u ("plane wave" with normal u).

These properties essentially characterize the exponentials and since they are geometric properties we shall see that they have analogs for the space D.

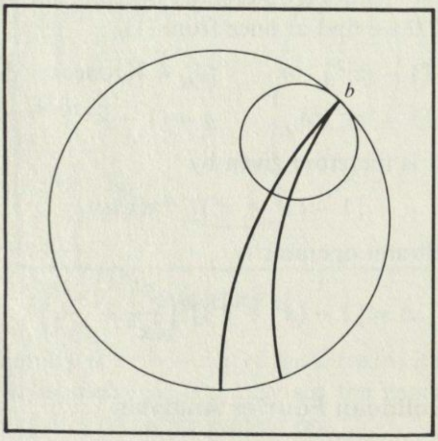

FIGURE 2

Parallel geodesics in D are by definition geodesics corresponding to the same point b on the boundary B of D. A *horocycle with normal b* is by definition an orthogonal trajectory to the family of all parallel geodesics corresponding to b. Thus a horocycle in D is the non-Euclidean analog of a hyperplane in $\boldsymbol{R}^n$. Since the inner product (x, w) in (3) is the distance from the origin to the hyperplane with normal w passing through x we define $\langle z, b\rangle$ for $z \in D$, $b \in B$, as the non-Euclidean distance from 0 to the horocycle $\xi(z, b)$ with normal b, passing through z. (Here $\langle z, b\rangle$ is taken negative in case 0 falls inside the horocycle.)

For $\mu \in \boldsymbol{C}$, $b \in B$ we consider the function

$$e_{\mu,\,b} : z \to e^{\mu\langle z,\,b\rangle} \qquad z \in D \tag{5}$$

These formal analogs of the exponential functions on $\boldsymbol{R}^n$ are also conceptual analogs for they satisfy the following non-Euclidean counterparts to (i) and (ii):

(i)′ $e_{\mu, b}$ is an eigenfunction of the Laplace–Beltrami operator on D (for example, use (9) in §1-2 and (11) below);

(ii)′ $e_{\mu, b}$ is constant on each horocycle with normal b.

Consequently, we define *Fourier analysis on D* to be decomposition of "arbitrary" functions into functions $e_{\mu, b}$ in (5).

Theorem 3.1. For $f \in C_c^\infty(D)$ set

$$\tilde{f}(\lambda, b) = \int_D f(z) e^{(-i\lambda+1)\langle z, b\rangle} \, dz \qquad \lambda \in \boldsymbol{R}, b \in B$$

where dz is the volume element on D. Then

$$f(z) = (2\pi)^{-2} \int_{\boldsymbol{R}} \int_B \tilde{f}(\lambda, b) e^{(i\lambda+1)\langle z, b\rangle} \lambda \tanh(\tfrac{1}{2}\pi\lambda) \, d\lambda \, db \tag{6}$$

where db is the usual angular measure on B.

We shall now indicate how (6) follows from classical facts. Denote the measure $(2\pi)^{-2}\lambda \tanh(\frac{1}{2}\pi\lambda) \, d\lambda \, db$ by $d\mu(\lambda, b)$ and define the operators T and S by

$$(Tf)(\lambda, b) = \tilde{f}(\lambda, b) \qquad f \in C_c^\infty(X)$$

$$(SF)(z) = \int_{\boldsymbol{R}\times B} F(\lambda, b) e^{(i\lambda+1)\langle z, b\rangle} \, d\mu(\lambda, b)$$

the function F restricted such that the integral converges absolutely. Then

$$\int_D f(z)\overline{(SF)(z)} \, dz = \int_{\boldsymbol{R}\times B} (Tf)(\lambda, b)\overline{F(\lambda, b)} \, d\mu(\lambda, b)$$

and by iteration

$$\int_D f(z)\overline{STg(z)} \, dz = \int_D (STf)(z)\overline{g(z)} \, dz \qquad f, g \in C_c^\infty(D) \tag{7}$$

because Tf and Tg satisfy the growth restrictions placed on F.

Lemma 3.2. Let τ be an isometry of D and if g is a function on D, put $g^\tau(z) = g(\tau^{-1} \cdot z)$. Then

$$STf^\tau = (STf)^\tau \qquad \text{for } f \in C_c^\infty(D)$$

Proof. Since τ preserves the volume element on D,

$$\widetilde{f^\tau}(\lambda, b) = \int_D f(z) e^{(-i\lambda+1)\langle \tau\cdot z, b\rangle} \, dz \tag{8}$$

But the isometry τ extends in an obvious way to the boundary B (*cf.* (4) §1-2), and we have

$$\langle \tau \cdot z, \tau \cdot b \rangle = \langle z, b \rangle + \langle \tau \cdot 0, \tau \cdot b \rangle \tag{9}$$

This identity is easily seen by observing that the horocycles $\xi(\tau \cdot 0, \tau \cdot b)$ and $\xi(\tau \cdot z, \tau \cdot b)$ cut segments of equal length off the parallel geodesics $(0, \tau \cdot b)$ and $(\tau \cdot 0, \tau \cdot b)$. Thus

$$\langle \tau \cdot z, b \rangle = \langle z, \tau^{-1} \cdot b \rangle + \langle \tau \cdot 0, b \rangle$$

so (8) becomes

$$\tilde{f}^{\tau}(\lambda, b) = e^{(-i\lambda + 1)\langle \tau \cdot 0, b \rangle} \tilde{f}(\lambda, \tau^{-1} \cdot b)$$

so

$$(STf^{\tau})(z) = \int_{R \times B} \tilde{f}(\lambda, \tau^{-1} \cdot b) e^{(-i\lambda + 1)\langle \tau \cdot 0, b \rangle} e^{(i\lambda + 1)\langle z, b \rangle} \, d\mu(\lambda, b)$$

Now we change variables; the Jacobian of the mapping $b \to \tau \cdot b$ satisfies

$$\left| \frac{d(\tau \cdot b)}{db} \right| = e^{2\langle \tau^{-1} \cdot 0, b \rangle} \qquad b \in B \tag{10}$$

In order to verify this observe that $\tau = \kappa_1 \sigma \kappa_2$, where κ_1, κ_2 are rotations around 0, and σ maps the x axis onto itself. We can thus assume τ of the form

$$\tau \cdot z = \frac{(\cosh t)z + \sinh t}{(\sinh t)z + \cosh t}$$

so if $b = e^{i\phi}$, the left-hand side of (10) equals $(\cosh 2t + \sinh 2t \cos \phi)^{-1}$. On the other hand, a simple computation using (3) §1-2 shows that if $z = |z| e^{i\theta}$, $b = e^{i\phi}$ then

$$e^{2\langle z, b \rangle} = \frac{1 - |z|^2}{1 - 2|z| \cos(\theta - \phi) + |z|^2} \tag{11}$$

so, in particular, (10) follows. Using also $\langle \tau^{-1} \cdot 0, b \rangle = -\langle \tau \cdot 0, \tau \cdot b \rangle$ [which follows from (9)], we obtain

$$(STf^{\tau})(z) = \int_{R \times B} \tilde{f}(\lambda, b) e^{(-i\lambda - 1)\langle \tau \cdot 0, \tau \cdot b \rangle} e^{(i\lambda + 1)\langle z, \tau \cdot b \rangle} \, d\mu(\lambda, b)$$

which again by (9) equals $(STf)(\tau^{-1} \cdot z)$, proving the lemma.

In order to prove Theorem 3.1, that $f = STf$, it suffices, by (7), to prove this for a sequence (f_n) where $f_n \to \delta_z$, the delta function at an arbitrary point $z \in D$. By Lemma 3.2 we can assume that z is the origin in D. But then the functions f_n could be taken to be radial functions. But if $f(z) = F(d(0, z))$, $F \in C_c^{\infty}(\boldsymbol{R})$ (F even), then $\tilde{f}(\lambda, b)$ is an even function $\tilde{F}(\lambda)$ of λ alone. If $r = d(0, z)$ then

$$z = |z| \, e^{i\theta} = (\tanh r) e^{i\theta}$$

In the coordinates (r, θ) the volume element (8) §1-2 becomes

$$dz = \tfrac{1}{2} \sinh 2r \, dr \, d\theta$$

If we now consider the Legendre function

$$P_\nu(\cosh r) = \frac{1}{2\pi} \int_0^{2\pi} (\cosh r + \sinh r \cos \theta)^\nu \, d\theta \qquad (\nu \in \mathbf{C})$$

the formulas in Theorem 3.1 become

$$\tilde{F}(\lambda) = \pi \int_0^\infty F(r) P_{-\frac{1}{2}+\frac{1}{2}i\lambda} \cosh (2r) \sinh (2r) \, dr \tag{12}$$

$$F(r) = \frac{1}{2\pi} \int_0^\infty \tilde{F}(\lambda) P_{-\frac{1}{2}-\frac{1}{2}i\lambda} \cosh (2r) \, \lambda \tanh (\tfrac{1}{2}\pi\lambda) \, d\lambda \tag{13}$$

After a harmless change of variables, (13) becomes simply the inversion formula for the Mehler transform (Erdélyi [17], Vol. I, p. 175, Fok [18] and Godement [22a]). Assuming this inversion formula, Theorem 3.1 is proved (*cf.* Helgason [35], [36]).

If we compare the formulas in Theorem 3.1 with (3) and (4) we note a factor $e^{2\langle z, b\rangle}$ which has no analog in the Euclidean case. But according to (11) this factor is just the classical Poisson kernel but expressed in non-Euclidean terms. Consequently, the classical Poisson integral formula for a harmonic function u on D with continuous boundary values $f(b)$ on B,

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos (\theta - \phi) + r^2} f(e^{i\phi}) \, d\phi$$

can be written

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{2\langle z, b\rangle} f(b) \, db \tag{14}$$

According to our stated conventions this is a formula in Fourier analysis on D.

Note that the Euclidean harmonic functions coincide with the non-Euclidean harmonic functions according to (9) §1-2. Thus (14) is entirely non-Euclidean.

1-4 Interpretation by Representation Theory

Let X be a space with a measure μ and let G be a transformation group of X leaving the measure μ invariant. To each $g \in G$ we associate the operator $T(g) : f \to f^g$ on the space $L^2(X)$ of square-integrable functions on X. (As in Lemma 3.2, f^g denotes the function $x \to f(g^{-1} \cdot x)$ on X.) Then the mapping $g \to T(g)$ is a unitary representation of G on the Hilbert space $L^2(X)$. Now arises the natural problem of decomposing this unitary representation T

into irreducible representations T_λ acting on Hilbert spaces $\mathfrak{H}_\lambda$ such that for a suitable measure ν

$$L^2(X) = \int \mathfrak{H}_\lambda \, d\nu(\lambda) \qquad T = \int T_\lambda \, d\nu(\lambda) \tag{1}$$

in the sense of direct integrals of Hilbert spaces (see, for example, Dixmier [15]). In §1-3 we have some examples of (1):

a. First let G denote the group of translations of $\boldsymbol{R}^n$. Then for each $u \in \boldsymbol{R}^n$ the space $\mathfrak{H}_u = Ce_u$ is invariant and irreducible under G; let T_u denote the representation of G on $\mathfrak{H}_u$ given by

$$[T_u(g)f](x) = f(g^{-1}x) \qquad \text{for } f \in \mathfrak{H}_u, \, g \in G, \, x \in \boldsymbol{R}^n.$$

Then (2) in §1-3 (together with the Plancherel formula

$$\int |f(x)|^2 \, dx = (2\pi)^{-n} \int |\tilde{f}(u)|^2 \, du)$$

can be written

$$L^2(\boldsymbol{R}^n) = \int \mathfrak{H}_u \, du^* \qquad T = \int T_u \, du^* \tag{2}$$

where $du^* = (2\pi)^{-n} \, du$.

b. Next let G denote the group of all transformations of $\boldsymbol{R}^n$ preserving orientation and distance. For each $\lambda \in \boldsymbol{R}^+$ consider the Hilbert space of functions on $\boldsymbol{R}^n$ given by

$$\mathfrak{H}_\lambda = \left\{ F_\lambda(x) = \int_{S^{n-1}} e^{i\lambda(x,\omega)} F(\omega) \, d\omega \mid F \in L^2(S^{n-1}) \right\} \tag{3}$$

(defining $\|F_\lambda\|$ as the L^2 norm of F) and let T_λ denote the representation of G on $\mathfrak{H}_\lambda$ given by

$$(T_\lambda(g)F_\lambda)(x) = F_\lambda(g^{-1}x) \qquad F_\lambda \in \mathfrak{H}_\lambda, \, g \in G, \, x \in \boldsymbol{R}^n$$

T_λ is in fact a unitary representation, because if $g = tk$ (t is the translation, k the rotation around 0), then

$$(T_\lambda(g)F_\lambda)(x) = \int_{S^{n-1}} e^{i\lambda(x,\omega)} e^{-i\lambda(g\cdot 0,\omega)} F(k^{-1}\omega) \, d\omega \tag{4}$$

and T_λ is in fact irreducible (*cf.* Itô [42] and Mackey [51, §14]) and different λ in $\boldsymbol{R}^+$ give inequivalent T_λ. Thus (4) in §1-3 together with the Plancherel formula

$$\int_{\boldsymbol{R}} |f(x)|^2 \, dx = (2\pi)^{-n} \int_{\boldsymbol{R}^+ \times S^{n-1}} |\tilde{f}(\lambda\omega)|^2 \lambda^{n-1} \, d\lambda \, d\omega$$

gives the direct integral decomposition

$$L^2(X) = \int_{R^+} \mathfrak{H}_\lambda \, d\lambda^* \qquad T = \int_{R^+} T_\lambda \, d\lambda^* \tag{5}$$

where $d\lambda^* = (2\pi)^{-n}\lambda^{n-1}\, d\lambda$.

c. Finally, we consider the case when G is the group $\boldsymbol{SU}(1, 1)$ operating on D. For each $\lambda \in \boldsymbol{R}$ consider the Hilbert space

$$\mathfrak{H}_\lambda = \left\{ h_\lambda(z) = \int_B e^{(i\lambda+1)\langle z, b\rangle} h(b)\, db \mid h \in L^2(B) \right\}$$

(defining $\|h_\lambda\|$ as the L^2 norm of h) and let T_λ denote the representation of G on $\mathfrak{H}_\lambda$ given by

$$[T_\lambda(g)h_\lambda](z) = h_\lambda(g^{-1}z)$$

Using formulas (9) and (10) in §1-3, we find

$$h_\lambda(g^{-1} \cdot z) = \int_B e^{(i\lambda+1)\langle z, b\rangle} e^{(-i\lambda+1)\langle g\cdot 0, b\rangle} h(g^{-1} \cdot b)\, db$$

[compare with (4)]; so using (10) again we see that T_λ is unitary; comparing with Bargmann [1], Thm. 1, p. 613, we see that T_λ is irreducible. Finally (6) in §1-3 and the Plancherel formula

$$\int_D |f(z)|^2\, dz = \int_{R\times B} |\tilde{f}(\lambda, b)|^2\, d\mu(\lambda, b)$$

show that

$$L^2(D) = \int_{R/Z_2} \mathfrak{H}_\lambda \, d\mu(\lambda) \qquad T = \int_{R/Z_2} T_\lambda \, d\mu(\lambda) \tag{6}$$

where $d\mu(\lambda) = 2(2\pi)^{-2}\lambda \tanh(\frac{1}{2}\pi\lambda)$ and integration is taken over $\boldsymbol{R}/\boldsymbol{Z}_2$ since T_λ and T_μ can be shown equivalent if and only if $\lambda = -\mu$.

1-5 The Eigenfunctions of the Laplacian on the Non-Euclidean Disk

Let $P(z, b)$ denote the Poisson kernel

$$P(z, b) = \frac{1 - |z|^2}{1 - 2|z| \cos(\theta - \phi) + |z|^2}$$

$$z = |z|\, e^{i\theta} \qquad b = e^{i\phi}$$

If $\lambda \in \boldsymbol{C}$ is any complex number it is clear from (i)′ and (11) in §1-3 that for for each $b \in B$ the power $P(z, b]^\lambda$ gives an eigenfunction of the non-Euclidean Laplacian Δ. A direct computation gives

$$\Delta_z(P(z, b)^\lambda) = 4\lambda(\lambda - 1)P(z, b)^\lambda$$

which shows that the eigenvalue is independent of b. Note that the eigenvalue is $\geqslant -1$ (and real) if and only if $\lambda \in \boldsymbol{R}$. We shall now consider the problem of constructing the most general eigenfunctions of Δ.

Let $A(B)$ denote the set of analytic function *on the boundary* B, considered as an analytic manifold. The space $A(B)$ carries an atural topology (see, for example, Köthe [48]). The continuous linear functions $A(B) \to \boldsymbol{C}$ are called *analytic functionals* on B; they constitute the dual space $A'(B)$ of $A(B)$. If $T \in A'(B)$, $f \in A(B)$ we write for $T(f)$ also $\int_B f(b)\, dT(b)$, since the elements of A are generalizations of measures. For the eigenfunctions of Δ we have the following result (unpublished):

Theorem 5.1. The functions

$$F(z) = \int_B P(z, b)^\lambda\, dT(b)$$

where $\lambda \in \boldsymbol{R}$ and T is an analytic functional on B constitute precisely the eigenfunctions of Δ with eigenvalue $\geqslant -1$.

CHAPTER 2: LIE GROUPS AND LIE ALGEBRAS

2-1 The Lie Algebra of a Lie Group

Let M be a manifold, p a point in M, and M_p the tangent space to M at p; this is a vector space over $\boldsymbol{R}$. In differential geometry one studies a manifold by means of its family of tangent spaces to which numerous objects are associated (vector fields, differential forms, arbitrary tensor fields).

If G is a Lie group, the tangent space G_g at an arbitrary point $g \in G$ is obtained from G_e (e is the identity element) by the left translation $L_g : x \to gx$ ($x \in G$), that is, $G_g = dL_g(G_e)$. This circumstance makes it possible to introduce an additional structure on G_e as follows:

Let X, $Y \in G_e$. Then we obtain vector fields $\tilde{X}$, $\tilde{Y}$ on G by left translations:

$$\tilde{X}_g = dL_g(X) \qquad \tilde{Y}_g = dL_g(Y) \qquad g \in G$$

The bracket $[\tilde{X}, \tilde{Y}] = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}$ is another vector field on G which is invariant under left translations so there exists a unique vector $Z \in G_e$ such that

$$[\tilde{X}, \tilde{Y}]_e = \tilde{Z}$$

We write $[X, Y]$ instead of Z. The vector space G_e with the rule of composition $(X, Y) \to [X, Y]$ is called the *Lie algebra* of G and will be denoted by $\mathfrak{g}$.

The bracket [,] has the following properties:

(a) $[X, Y] = -[Y, X]$

(b) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

A vector space $\mathfrak{a}$ with a bilinear map $(X, Y) \to [X, Y]$ of $\mathfrak{a} \times \mathfrak{a}$ into $\mathfrak{a}$ satisfying (a) and (b) above is called a *Lie algebra*. For Lie algebras one can in an obvious manner define *subalgebras*, *ideals*, *homomorphisms*, *isomorphisms*, and *automorphisms*.

If V is a vector space let $\mathfrak{gl}(V)$ denote the vector space of all linear transformations of V into V, with the bracket operation $[A, B] = AB - BA$. Then $\mathfrak{gl}(V)$ is a Lie algebra. A homomorphism of a Lie algebra $\mathfrak{a}$ into $\mathfrak{gl}(V)$ is called a *representation* of $\mathfrak{a}$ on V. In particular, if for a given $X \in \mathfrak{a}$, the mapping $Y \to [X, Y]$ is denoted ad X, the mapping ad : $X \to$ ad X is a representation of $\mathfrak{a}$ on $\mathfrak{a}$. The kernel of ad is called the *center* of $\mathfrak{a}$; $\mathfrak{a}$ is called *Abelian* if its center is $\mathfrak{a}$; that is, if $[X, Y] = 0$ for all $X, Y \in \mathfrak{a}$.

2-2 The Exponential Mapping

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$ and let $\tilde{X}$ be the left invariant vector field on G such that $\tilde{X}_e = X$. Let $\phi(t)$ $(t \in \boldsymbol{R})$ be the integral curve to $\tilde{X}$ passing through e, that is,

$$\dot{\phi}(t) = d\phi\left(\frac{d}{dt}\right) = \tilde{X}_{\phi(t)} \qquad \phi(0) = e \tag{1}$$

For small t, $\phi(t)$ exists and is unique because (1) is a first-order system of ordinary differential equations. For the global statement one uses the group property to continue the solution. The mapping exp : $\mathfrak{g} \to G$ is now defined by

$$\exp X = \phi(1)$$

and is called the exponential mapping. It sets up a very far-reaching relationship between $\mathfrak{g}$ and G; some of the main results will be summarized below.

First we have

(i) $\exp sX \exp tX = \exp (s + t)X \qquad (s, t \in \boldsymbol{R})$

that is, the curve $t \to \exp tX$ is a one-parameter subgroup of G. In fact, if $s \in \boldsymbol{R}$, then $L_{\exp sX}$ maps $\tilde{X}$ into itself so it maps the integral curve through e into the integral curve through $\exp sX$. Thus, $L_{\exp sX}(\phi(t)) = \phi(s + t)$ which is (i).

By the definition of $\tilde{X}$,

$$\tilde{X}_g f = \left\{\frac{d}{dt} f(g \exp tX)\right\}_{t=0} \qquad f \in C^\infty(G),\ g \in G$$

Thus the value of the function $\tilde{X}f$ at $g \exp sX$ is

$$(\tilde{X}f)(g \exp sX) = \left\{\frac{d}{dt} f(g \exp sX \exp tX)\right\}_{t=0} = \frac{d}{ds} f(g \exp sX)$$

and by induction, if $n \in \mathbf{Z}^+$,

$$(\tilde{X}^n f)(g \exp sX) = \frac{d^n}{ds^n} f(g \exp sX) \tag{2}$$

(ii) If a function f is analytic in a neighborhood of a point $g \in G$, then

$$f(g \exp X) = \sum_0^\infty \frac{1}{n!} (\tilde{X}^n f)(g) \tag{3}$$

for all X in some neighborhood of 0 in $\mathfrak{g}$.

This relation follows by using (2) in Taylor's formula for the function $s \to f(g \exp sX)$.

(iii) The mapping $X \to \exp X$ is a diffeomorphism of an open neighborhood of 0 in $\mathfrak{g}$ onto an open neighborhood of e in G.

This is a direct consequence of the fact that the mapping $X \to \exp X$ has Jacobian $\neq 0$ at the origin $X = 0$.

(iv) If $X, Y \in \mathfrak{g}$ then

$$\exp tX \exp tY = \exp \{t(X+Y) + \tfrac{1}{2}t^2[X, Y] + 0(t^3)\} \tag{4}$$

where $0(t^3)$ denotes a vector such that $t^{-3}\, 0(t^3)$ is bounded near $t = 0$.

In fact, by (iii), we have for small t,

$$\exp tX \exp tY = \exp Z(t) \tag{5}$$

where $t \to Z(t)$ is a curve in $\mathfrak{g}$, analytic at $t = 0$ and

$$Z(t) = tZ_1 + t^2 Z_2 + 0(t^3) \qquad (Z_1, Z_2 \in \mathfrak{g})$$

But by (2) and (3) we have for f analytic at e,

$$f(\exp tX \exp tY) = \sum_{m,n \geq 0} \frac{t^{m+n}}{m!\,n!} (\tilde{X}^m \tilde{Y}^n f)(e)$$

whereas

$$f(\exp Z(t)) = \sum_{m=0}^\infty \frac{1}{m!} [(t\tilde{Z}_1 + t^2 \tilde{Z}_2 + 0(t^3))^m f](e)$$

Comparing coefficients we get $Z_1 = X + Y$, $\frac{1}{2}\tilde{Z}_1{}^2 + \tilde{Z}_2 = \frac{1}{2}\tilde{X}^2 + \tilde{X}\tilde{Y} + \frac{1}{2}\tilde{Y}^2$, whence $Z_2 = \frac{1}{2}[X, Y]$, proving (4).

From (4) we deduce that

$$\exp(-tX)\exp(-tY)\exp tX \exp tY = \exp\{t^2[X, Y] + 0(t^3)\}$$

which shows that $[X, Y]$ is the tangent vector at e to the curve

$$t \to \exp(-\sqrt{t}\,X)\exp(-\sqrt{t}\,Y)\exp(\sqrt{t}\,X)\exp(\sqrt{t}\,Y)$$

(v) Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.

The "only if" part is immediate from (4). On the other hand, it is possible to carry further the computation above and express $Z(t)$ in (5) completely in terms of t, X, Y and their repeated brackets. (The resulting formula is the so-called Campbell–Hausdorff formula, see for example, Jacobson [43].) The "if" part of (v) is an immediate consequence.

A Fundamental Example

Let $\boldsymbol{GL}(n, \boldsymbol{R})$ denote the group of real $n \times n$ matrices of determinant $\neq 0$ and $\mathfrak{gl}(n, \boldsymbol{R})$ the Lie algebra of all real $n \times n$ matrices, the bracket being $[A, B] = AB - BA$. If $\sigma = (x_{ij}(\sigma))$ is a matrix in $\boldsymbol{GL}(n, \boldsymbol{R})$ we consider the matrix elements $x_{ij}(\sigma)$ as coordinates of σ whereby $\boldsymbol{GL}(n, \boldsymbol{R})$ is a manifold; if we express $x_{ij}(\sigma\tau^{-1})$ $(\sigma, \tau \in \boldsymbol{GL}(n, \boldsymbol{R}))$ in terms of $x_{kl}(\sigma)$, $x_{pq}(\tau)$ by ordinary matrix multiplication we see that $\boldsymbol{GL}(n, \boldsymbol{R})$ is a Lie group. Let $\mathfrak{g}$ denote its Lie algebra and if $X \in \mathfrak{g}$ let $\tilde{X}$ denote the left invariant vector field on $\boldsymbol{GL}(n, \boldsymbol{R})$ satisfying $\tilde{X}_e = X$. Let (X_{ij}) denote the matrix $(\tilde{X}_e x_{ij})$ and consider the mapping $\phi : X \to (X_{ij})$ of $\mathfrak{g}$ into $\mathfrak{gl}(n, \boldsymbol{R})$. The mapping ϕ is linear, one-to-one and onto. Furthermore if L_σ denotes the left translation $\tau \to \sigma\tau$ we have by the left invariance of $\tilde{X}$,

$$(\tilde{X}x_{ij})(\sigma) = X(x_{ij} \circ L_\sigma)$$

But

$$(x_{ij} \circ L_\sigma)(\tau) = x_{ij}(\sigma\tau) = \sum_k x_{ik}(\sigma)x_{kj}(\tau)$$

so

$$(\tilde{X}x_{ij})(\sigma) = \sum_k x_{ik}(\sigma)X_{kj} \tag{6}$$

It follows that

$$(\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X})_e x_{ij} = \sum_k (X_{ik}Y_{kj} - Y_{ik}X_{kj}) = [\phi(X), \phi(Y)]_{ij}$$

so ϕ is a Lie algebra isomorphism. Thus the Lie algebra of $\boldsymbol{GL}(n, \boldsymbol{R})$ is identified with $\mathfrak{gl}(n, \boldsymbol{R})$. In this statement one can replace the real field $\boldsymbol{R}$ by the field $\boldsymbol{C}$. In view of (2) and (6) we have

$$\frac{d}{dt} x_{ij}(\exp tX) = \sum_k x_{ik}(\exp tX) X_{kj}$$

so the matrix function $Y(t) = \exp tX$ satisfies

$$\frac{d}{dt} Y(t) = Y(t)X \quad Y(0) = I \tag{7}$$

But this equation is also satisfied by the matrix exponential function

$$e^{tX} = I + tX + \tfrac{1}{2}t^2X^2 + \cdots$$

so $\exp X = e^X$ for all $X \in \mathfrak{gl}(n, \boldsymbol{R})$. Thus the exponential mapping for Lie groups generalizes the exponential function for matrices.

Let G be any Lie group. A Lie group H is called a *Lie subgroup* of G if it is a subgroup of G and a submanifold of G. If this is the case the Lie algebra $\mathfrak{h}$ of H is a subalgebra of the Lie algebra $\mathfrak{g}$ of G, and the exponential maps for $\mathfrak{h}$ and $\mathfrak{g}$ coincide on $\mathfrak{h}$.

(vi) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then there exists exactly one connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.

This important fact is proved along the following lines: Consider the (abstract) subgroup H of G generated by the set $\exp \mathfrak{h}$. Using (iii), one introduces a topology in H (this is not necessarily the relative topology of G) as well as a coordinate system near the identity of H. By left translations on H this gives a coordinate system in some neighborhood of an arbitrary point of H and one must finally prove that this manifold structure on H has the required properties. A connected Lie subgroup is usually called *analytic subgroup*.

(vii) Let G be a Lie group and H a subgroup of G which is closed as a subset of G. Then there exists a unique manifold structure on H such that H is a topological Lie subgroup of G.

If $\mathfrak{h}$ and $\mathfrak{g}$ are the respective Lie algebras of H and G then

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp tX \in H \text{ for all } t \in \boldsymbol{R}\} \tag{8}$$

Example

Let us use (8) to find the Lie algebra of the group $\boldsymbol{SU}(1, 1)$ considered in Chapter 1. First note that $\boldsymbol{SU}(1, 1)$ is the group of matrices of determinant

1 leaving invariant the Hermitian form $-z_1\bar{z}_1 + z_2\bar{z}_2$, that is, a matrix A belongs to $\boldsymbol{SU}(1, 1)$ if and only if

$$^tAJ\bar{A} = J, \qquad \det A = 1$$

where ^tA is the transpose of A and

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\mathfrak{gl}(2, \boldsymbol{C})$ is the Lie algebra of $\boldsymbol{GL}(2, \boldsymbol{C})$ we see that X belongs to the Lie algebra $\mathfrak{su}(1, 1)$ of $\boldsymbol{SU}(1, 1)$ if and only if

$$^t(\exp sX)J\,\overline{\exp sX} = J \qquad \det(\exp sX) = 1 \qquad (s \in \boldsymbol{R})$$

But $\exp(^tX) = {}^t(\exp X)$, so the first relation can be written

$$\begin{aligned} \exp s\bar{X} &= J\exp(-s\,^tX)J^{-1} \\ &= \exp s(-J\,^tXJ^{-1}) \qquad (s \in \boldsymbol{R}) \end{aligned}$$

Thus $X \in \mathfrak{su}(1, 1)$ if and only if $\bar{X} = -J\,^tXJ^{-1}$ and Trace $X = 0$. This is equivalent to

$$\mathfrak{su}(1, 1) = \left\{ X = \begin{pmatrix} i\alpha & \beta \\ \bar{\beta} & -i\alpha \end{pmatrix} \middle| \, \alpha \in \boldsymbol{R}, \beta \in \boldsymbol{C} \right\}$$

Property (v) shows that local properties of a Lie group are completely determined by the Lie algebra. This is of great consequence because all the machinery of linear algebra (theory of linear transformations of a vector space) can be applied to Lie algebras. In particular, let us see how the left invariant Haar measure on a Lie group can be written in Lie algebra terms.

Consider a Lie group G with Lie algebra $\mathfrak{g}$. If $X \in \mathfrak{g}$ the differential of the exponential map at X maps the tangent space $\mathfrak{g}_X$ onto the tangent space $G_{\exp X}$, which is $dL_{\exp X}(\mathfrak{g})$ (since $\mathfrak{g} = G_e$). We identify $\mathfrak{g}_X$ with $\mathfrak{g}$ via the ordinary parallelism of vectors. Thus if $Y \in \mathfrak{g}$ there exists a unique vector $Z \in \mathfrak{g}$ such that

$$d\exp_X(Y) = (dL_{\exp X})(Z)$$

Let us compute Z. By the definition of the differential of a map we have if f is differentiable at $\exp X$,

$$d\exp_X(Y)f = Y_X(f \circ \exp) \tag{9}$$

where Y_X is the vector Y viewed as a tangent vector to $\mathfrak{g}$ at X. But

$$Y_X(f \circ \exp) = \left\{ \frac{d}{dt} f(\exp(X + tY)) \right\}_{t=0} \tag{10}$$

Now take f to be analytic at e. Then if X and t are sufficiently small,

$$f(\exp(X+tY)) = \sum_0^\infty \frac{1}{n!}[(\tilde{X}+t\tilde{Y})^n f](e)$$

so by (9) and (10),

$$d\exp_X(Y)f = \sum_0^\infty \frac{1}{(n+1)!}[(\tilde{X}^n\tilde{Y}+\tilde{X}^{n-1}\tilde{Y}\tilde{X}+\cdots+\tilde{Y}\tilde{X}^n)f](e)$$

Now consider the algebra generated by the left invariant vector fields on G and the operators

$$L(\tilde{X}): A \to \tilde{X}A \qquad R(\tilde{X}): A \to A\tilde{X} \qquad \theta(\tilde{X}): A \to \tilde{X}A - A\tilde{X}$$

of this algebra. Then $\theta(\tilde{X}) = L(\tilde{X}) - R(\tilde{X})$ and $L(\tilde{X})$ and $\theta(\tilde{X})$ commute so

$$R(\tilde{X})^m = (L(\tilde{X}) - \theta(\tilde{X}))^m = \sum_{p=0}^m (-1)^p \binom{m}{p} L(\tilde{X})^{m-p}\theta(\tilde{X})^p$$

and

$$\tilde{X}^n\tilde{Y}+\cdots+\tilde{Y}\tilde{X}^n = \sum_{p=0}^n \tilde{X}^p \sum_{k=0}^{n-p}(-1)^k\binom{n-p}{k}\tilde{X}^{n-p-k}\theta(\tilde{X})^k(\tilde{Y})$$

which by the elementary formula

$$\sum_{p=0}^{n-k}\binom{n-p}{k} = \binom{n+1}{k+1}$$

equals

$$\sum_{k=0}^n \binom{n+1}{k+1}\tilde{X}^{n-k}\theta(-\tilde{X})^k(\tilde{Y})$$

Hence,

$$d\exp_X(Y)f = \sum_{n=0}^\infty \left[\sum_{k=0}^n \left\{\frac{\tilde{X}^{n-k}}{(n-k)!}\frac{\theta(-\tilde{X})^k}{(k+1)!}(\tilde{Y})\right\}(f)\right](e) \tag{11}$$

For sufficiently small X one can use the analyticity of f to interchange the two summations and use the formula

$$\sum_{n=0}^\infty \sum_{k=0}^n = \sum_{k=0}^\infty \sum_{n=k}^\infty$$

to equate the right-hand side with

$$\sum_{r=0}^\infty \left[\frac{\tilde{X}^r}{r!}\sum_{k=0}^\infty \frac{\theta(-\tilde{X})^k}{(k+1)!}(\tilde{Y})f\right](e) \tag{12}$$

which by (3) equals

$$\sum_0^\infty \left[\frac{\theta(-\tilde{X})^k}{(k+1)!}(\tilde{Y})f\right](\exp X)$$

But $\theta(-\tilde{X})^k(\tilde{Y})$ is the left invariant vector field corresponding to the vector $\operatorname{ad}(-X)^k(Y)$ in $\mathfrak{g}$ so we have proved

$$d \exp_X(Y) = dL_{\exp X}\left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right)(Y) \tag{13}$$

at least if X is sufficiently small. Because of the analyticity of both sides (13) holds actually for all $X \in \mathfrak{g}$.

Note that in the right-hand side of (13),

$$\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} = \int_0^1 e^{-t \operatorname{ad} X}\, dt$$

Now let $\exp : V_0 \to V_e$ be a diffeomorphism, V_0 and V_e being open sets in $\mathfrak{g}$ and G, respectively. Let $f \in C_c^\infty(G)$ have support contained in V_e. If dx denotes a left invariant Haar measure on G we have

$$\int_G f(x)\, dx = \int_{\mathfrak{g}} f(\exp X) J(X)\, dX$$

dX being a Euclidean volume element on $\mathfrak{g}$ and J the Jacobian of the exponential map. In view of (13) we have

$$\int_G f(x)\, dx = c \int_{\mathfrak{g}} f(\exp X) \det\left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right) dX \tag{14}$$

where c is a constant. For a formulation of (13) for differential forms see [11] p. 21 and [12] p. 157. For a generalization to Riemannian manifolds see [30].

(ix) Given a Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ there exists a Lie group G with Lie algebra $\mathfrak{g}$.

The local result is called the third fundamental theorem of Lie; the global statement was later proved by É. Cartan. One proof of (ix) uses Ado's theorem that there exists an isomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(n, \boldsymbol{R})$. Then the desired G can by (vi) be taken as a suitable subgroup of $\boldsymbol{GL}(n, \boldsymbol{R})$. Another proof will be indicated later.

CHAPTER 3: STRUCTURE THEORY OF LIE GROUPS

3-1 Solvable and Semisimple Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra and as before let $\operatorname{ad} X$ denote the linear transformation $Y \to [X, Y]$ of $\mathfrak{g}$. Lie algebra theory is concerned with this family of linear transformations.

The vector space spanned by all elements $[X, Y]$ is an ideal in $\mathfrak{g}$, called the *derived algebra* of $\mathfrak{g}$ and denoted $\mathfrak{D}\mathfrak{g}$. The nth derived algebra $\mathfrak{D}^n\mathfrak{g}$ of $\mathfrak{g}$ is defined inductively by $\mathfrak{D}^0\mathfrak{g} = \mathfrak{g}$, $\mathfrak{D}^n\mathfrak{g} = \mathfrak{D}(\mathfrak{D}^{n-1}\mathfrak{g})$. A Lie algebra is called *solvable* if $\mathfrak{D}^n\mathfrak{g} = \{0\}$ for some $n \geqslant 0$. A Lie group is called solvable if its Lie algebra is solvable.

A Lie algebra is called *nilpotent* if for each $X \in \mathfrak{g}$, ad X is nilpotent. It can be proved that a Lie algebra is solvable if and only if its derived algebra is nilpotent. In particular we see that a nilpotent Lie algebra is solvable.

Example

Let $\mathfrak{t}(n)$ denote the Lie subalgebra of $\mathfrak{gl}(n, \boldsymbol{R})$ formed by the upper triangular matrices and let $\mathfrak{n}(n)$ denote the subalgebra of matrices in $\mathfrak{t}(n)$ with diagonal 0. Then $\mathfrak{t}(n)$ is solvable, $\mathfrak{n}(n)$ nilpotent and coincides with the derived algebra of $\mathfrak{t}(n)$.

Let $\mathfrak{g}$ be a Lie algebra. The *Killing form* of $\mathfrak{g}$ is defined as the bilinear form $B(X, Y) = \mathrm{Tr}\,(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ (Tr = trace); $\mathfrak{g}$ is called *semisimple* if B is nondegenerate and $\mathfrak{g}$ is called *simple* if in addition it has no ideals except 0 and $\mathfrak{g}$.

Example

Let $\boldsymbol{SL}(n, \boldsymbol{R})$ denote the group of $n \times n$ real matrices of determinant 1. It is a closed subgroup of $\boldsymbol{GL}(n, \boldsymbol{R})$, hence a Lie subgroup [*cf.* (vii) §2-2] and since the relation $\det(e^A) = e^{\mathrm{Tr}\, A}$ holds for any matrix A, we see from (8) §2-2 that the Lie algebra $\mathfrak{sl}(n, \boldsymbol{R})$ of the subgroup $\boldsymbol{SL}(n, \boldsymbol{R})$ of $\boldsymbol{GL}(n, \boldsymbol{R})$ is the subalgebra of $\mathfrak{gl}(n, \boldsymbol{R})$ consisting of all $n \times n$ matrices of trace 0. This statement holds also with $\boldsymbol{R}$ replaced with the complex field $\boldsymbol{C}$. Let us compute the Killing form of $\mathfrak{sl}(n, \boldsymbol{C})$. Let $\mathfrak{d}(n)$ denote the set of diagonal matrices in $\mathfrak{sl}(n, \boldsymbol{C})$. If $H \in \mathfrak{d}(n)$ each matrix E_{ij} with 1 at the ith row and the jth column, 0 elsewhere,

$$E_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & 1 & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

is an eigenvector for ad H and we find easily that

$$\mathrm{Tr}\,(\mathrm{ad}\, H\, \mathrm{ad}\, H) = 2n\, \mathrm{Tr}\,(HH) \tag{1}$$

The mapping $X \to gXg^{-1}$ ($g \in \boldsymbol{GL}(n, \boldsymbol{C})$) is an automorphism of $\mathfrak{sl}(n, \boldsymbol{C})$ and any automorphism of a Lie algebra leaves the Killing form invariant. If $gXg^{-1} \in \mathfrak{d}(n)$ we have therefore

$$\mathrm{Tr}\,(\mathrm{ad}\, X\, \mathrm{ad}\, X) = \mathrm{Tr}\,(\mathrm{ad}\,(gXg^{-1})\, \mathrm{ad}\,(gXg^{-1})) = 2n\, \mathrm{Tr}\,(gXXg^{-1}) \tag{2}$$

so

$$\mathrm{Tr}\,(\mathrm{ad}\,X\;\mathrm{ad}\,X) = 2n\,\mathrm{Tr}\,(XX) \tag{3}$$

The matrices which are conjugate to a diagonal matrix in $\mathfrak{d}(n)$ form a dense subset of $\mathfrak{sl}(n, \boldsymbol{C})$ so (3) holds for all $X \in \mathfrak{sl}(n, \boldsymbol{C})$. Hence by "polarization"

$$B(X, Y) = 2n\,\mathrm{Tr}\,(XY) \qquad \text{for } X, Y \in \mathfrak{sl}(n, \boldsymbol{C}) \tag{4}$$

It is a trivial matter to verify that B in (4) is nondegenerate so $\mathfrak{sl}(n, \boldsymbol{C})$ is semisimple.

A fundamental result in Lie algebra theory (the *Levi decomposition*) states that every Lie algebra $\mathfrak{g}$ is the direct vector space sum

$$\mathfrak{g} = \mathfrak{r} + \mathfrak{s} \tag{5}$$

where $\mathfrak{r}$ is the maximal solvable ideal in $\mathfrak{g}$ and $\mathfrak{s}$ is a semisimple subalgebra. To a large extent this result splits Lie group theory into two branches—one for solvable Lie groups, the other for semisimple Lie groups. The latter branch is further developed and has had more contact with physics and geometry and is therefore emphasized in these lectures. (Of course the two branches are related because semisimple Lie algebras always have solvable subalgebras.)

The Levi decomposition can for example be used as a basis of an alternative proof of (ix) §2-2. Let Aut $(\mathfrak{s})$ denote the group of all automorphisms of $\mathfrak{s}$. This is a closed subgroup of $\boldsymbol{GL}(\mathfrak{s})$, hence a Lie subgroup, and by (8) §2-2 its Lie algebra is given by the set of endomorphisms A of $\mathfrak{s}$ for which $e^{tA} \in \mathrm{Aut}\,(\mathfrak{s})$ for all $t \in \boldsymbol{R}$. But the relation

$$e^{tA}[X, Y] = [e^{tA}X, e^{tA}Y]$$

implies (by differentiation)

$$A[X, Y] = [AX, Y] + [X, AY] \tag{6}$$

and vice versa. A linear transformation A satisfying (6) for all $X, Y \in \mathfrak{s}$ is called a *derivation* of $\mathfrak{s}$ so we see that the Lie algebra of Aut $(\mathfrak{s})$ is the set of derivations of $\mathfrak{s}$. On the other hand, if $X \in \mathfrak{s}$, ad X is obviously a derivation of $\mathfrak{s}$. Using the semisimplicity, one can prove that all derivations of $\mathfrak{s}$ are of this form. Thus ad $(\mathfrak{s})$ is the Lie algebra of Aut $(\mathfrak{s})$; but the semisimplicity of $\mathfrak{s}$ shows that $X \to \mathrm{ad}\,X$ is an isomorphism so we have verified that any semisimple Lie algebra is the Lie algebra of a Lie group. For solvable Lie algebras the statement can be proved by induction and by the Levi decomposition (5) the theorem can be proved in general by taking appropriate semidirect products.

For any Lie algebra $\mathfrak{g}$, let Int $(\mathfrak{g})$ denote the connected Lie subgroup of $\boldsymbol{GL}(\mathfrak{g})$ with Lie algebra ad $(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$; Int $(\mathfrak{g})$ is called the *adjoint group of* $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple then Int $(\mathfrak{g})$ is the identity component of Aut $(\mathfrak{g})$. If G is a

Lie group with Lie algebra $\mathfrak{g}$, and $g \in G$, the inner automorphism $x \to gxg^{-1}$ of G induces an automorphism of $\mathfrak{g}$, denoted Ad (g). If G is connected, $\mathrm{Ad}(G) = \mathrm{Int}(\mathfrak{g})$. In fact if $X, Y \in \mathfrak{g}$ we obtain by iterating (4) §2-2

$$\begin{aligned}\exp(\mathrm{Ad}(\exp tX)tY) &= \exp tX \exp tY \exp(-tX) &(7)\\ &= \exp(tY + t^2[X, Y] + 0(t^3))\end{aligned}$$

so

$$\mathrm{Ad}(\exp tX)Y = Y + t[X, Y] + 0(t^2) \tag{8}$$

On the other hand, the mapping $g \to \mathrm{Ad}(g)$ is a homomorphism of G into $\boldsymbol{GL}(\mathfrak{g})$. Hence $t \to \mathrm{Ad}(\exp tX)$ is a one-parameter subgroup of $\boldsymbol{GL}(\mathfrak{g})$, thus by the fundamental example in Ch. 2 of the form

$$\mathrm{Ad}(\exp tX) = e^{tA}$$

But then (8) shows $A = \mathrm{ad}\, X$ so

$$\mathrm{Ad}(\exp X) = e^{\mathrm{ad}\, X} \tag{9}$$

and the relation $\mathrm{Ad}(G) = \mathrm{Int}(\mathfrak{g})$ follows.

The homomorphism $g \to \mathrm{Ad}(g)$ is called the *adjoint representation* of G. For clarity it is sometimes written Ad_G.

3-2 Structure of Semisimple Lie Algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra, B its Killing form. If $\boldsymbol{O}(B)$ denotes the group of linear transformations of $\mathfrak{g}$ leaving B invariant, we have $\mathrm{Aut}(\mathfrak{g}) \subset \boldsymbol{O}(B)$; also

$$B(X, \mathrm{ad}\, Y(Z)) = -B(\mathrm{ad}\, Y(X), Z)$$

for $X, Y, Z \in \mathfrak{g}$, so each ad Y is skew-symmetric with respect to B.

Definition. A Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ is called *compact* if its adjoint group $\mathrm{Int}(\mathfrak{g})$ is compact.

Proposition 2.1

(i) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\boldsymbol{R}$. Then $\mathfrak{g}$ is compact if and only if the Killing form of $\mathfrak{g}$ is negative definite.

(ii) Every compact Lie algebra is the direct sum $\mathfrak{g} = \mathfrak{z} + [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and the ideal $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and compact.

PROOF OF (i). If the Killing form is negative definite $\boldsymbol{O}(B)$ is compact and so are the groups $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g})$. On the other hand, if $\mathrm{Int}(\mathfrak{g})$ is compact it leaves invariant a positive definite quadratic form Q on $\mathfrak{g}$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ such that

$$Q(X) = \sum_1^n x_i^2 \qquad \text{if} \qquad X = \sum_1^n x_i X_i$$

By means of this basis each $\sigma \in \operatorname{Int}(\mathfrak{g})$ is given by an orthogonal matrix, so if $X \in \mathfrak{g}$ each ad X is skew-symmetric, that is, ${}^t(\operatorname{ad} X) = -\operatorname{ad} X$, where t denotes transpose. But then,

$$\begin{aligned} B(X, X) &= \operatorname{Tr}(\operatorname{ad} X \operatorname{ad} X) = -\operatorname{Tr}(\operatorname{ad} X\, {}^t(\operatorname{ad} X)) \\ &= -\sum_{i,j} x_{ij}^2 \qquad \text{if} \qquad \operatorname{ad} X = (x_{ij}) \end{aligned}$$

This proves (i); the second part is proved similarly.

Since the study of Lie algebras amounts to a study of the linear transformations ad X ($X \in \mathfrak{g}$), the first problem is, of course, diagonalization. Here one gets further by working with $\boldsymbol{C}$ as the base field, so we make the following definition.

Definition. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\boldsymbol{C}$. A *Cartan subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ such that (1) $\mathfrak{h} \subset \mathfrak{g}$ is a maximal abelian subalgebra; and (2) for each $H \in \mathfrak{h}$, ad H is a semisimple endomorphism of $\mathfrak{g}$ (that is, it can be put into diagonal form by means of a suitable basis).

The idea behind this definition is: If $X_1, X_2 \in \mathfrak{g}$ are such that ad X_1 and ad X_2 have simultaneous diagonalization then $[\operatorname{ad} X_1, \operatorname{ad} X_2] = 0$ so $[X_1, X_2] = 0$; thus the set ad $(\mathfrak{h})$ is a maximal family of simultaneously diagonalizable endomorphisms of $\mathfrak{g}$. Although our objective is the study of semisimple Lie algebras $\mathfrak{a}$ over $\boldsymbol{R}$ the definition above is useful because the complexification $\mathfrak{g} = \mathfrak{a} + i\mathfrak{a}$ is also semisimple. If $\mathfrak{g}$ is any Lie algebra over $\boldsymbol{C}$ a *real form* of $\mathfrak{g}$ is a *real* linear subspace $\mathfrak{b}$ of $\mathfrak{g}$ (that is, $r \in \boldsymbol{R}$, $X, Y \in \mathfrak{b} \Rightarrow rX$, $X + Y \in \mathfrak{b}$) which is closed under the bracket operation and satisfies $\mathfrak{g} = \mathfrak{b} + i\mathfrak{b}$ (direct sum). The mapping $X + iY \to X - iY$ ($X, Y \in \mathfrak{b}$) is called the *conjugation* of $\mathfrak{g}$ with respect to $\mathfrak{b}$. A Lie algebra $\mathfrak{g}$ over $\boldsymbol{C}$ may have many real forms.

Examples

(i) $\mathfrak{sl}(n, \boldsymbol{R})$ is a real form of $\mathfrak{sl}(n, \boldsymbol{C})$. The diagonal matrices in $\mathfrak{sl}(n, \boldsymbol{C})$ form a Cartan subalgebra.

(ii) The Lie algebra $\mathfrak{su}(1, 1)$ is a real form of $\mathfrak{sl}(2, \boldsymbol{C})$. In fact, if

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathfrak{sl}(2, \boldsymbol{C})$$

we can write (since $z_{22} = -z_{11}$)

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} i\alpha_1 & \beta_1 \\ \bar{\beta}_1 & -i\alpha_1 \end{pmatrix} + i\begin{pmatrix} i\alpha_2 & \beta_2 \\ \bar{\beta}_2 & -i\alpha_2 \end{pmatrix}$$

for $\alpha_1, \alpha_2 \in \boldsymbol{R}$, $\beta_1, \beta_2 \in \boldsymbol{C}$.

(iii) The Lie algebra $\mathfrak{su}(2)$ of skew-Hermitian matrices of trace 0,

$$X = \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} \qquad \alpha \in \boldsymbol{R},\ \beta \in \boldsymbol{C}$$

is obviously a real form of $\mathfrak{sl}(2, \boldsymbol{C})$. Since the Killing form of a real form is in general obtained by restriction we see from (4) §3-1 that

$$B(X, X) = 4 \operatorname{Trace}(XX) = -8(\alpha^2 + |\beta|^2)$$

so $\mathfrak{su}(2)$ is a compact real form of $\mathfrak{sl}(2, \boldsymbol{C})$.

The following two results are of fundamental importance.

Theorem 2.2. Every semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{C}$ contains a Cartan subalgebra $\mathfrak{h}$.

Theorem 2.3. Every semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{C}$ has a real form $\mathfrak{u}$ which is compact.

Ordinarily Theorem 2.2 is proved first using theorems on solvable Lie algebras (Lie's theorem that a solvable Lie algebra of complex matrices has a common eigenvector). The simultaneous diagonalization of the endomorphisms ad $\mathfrak{h}$ leads to a detailed structure theory for $\mathfrak{g}$ by which the compact real form $\mathfrak{u}$ is constructed. The details are as follows:

Assume $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Given a linear form $\alpha \neq 0$ on $\mathfrak{h}$ let

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid \operatorname{ad} H(X) = \alpha(H)X \qquad \text{for all } H \in \mathfrak{h}\}$$

This linear form α is called a *root* if $\mathfrak{g}^\alpha \neq \{0\}$. Let Δ denote the set of all roots. Then

$\in \Delta \Rightarrow -\alpha \in \Delta$

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha \qquad \text{(direct sum)} \tag{1}$$

and it can be proved that

$$\dim \mathfrak{g}^\alpha = 1 \qquad (\alpha \in \Delta) \tag{2}$$

Let $\mathfrak{h}^*$ denote the subset (real-linear subspace) of $\mathfrak{h}$, where all the roots have real values. Then for a suitable choice of vectors $X_\alpha \in \mathfrak{g}^\alpha$ the set

$$\mathfrak{u} = i\mathfrak{h}^* + \sum_{\alpha \in \Delta} \boldsymbol{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Delta} \boldsymbol{R}(i(X_\alpha + X_{-\alpha})) \tag{3}$$

is a compact real form of $\mathfrak{g}$.

$N_{\alpha,\beta} = -N_{-\alpha,-\beta}$

Example

Consider again the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, C)$ and its Cartan subalgebra $\mathfrak{h}$ of diagonal matrices of trace 0. Let again E_{ij} denote the matrix

$$(\delta_{ai}\, \delta_{bj})_{1 \leqslant a,\, b \leqslant n}$$

and for each $H \in \mathfrak{h}$ let $e_i(H)$ denote the ith diagonal element in H. Then

$$[H, E_{ij}] = (e_i(H) - e_j(H))E_{ij}$$

for all $H \in \mathfrak{h}$ so the linear form $\alpha_{ij}(H) = e_i(H) - e_j(H)$ is a root for $i \neq j$ and by (1) this does give all the roots. The space $\mathfrak{h}^*$ consists of all real diagonal matrices of trace 0. Let us put $X_{\alpha_{ij}} = E_{ij}\ (i \neq j)$. Then it is easily seen that the space (3) is the set $\mathfrak{su}(n)$ of all skew-Hermitian $n \times n$ matrices, which is indeed a compact real form of $\mathfrak{sl}\,(n, C)$ (*cf.* example above).

It is tempting to try to prove Theorem 2.3 directly, because then Theorem 2.2 would be an immediate corollary. In fact, for each $X \in \mathfrak{u}$, ad X can be diagonalized, so if $\mathfrak{t} \subset \mathfrak{u}$ is any maximal Abelian subalgebra, the space $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

A direct and elementary proof of Theorem 2.3 (without the use of Theorem 2.2) does not seem to be available. However, Cartan has proposed an idea for this purpose (*J. Math. Pures Appl.* **8** (1929), p. 23), which I shall describe here.

Since the Killing form of $\mathfrak{g}$ is nondegenerate, there exists a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ such that

$$B(Z, Z) = -\sum_1^n {z_i}^2 \qquad \text{if} \quad Z = \sum_1^n z_i\, e_i \tag{4}$$

Let the structural constants $c_{ijk} \in C$ be determined by

$$[e_i, e_j] = \sum_1^n c_{ijk}\, e_k$$

Then

$$B(Z, Z) = \operatorname{Tr}\,(\operatorname{ad} Z \operatorname{ad} Z) = \sum_{i,j} \left(\sum_{h,k} c_{ikh}\, c_{jhk} \right) z_i\, z_j$$

so by (4)

$$\sum_{h,k} c_{ikh}\, c_{jhk} = -\delta_{ij} \tag{5}$$

Also,

$$B([X_i, X_j], X_k) + B(X_j, [X_i, X_k]) = 0$$

so

$$c_{ijk} + c_{ikj} = 0$$

and by (5)

$$\sum_{i,h,k} c_{ihk}^2 = n$$

The space

$$\mathfrak{u} = \sum_1^n \boldsymbol{R} e_i$$

is a real form of $\mathfrak{g}$ if and only if all the c_{ijk} are real.

Consider now the set $\mathfrak{F}$ of all bases $(e_1, \ldots, e_n)$ of $\mathfrak{g}$ such that (4) holds. Consider the function f on $\mathfrak{F}$ given by

$$f(e_1, \ldots, e_n) = \sum_{i,j,k} |c_{ijk}|^2$$

Then we have seen that

$$\sum_{i,j,k} |c_{ijk}|^2 \geqslant \left| \sum_{i,j,k} c_{ijk}^2 \right| = \sum_{i,j,k} c_{ijk}^2 = n \tag{6}$$

and the equality sign holds if and only if all the c_{ijk} are real, that is, if and only if

$$\mathfrak{u} = \sum_1^n \boldsymbol{R} e_i$$

is a real form. In this case it is a compact real form in view of (4) and Prop. 2.1.

Thus Theorem 2.3 follows if one can prove: (I) The function f on $\mathfrak{F}$ has a minimum value; and (II) this minimum value is attained at a point $(e_1{}^0, \ldots, e_n{}^0) \in \mathfrak{F}$ for which the structural constants are real. Note that (II) is equivalent to (II′): The minimum of f is n.

3-3 Cartan Decompositions

We now go back to considering a semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ and as usual we denote by B the Killing form of $\mathfrak{g}$. There are of course many possible ways to find a direct vector space decomposition $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ such that B is positive definite on $\mathfrak{g}^+$ and negative definite on $\mathfrak{g}^-$. However, we should like to find a decomposition which is directly related to the Lie algebra structure of $\mathfrak{g}$.

Definition. A *Cartan decomposition* of $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that (i) $B < 0$ on $\mathfrak{k}$, $B > 0$ on $\mathfrak{p}$; and (ii) The mapping $\theta : T + X \to T - X$ $(T \in \mathfrak{k},\ X \in \mathfrak{p})$ is an automorphism of $\mathfrak{g}$.

In this case θ is called a *Cartan involution* of $\mathfrak{g}$ and the positive definite bilinear form $(X, Y) \to -B(X, \theta Y)$ is denoted by B_θ. We shall now establish the existence of Cartan decompositions, using compact real forms for semisimple Lie algebras over $\boldsymbol{C}$.

Theorem 3.1. Suppose θ is a Cartan involution of a semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ and σ an arbitrary involutive automorphism of $\mathfrak{g}$. There then exists an automorphism ϕ of $\mathfrak{g}$ such that the Cartan involution $\phi\theta\phi^{-1}$ commutes with σ.

PROOF. The product $N = \sigma\theta$ is an automorphism of $\mathfrak{g}$ and if $X, Y \in \mathfrak{g}$,

$$-B_\theta(NX, Y) = B(NX, \theta Y) = B(X, N^{-1}\theta Y) = B(X, \theta NY)$$

so

$$B_\theta(NX, Y) = B_\theta(X, NY)$$

that is, N is symmetric with respect to the positive definite bilinear form B_θ. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ diagonalizing N. Then $P = N^2$ has a positive diagonal, say, with elements $\lambda_1, \dots, \lambda_n$. Take P^t ($t \in \boldsymbol{R}$) with diagonal elements $\lambda_1{}^t, \dots, \lambda_n{}^t$ and define the structural constants c_{ijk} by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k$$

Since P is an automorphism, we conclude

$$\lambda_i \lambda_j c_{ijk} = \lambda_k c_{ijk}$$

which implies

$$\lambda_i{}^t \lambda_j{}^t c_{ijk} = \lambda_k{}^t c_{ijk} \qquad (t \in \boldsymbol{R})$$

so P^t is an automorphism. Put $\theta_t = P^t\theta P^{-t}$. Since $\theta N\theta^{-1} = N^{-1}$, we have $\theta P\theta^{-1} = P^{-1}$, that is $\theta P = P^{-1}\theta$. In matrix terms (using still the basis $X_1, \dots, X_n$) this means (since θ is symmetric with respect to B_θ)

$$\theta_{ij}\lambda_j = \lambda_i^{-1}\theta_{ij}$$

so

$$\theta_{ij}\lambda_j^t = \lambda_i^{-t}\theta_{ij}$$

that is, $\theta P^t\theta^{-1} = P^{-t}$. Hence,

$$\sigma\theta_t = \sigma P^t\theta P^{-t} = \sigma\theta P^{-2t} = NP^{-2t}$$

$$\theta_t\sigma = (\sigma\theta_t)^{-1} = P^{2t}N^{-1} = N^{-1}P^{2t}$$

so it suffices to put $\phi = P^{1/4}$ $(= \sqrt{\sigma\theta})$. (*cf.* [3], p. 100, [31], p. 156, [47], p. 884). The following result is given in Mostow [54].

Corollary 3.2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\boldsymbol{R}$, $\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}$ its complexification, $\mathfrak{u}$ any compact real form of $\mathfrak{g}_c$, σ and τ the conjugations of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ and $\mathfrak{u}$, respectively. Then there exists an automorphism ϕ of $\mathfrak{g}_c$ such that $\phi \cdot \mathfrak{u}$ is invariant under σ.

PROOF. Let $\mathfrak{g}_c^{\boldsymbol{R}}$ denote the Lie algebra $\mathfrak{g}_c$ considered as a Lie algebra over $\boldsymbol{R}$, $B^{\boldsymbol{R}}$ the Killing form. It is not hard to show that $B^{\boldsymbol{R}}(X, Y) = 2\mathrm{Re}\,(B_c(X, Y))$ if B_c is the Killing form of $\mathfrak{g}_c$. Thus σ and τ are Cartan involutions of $\mathfrak{g}_c^{\boldsymbol{R}}$ and the corollary follows (note that since $\sigma\tau$ is a (complex) automorphism of $\mathfrak{g}_c$, ϕ is one as well).

Corollary 3.3. Each semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ has Cartan decompositions and any two such are conjugate under an automorphism of $\mathfrak{g}$.

PROOF. Let $\mathfrak{g}_c$ denote the complexification of $\mathfrak{g}$, σ the corresponding conjugation, and $\mathfrak{u}$ a compact real form of $\mathfrak{g}_c$ invariant under σ (Theorem 2.3 and Cor. 3.2). Then put $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$, $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$. Then $B < 0$ on $\mathfrak{k}$, $B > 0$ on $\mathfrak{p}$, and since $\theta : T + X \to T - X\,(T \in \mathfrak{k}, X \in \mathfrak{p})$ is an automorphism, $B(\mathfrak{k}, \mathfrak{p}) = 0$. It follows that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition.

Consider now two Cartan decompositions,

$$\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{p}_1 \qquad \mathfrak{g} = \mathfrak{k}_2 + \mathfrak{p}_2$$

Then $\mathfrak{u}_1 = \mathfrak{k}_1 + i\mathfrak{p}_1$ and $\mathfrak{u}_2 = \mathfrak{k}_2 + i\mathfrak{p}_2$ are compact real forms of $\mathfrak{g}_c$. Let τ_1 and τ_2 denote the corresponding conjugations. By Cor. 3.2 there exists an automorphism ϕ of $\mathfrak{g}_c$ such that $\phi \cdot \mathfrak{u}_2$ is invariant under τ_1. Thus $\phi \cdot \mathfrak{u}_2$ is equal to the direct sum of its intersections with $\mathfrak{u}_1$ and $i\mathfrak{u}_1$. Now $B > 0$ on $i\mathfrak{u}_1$ and $B < 0$ on $\phi \cdot \mathfrak{u}_2$. Hence $i\mathfrak{u}_1 \cap \phi \cdot \mathfrak{u}_2 = \{0\}$ so $\mathfrak{u}_1 = \phi \cdot \mathfrak{u}_2$. But τ_1 and τ_2 both leave $\mathfrak{g}$ invariant and ϕ can (according to the proof of Theorem 3.1) be taken as a power of $\tau_1\tau_2$ so it also leaves $\mathfrak{g}$ invariant. Thus $\phi(\mathfrak{g} \cap \mathfrak{u}_2) = \mathfrak{g} \cap \mathfrak{u}_1$ so ϕ gives the desired automorphism of $\mathfrak{g}$.

Examples

Let $\mathfrak{g} = \mathfrak{sl}\,(n, \boldsymbol{R})$, the Lie algebra of the group $\boldsymbol{SL}(n, \boldsymbol{R})$. The group $\boldsymbol{SO}(n)$ of orthogonal matrices is a closed subgroup, hence a Lie subgroup, and by (8) §2-2, its Lie algebra, denoted $\mathfrak{so}(n)$, consists of those matrices $X \in \mathfrak{sl}(n, \boldsymbol{R})$ for which $\exp tX \in \boldsymbol{SO}(n)$ for all $t \in \boldsymbol{R}$. But

$$\exp tX \in \boldsymbol{SO}(n) \Leftrightarrow \exp tX \exp t({}^tX) = 1 \qquad \det(\exp tX) = 1$$

so

$$\mathfrak{so}(n) = \{X \in \mathfrak{sl}(n, \boldsymbol{R}) \mid X + {}^tX = 0\}$$

the set of skew-symmetric $n \times n$ matrices (which are automatically of trace 0).

The mapping $\theta : X \to -{}^tX$ is an automorphism of $\mathfrak{sl}(n, \boldsymbol{R})$ and $\theta^2 = 1$. Since $B(X, X) = 2n\,\mathrm{Tr}\,(XX)$, $B(X, \theta X) < 0$ so θ is a Cartan involution and

$$\mathfrak{sl}(n, \boldsymbol{R}) = \mathfrak{so}(n) + \mathfrak{p} \tag{1}$$

where $\mathfrak{p}$ is the set of $n \times n$ symmetric matrices of trace 0, is the corresponding

Cartan decomposition. Now it is known that every positive definite matrix can be written uniquely e^X ($X =$ symmetric) and every nonsingular matrix g can be written uniquely $g = op$ ($o =$ orthogonal, $p =$ positive definite). Thus we have a global analog of (1),

$$\boldsymbol{SL}(n, \boldsymbol{R}) = \boldsymbol{SO}(n)P \tag{2}$$

where $P = \exp \mathfrak{p}$, the set of positive definite matrices of determinant 1.

We shall now state a generalization of (2).

Theorem 3.4. Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition ($\mathfrak{k}$ the algebra), K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then the mapping

$$(X, k) \to (\exp X)k$$

is a diffeomorphism of $\mathfrak{p} \times K$ onto G.

In Theorem 3.4, the center $\mathfrak{z}$ of $\mathfrak{g}$ is $\{0\}$, (immediate from the definition) so the center Z of G is discrete. One can prove $Z \subset K$ and that K is compact if and only if Z is finite. In this case K is a maximal compact subgroup of G, and every compact subgroup is conjugate to a subgroup of K.

Proposition 3.5. In terms of the notation of Theorem 3.4, the mapping

$$(\exp X)k \to \exp(-X)k \tag{3}$$

is an automorphism of G.

In fact let $\tilde{G}$ be the universal covering group of G. Since all simply connected Lie groups with the same Lie algebra are isomorphic (*cf.* (v) §2-2) the automorphism θ of $\mathfrak{g}$ induces an automorphism $\tilde{\theta}$ of $\tilde{G}$ such that $d\tilde{\theta}_e = \theta$. By the remarks above, the center $\tilde{Z}$ of $\tilde{G}$ is contained in the analytic subgroup $\tilde{K}$ of $\tilde{G}$ corresponding to $\mathfrak{k}$. But $G = \tilde{G}/N$, where $N \subset \tilde{Z}$ so $\tilde{\theta}$ induces an automorphism of G which is (3).

Consider now the set G/K of left cosets gK ($g \in G$). This set has a unique manifold structure such that the map $X \to (\exp X)K$ is a diffeomorphism of $\mathfrak{p}$ onto G/K. (More generally if K is a closed subgroup of a Lie group G, G/K is a manifold in a natural way.) The group G operates on G/K: each $g \in G$ gives rise to a diffeomorphism $\tau(g) : xK \to gxK$ of G/K. Since $Z \subset K$ we have $G/K = (G/Z)/(K/Z)$ and $G/Z = \operatorname{Int}(\mathfrak{g})$ *so the space G/K is independent of the choice of the Lie group G with Lie algebra* $\mathfrak{g}$. In view of Cor. 3.3 the different possibilities for K are all conjugate so the space G/K is in a canonical way associated with $\mathfrak{g}$. Let o denote the point $\{K\}$ in G/K (the origin) and $(G/K)_o$ the tangent space. The mapping $\pi : g \to gK$ has a differential $d\pi$ mapping $\mathfrak{g}$ onto $(G/K)_o$ with a kernel which contains $\mathfrak{k}$. By reasons of dimensionality, we see therefore that the mapping

$$d\pi : \mathfrak{p} \to (G/K)_o \tag{4}$$

is an isomorphism and if $k \in K$ we have for $X \in \mathfrak{p}$, $t \in \boldsymbol{R}$

$$\pi(\exp \operatorname{Ad}(k)tX) = \pi(k \exp tX\, k^{-1}) = \tau(k)\pi(\exp tX)$$

so

$$d\pi\,(\operatorname{Ad}(k)X) = d\tau(k)\, d\pi(X). \tag{5}$$

Now the form B is > 0 on $\mathfrak{p}$ so by (4) and (5) we obtain a positive definite quadratic form Q_o on $(G/K)_o$ invariant under $d\tau(k)$ $(k \in K)$. If $p \in G/K$ is arbitrary there exists a $g \in G$ such that $p = gK$ and $d\tau(g) : (G/K)_o \to (G/K)_p$ is an isomorphism giving rise to a quadratic form Q_p on $(G/K)_p$. If $g' \in G$ satisfies $g'K = gK$, $d\tau(g')$ gives the same quadratic form Q_p on $(G/K)_p$ because of the K-invariance of Q_o. Thus we have a Riemannian structure Q on G/K induced by B.

Proposition 3.6. The manifold G/K with the Riemannian structure induced by B is a symmetric space.

PROOF. Let θ denote the automorphism (3) and s_o the mapping $gK \to \theta(g)K$ of G/K onto itself. Then s_o is a diffeomorphism and $s_o{}^2 = I$, $(ds_o)_o = -I$. To see that s_o is an isometry let $p = gK\,(g \in G)$ and $X \in (G/K)_p$. Then the vector $X_o = d\tau(g^{-1})X$ belongs to $(G/K)_o$. But if $x \in G$ we have

$$s_o(gxK) = \theta(gx)K = \tau(\theta(g))(s_o(xK))$$

so $s_o \circ \tau(g) = \tau(\theta(g)) \circ s_o$ and therefore

$$\begin{aligned} Q(ds_o(X), ds_o(X)) &= Q(ds_o \circ d\tau(g)(X_o), ds_o \circ d\tau(g)(X_o)) \\ &= Q(d\tau(\theta(g)) \circ ds_o(X_o), d\tau(\theta(g)) \circ ds_o(X_o)) \\ &= Q(X_o, \cancel{Y_o}) = Q(X, \cancel{Y}) \end{aligned}$$

Thus s_o is an isometry and since $(ds_o)_o = -I$, it reverses the geodesics through o. The geodesic symmetry with respect to $p = gK$ is given by

$$s_p = \tau(g) \circ s_o \circ \tau(g^{-1})$$

which is an isometry, so the proposition follows.

Proposition 3.7. The geodesics through the origin in G/K are the curves $t \to \exp tX \cdot o$ $(X \in \mathfrak{p})$.

Although the proof is not difficult we shall omit it. Instead let us take a second look at the example $G = \boldsymbol{SU}(1, 1)$. The decomposition

$$\begin{pmatrix} i\alpha & \beta \\ \bar{\beta} & -i\alpha \end{pmatrix} = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix} \tag{6}$$

gives a Cartan decomposition of $\mathfrak{su}(1, 1)$. We have also if

$$X_\beta = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}$$

$$\exp(tX_\beta) = \cosh(t\,|\beta|)I + \frac{1}{|\beta|}\sinh(t\,|\beta|)X_\beta$$

so

$$\exp(tX_\beta)\cdot o = (\tanh t\,|\beta|)\frac{\beta}{|\beta|}$$

verifying the proposition in this case.

3-4 Discussion of Symmetric Spaces

We shall now summarize some basic results in the general theory of symmetric spaces and indicate how the coset spaces G/K from the last section fit into this general theory.

Let M be a symmetric space as defined in Ch. 1. The group $I(M)$ of all isometries of M is transitive on M. (In fact, if $p, q \in M$ they can be joined by a broken geodesic and the product of the symmetries in the midpoints of these geodesics gives the desired isometry.) One can now parametrize the group $I(M)$ in a natural way turning it into a Lie group. The identity component $G = I_o(M)$ is still transitive on M. Fix a point $o \in M$ and let K be the group of elements in G which leaves o fixed. Then the mapping $gK \to g \cdot o$ is a diffeomorphism of G/K onto M. If s_o is the geodesic symmetry with respect to o the mapping $\sigma : g \to s_o\, g s_o$ is an involutive automorphism of G and $(K_\sigma)_o \subset K \subset K_\sigma$, where K_σ is the set of fixed points of σ and $(K_\sigma)_o$ its identity component. In order to verify these inclusions let $k \in K$. Then the maps k and $s_o k s_o$ are isometries leaving o fixed and inducing the same linear map of the tangent space M_o. Considering the geodesics starting at o we see that k and $s_o k s_o$ must coincide so $K \subset K_\sigma$. On the other hand, suppose X in the Lie algebra $\mathfrak{g}$ of G is fixed under the differential $(d\sigma)_e$. Then $s_o \exp tX\, s_o = \exp tX$ for all $t \in \boldsymbol{R}$, so applying both sides to the point o we see that $\exp tX \cdot o$ is fixed under s_o. But o is an isolated fixed point of s_o so $\exp tX \cdot o = o$ for all sufficiently small t. But then $X \in \mathfrak{k}$, the Lie algebra of K, whence $(K_\sigma)_o \subset K$. Note finally that the group $\mathrm{Ad}_G(K)$ is compact, being a continuous image of the compact group K.

Conversely, let G be a connected Lie group, K a closed subgroup, $\mathrm{Ad}_G(K)$compact. Suppose there exists an involutive automorphism σ of G such that $(K_\sigma)_o \subset K \subset K_\sigma$. Then there exists a Riemannian structure on G/K invariant under G, and for every such Riemannian structure, G/K is a symmetric space.

Consider now M as above and $G = I_o(M)$; M is said to be of the *noncompact type* if G is noncompact, semisimple without a compact normal subgroup $\neq \{e\}$, and of the *compact type* if G is compact and semisimple.

Proposition 4.1. Let M be a symmetric space, which is simply connected. Then M is a product

$$M = M_o \times M_c \times M_n$$

where M_0 is a Euclidean space and M_c and M_n are symmetric spaces of the compact type and the noncompact type, respectively.

Proposition 4.2. A symmetric space of the compact type (noncompact type) has sectional curvature everywhere $\geqslant 0$ (respectively $\leqslant 0$).

There is a very interesting *duality* between the compact type and the noncompact type. Let $M = G/K$ be a symmetric space of the noncompact type where $G = I_o(M)$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}$ the complexification of $\mathfrak{g}$. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, the subspace $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ of $\mathfrak{g}_c$ is actually a Lie algebra and another real form of $\mathfrak{g}_c$. Since the Killing form of $\mathfrak{g}_c$ is < 0 on $\mathfrak{k}$, and > 0 on $\mathfrak{p}$, it is < 0 on $\mathfrak{u}$, so $\mathfrak{u}$ is a compact real form. If U is a connected Lie group with Lie algebra $\mathfrak{u}$ and K' is the connected Lie subgroup with Lie algebra $\mathfrak{k}$, the space U/K' is a symmetric space of the compact type. This process can be reversed, that is, G/K can be constructed with U/K as a starting point.

Examples

(i) Consider the symmetric space G/K, where $G = \boldsymbol{SU}(1, 1)$ and K the subgroup of matrices $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $|t| = 1$. In this case the Cartan decomposition (6) in §3-3 shows that $\mathfrak{u}$ is the set of all matrices of the form

$$\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} + \begin{pmatrix} 0 & i\beta \\ i\beta & 0 \end{pmatrix}$$

so $\mathfrak{u} = \mathfrak{su}(2)$, the algebra of all 2×2 skew symmetric matrices of trace 0. For the space U/K' we can therefore take the space $\boldsymbol{SU}(2)/K$. [$\boldsymbol{SU}(n)$ denotes the special unitary group.] It is not hard to show that when the unit sphere S^2 is projected stereographically onto the complex plane the rotations of the sphere correspond to the transformations

$$z \rightarrow \frac{az + \bar{b}}{-bz + \bar{a}} \qquad |a|^2 + |b|^2 = 1$$

that is, to the members of $\boldsymbol{SU}(2)$. In this manner $\boldsymbol{SU}(2)$ acts transitively on

S^2 and the subgroup leaving the point $z = 0$ fixed is K. Thus $U/K = S^2$ so the non-Euclidean disk D (Ch. 1) and the sphere S^2 correspond under the general duality indicated. The formulas $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ can be regarded as an explanation of the phenomenon that the triangle formulas in non-Euclidean trigonometry are obtained from the triangle formulas in spherical trigonometry by replacing the sides a, b, c by ia, ib, ic and using the relations $\sinh(ia) = i \sin a$, $\cosh(ia) = \cos a$. Lobatschevsky did indeed speak of his non-Euclidean trigonometry as spherical trigonometry on a sphere of imaginary radius.

(ii) Let U be a connected, compact Lie group with Lie algebra $\mathfrak{u}$. If Q is any positive definite quadratic form on $\mathfrak{u}$, we obtain by left translations such quadratic forms on each tangent space to U and therefore a Riemannian metric on U which is invariant under all left translations. If Q is chosen invariant under $\mathrm{Ad}(U)$ then the Riemannian metric is invariant under right translations as well. One can prove that the geodesics through e are the one-parameter subgroups and the symmetry $s_e : x \to x^{-1}$ is an isometry so U is a symmetric space. If U^* denotes the diagonal in $U \times U$ one has a diffeomorphism $(u_1, u_2)U^* \to u_1 u_2^{-1}$ of $(U \times U)/U^*$ onto U. The group involution $(u_1, u_2) \to (u_2, u_1)$ of $U \times U$ leaves U^* pointwise fixed and induces the symmetry s_e of U, via the diffeomorphism indicated.

If U is in addition semisimple, the symmetric space $(U \times U)/U^*$ has in the above sense a noncompact dual G/U', where U' has Lie algebra $\mathfrak{u}$ and the Lie algebra $\mathfrak{g}$ of G is a certain real form of the complexification of the product algebra $\mathfrak{u} \times \mathfrak{u}$. One can prove that as $\mathfrak{u}$ runs through the compact semisimple Lie algebras, $\mathfrak{g}$ runs through the *complex* semisimple Lie algebras (regarded as Lie algebras over $\boldsymbol{R}$).

3-5 The Iwasawa Decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition. The operators $\mathrm{ad}\, X$ $(X \in \mathfrak{p})$ are all symmetric with respect to the positive definite form B_θ and each of them can therefore be diagonalized, and a commutative family can be simultaneously diagonalized. Hence let $\mathfrak{a}$ denote a maximal Abelian subspace of $\mathfrak{p}$ and if α is a real-valued linear function on $\mathfrak{a}$ put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\} \tag{1}$$

If $\mathfrak{g}_\alpha \neq \{0\}$, $\alpha \neq 0$, α is called a *restricted root*. Clearly, if Σ denotes the set of restricted roots,

$$\mathfrak{g} = \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha + \mathfrak{g}_o \tag{2}$$

The dimension $\dim(\mathfrak{g}_\alpha)$ is called the *multiplicity* of α. Let $\mathfrak{a}'$ denote the set of elements in $\mathfrak{a}$, where all roots are $\neq 0$. The connected components of $\mathfrak{a}'$

are intersections of half spaces; hence they are convex open sets. They are called *Weyl chambers*. Fix any Weyl chamber $\mathfrak{a}^+$ and call a restricted root *positive* if its values on $\mathfrak{a}^+$ are positive.

Let Σ^+ denote the set of positive restricted roots and put

$$\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}_\alpha \qquad \rho = \frac{1}{2} \sum_{\alpha > 0} (\dim \mathfrak{g}_\alpha)\alpha \tag{3}$$

Then $\mathfrak{n}$ is a nilpotent Lie algebra. The following result is called the Iwasawa decomposition.

Theorem 5.1. $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ (direct vector space sum). Let G be any connected Lie group with Lie algebra $\mathfrak{g}$, and let K, A, N denote the analytic subgroups corresponding to $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Then the mapping

$$(k, a, n) \to kan$$

is a diffeomorphism of $K \times A \times N$ onto G.

Rather than give the proof we consider some examples. Consider the Cartan decomposition (1) §3-3,

$$\mathfrak{sl}(n, \boldsymbol{R}) = \mathfrak{so}(n) + \mathfrak{p} \tag{4}$$

The diagonal matrices of trace 0 form a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and as in §3-2 we find that the corresponding restricted roots are the linear forms $\alpha_{ij}(H) = e_i(H) - e_j(H)$ $(H \in \mathfrak{a})$, $e_i(H)$ being the ith diagonal element in H. Hence $\mathfrak{a}'$ consists of those H for which all $e_i(H)$ are different. The set

$$\{H \in \mathfrak{a} \mid e_1(H) > e_2(H) > \cdots > e_n(H)\} \tag{5}$$

is clearly a connected component of $\mathfrak{a}'$ and we take this as the Weyl chamber $\mathfrak{a}^+$. Then Σ^+ consists of the roots α_{ij} $(i < j)$ and $\mathfrak{n}$ is easily found to be the set of upper triangular matrices with 0 in the diagonal. An Iwasawa decomposition of the group $\boldsymbol{SL}(n, \boldsymbol{R})$ is therefore $g = oan$, where $o \in \boldsymbol{SO}(n)$, a is a diagonal matrix of determinant 1 and diagonal > 0, and n is an upper triangular matrix with all diagonal elements 1.

For another example consider the Cartan decomposition of $\mathfrak{su}(1, 1)$ given by

$$\begin{pmatrix} ix & y \\ \bar{y} & -ix \end{pmatrix} = \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} + \begin{pmatrix} 0 & y \\ \bar{y} & 0 \end{pmatrix}$$

where $x \in \boldsymbol{R}$, $y \in \boldsymbol{C}$. As the space $\mathfrak{a}$ we can take

$$\boldsymbol{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and since

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} ix & y \\ \bar{y} & -ix \end{pmatrix}\right] = \begin{pmatrix} \bar{y} - y & -2ix \\ 2ix & y - \bar{y} \end{pmatrix}$$

we see that the decomposition (2) equals

$$\mathfrak{g} = \boldsymbol{R}\begin{pmatrix} i & -i \\ i & -i \end{pmatrix} + \boldsymbol{R}\begin{pmatrix} i & i \\ -i & -i \end{pmatrix} + \boldsymbol{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the restricted roots are α and $-\alpha$, where

$$\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$$

Thus $\mathfrak{a}'$ consists of the nonzero elements in $\mathfrak{a}$ and for $\mathfrak{a}^+$ we take for example

$$\boldsymbol{R}^+\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so

$$\mathfrak{n} = \boldsymbol{R}\begin{pmatrix} i & -i \\ i & -i \end{pmatrix}$$

and $N = \exp \mathfrak{n}$ equals the group of matrices

$$\begin{pmatrix} 1 + in & -in \\ in & 1 - in \end{pmatrix} \in \boldsymbol{SU}(1, 1)$$

The Iwasawa decomposition of a semisimple Lie algebra $\mathfrak{g}$ involves some free choices, namely, that of $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{a}^+$. We have seen that $\mathfrak{k}$ is unique up to conjugacy, and now we shall see that $\mathfrak{a}$ and $\mathfrak{a}^+$ are uniquely determined up to conjugacy by elements of K. We begin with a result which goes back to Weyl and Cartan with a proof given by Hunt [41].

Theorem 5.2. Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two maximal Abelian subspaces of $\mathfrak{p}$. Then there exists an element $k \in K$ such that $\operatorname{Ad}_G(k)\, \mathfrak{a} = \mathfrak{a}'$. Also

$$\mathfrak{p} = \bigcup_{k \in K} \operatorname{Ad}_G(k)\, \mathfrak{a}$$

PROOF. Select $H \in \mathfrak{a}$ such that its centralizer in $\mathfrak{p}$ equals $\mathfrak{a}$. (It suffices to take H such that $\alpha(H) \neq 0$ for all restricted roots α.) Put $K^* = \operatorname{Ad}_G(K)$ and let $X \in \mathfrak{p}$ be arbitrary. The function

$$k^* \to B(H, k^* \cdot X) \qquad (k^* \in K^*)$$

has a minimum, say, for $k^* = k_0$. If $T \in \mathfrak{k}$ we have therefore

$$\left\{\frac{d}{dt} B(H, \operatorname{Ad}(\exp tT)k_0 \cdot X)\right\}_{t=0} = 0$$

so

$$B(H, [T, k_o \cdot X]) = 0 \qquad T \in \mathfrak{k}$$

Thus

$$B(T, [H, k_o \cdot X]) = 0 \qquad \text{for all } T \in \mathfrak{k}$$

and since $[H, k_0 \cdot X] \in \mathfrak{k}$ we deduce $[H, k_0 \cdot X] = 0$ so by the choice of $H, k_0 \cdot X \in \mathfrak{a}$.

In particular, there exists a $k_1 \in K$ such that $H \in \mathrm{Ad}\,(k_1)\mathfrak{a}'$. Thus each element in $\mathrm{Ad}\,(k_1)\mathfrak{a}'$ commutes with H so $\mathrm{Ad}\,(k_1)\mathfrak{a}' \subset \mathfrak{a}$. This proves the theorem.

3-6 The Weyl Group

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition, G any connected Lie group with Lie algebra $\mathfrak{g}$, K the analytic subgroup with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Consider as before a maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and let M' and M denote, respectively, the *normalizer* and *centralizer* of $\mathfrak{a}$ in K; that is,

$$M' = \{k \in K \mid \mathrm{Ad}\,(k)\mathfrak{a} \subset \mathfrak{a}\}$$

$$M = \{k \in K \mid \mathrm{Ad}\,(k)H = H \text{ for all } H \in \mathfrak{a}\}$$

Clearly M is a normal subgroup of M' and the factor group M'/M can obviously be viewed as a group of linear transformations of $\mathfrak{a}$. It is called the *Weyl group* and denoted W. In view of Theorem 5.2 it is (up to isomorphism) independent of the choice of $\mathfrak{a}$.

Now M and M' are Lie subgroups of K and their Lie algebras $\mathfrak{m}$ and $\mathfrak{m}'$ are given by (*cf.* (8) §2-2, (7) §3-1),

$$\mathfrak{m} = \{T \in \mathfrak{k} \mid [H, T] = 0 \text{ for all } H \in \mathfrak{a}\}$$

$$\mathfrak{m}' = \{T \in \mathfrak{k} \mid [H, T] \subset \mathfrak{a} \text{ for all } H \in \mathfrak{a}\}$$

Note, however, that if $T \in \mathfrak{m}'$ then for $H \in \mathfrak{a}$,

$$B([H, T], [H, T]) = -B([H, [H, T]], T) = 0$$

so $T \in \mathfrak{m}$, whence $\mathfrak{m} = \mathfrak{m}'$. Thus M'/M is a discrete group and being also compact, must be finite.

If λ is a complex-valued linear function on $\mathfrak{a}$ let H_λ denote the vector in $\mathfrak{a} + i\mathfrak{a}$ determined by $B(H, H_\lambda) = \lambda(H)$ for all $H \in \mathfrak{a}$. For $\alpha \in \Sigma$ let s_α denote the symmetry in the hyperplane $\alpha(H) = 0$:

$$s_\alpha(H) = H - 2\,\frac{\alpha(H)}{\alpha(H_\alpha)}\,H_\alpha \qquad H \in \mathfrak{a}, \tag{1}$$

(Remember $\mathfrak{p}$ and hence $\mathfrak{a}$ have a Euclidean metric given by B.)

Theorem 6.1. $s_\alpha \in W$ for each $\alpha \in \Sigma$.

PROOF. Pick $Z_\alpha \in \mathfrak{g}$ such that $[H, Z_\alpha] = \alpha(H)Z_\alpha$. Decomposing $Z_\alpha = T_\alpha + X_\alpha$ $(T_\alpha \in \mathfrak{k},\ X_\alpha \in \mathfrak{p})$ the relations $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ imply that $(\text{ad}\, H)^2 T_\alpha = T_\alpha$. Multiplying Z_α by a real factor if necessary we may assume $B(T_\alpha, T_\alpha) = -1$. Now if $\alpha(H) = 0$ we have $[H, T_\alpha] = 0$ so

$$\text{Ad}\,(\exp tT_\alpha)H = e^{\text{ad}\,(tT_\alpha)}(H) = H \qquad \text{if } \alpha(H) = 0$$

A simple computation shows that

$$e^{\text{ad}\,(t_o T_\alpha)} H_\alpha = -H_\alpha$$

provided $t_o(\alpha(H_\alpha))^{1/2} = \pi$. Thus s_α coincides with the restriction of $\text{Ad}\,(\exp t_o T_\alpha)$ to $\mathfrak{a}$.

If $s \in W$ and $\alpha \in \Sigma$ it is clear from the definitions that the linear function $\alpha^s : H \to \alpha(s^{-1}H)$ on $\mathfrak{a}$ is a restricted root. Consequently, s permutes the Weyl chambers. Now let C_1 and C_2 be two Weyl chambers and let $H_1 \in C_1$, $H_2 \in C_2$. If the segment $\overrightarrow{H_1 H_2}$ intersects a hyperplane $\alpha(H) = 0$ $(\alpha \in \Sigma)$ then clearly the norm $|\ \ |$ in $\mathfrak{a}$ satisfies

$$|H_1 - H_2| > |H_1 - s_\alpha H_2| \tag{2}$$

As s runs through the finite group W the function $|H_1 - sH_2|$ takes a minimum, say for $s = s_0$. By (2) the segment from H_1 to $s_o H_2$ intersects no hyperplane $\alpha(H) = 0$ $(\alpha \in \Sigma)$ so H_1 and $s_o H_2$ lie in the same Weyl chamber and thus $C_1 = s_o C_2$. This proves:

Corollary 6.2. Any two Weyl chambers in $\mathfrak{a}$ are conjugate under some element of $\text{Ad}_G(K)$ which leaves $\mathfrak{a}$ invariant.

For orientation we state without proof a somewhat deeper result on the Weyl group.

Theorem 6.3. The Weyl group W is generated by the symmetries s_α $(\alpha \in \Sigma)$ and it is simply transitive on the set of Weyl chambers in $\mathfrak{a}$.

3.7 Boundary and Polar Coordinates on the Symmetric Space G/K

For the non-Euclidean disk D we have a natural notion of boundary, namely, the unit circle $|z| = 1$. However, this boundary notion refers to the position of D in $\boldsymbol{R}^2$. In order to make this definition more intrinsic we can define the boundary of D as the set of all rays (half-lines) from the origin in D. This motivates the following definition of the boundary of the symmetric space G/K. First, we recall the isomorphism $d\pi : \mathfrak{p} \to (G/K)_o$ from §3-3, which permits us to think of $\mathfrak{p}$ as the tangent space to G/K at o. Then we understand by a *Weyl chamber in* $\mathfrak{p}$ a Weyl chamber in some maximal Abelian subspace pf $\mathfrak{p}$. The *boundary* of G/K is now defined as the set of all Weyl chambers in $\mathfrak{p}$. Now fix $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{a}^+$ a Weyl chamber in $\mathfrak{a}$. Then according to Theorem 5.2 and Cor. 6.2, $\mathrm{Ad}(k)\mathfrak{a}^+$ $(k \in K)$ runs through the boundary and if $\mathrm{Ad}(k)\mathfrak{a}^+ = \mathfrak{a}^+$, then $k \in M'$ so $\mathrm{Ad}(k)$ on $\mathfrak{a}$ is a member of the Weyl group. Using Theorem 6.3 we see that $k \in M$. Thus the mapping

$$kM \to \mathrm{Ad}(k)\mathfrak{a}^+$$

identifies K/M with the boundary of G/K. In view of the Iwasawa decomposition $G = KAN$ and the fact that M normalizes AN we have a diffeomorphism

$$kM \to kMAN$$

of K/M onto G/MAN. In his paper [19], Furstenberg defines a boundary of G to be a compact coset space G/H of G such that for each probability measure μ on G/H there exists a sequence $(g_n) \subset G$ such that the transformed measures $g_n \cdot \mu$ converge weakly to the delta function on G/H. It was proved by Furstenberg [19] and Moore [53] that a "maximal" boundary of this sort is given by G/MAN which, as we saw, coincides with the geometrically defined boundary above. The relation $K/M = G/MAN$ shows in particular that G acts as a transformation group on the boundary; in an explicit manner

$$g(kM) = k(gk)M$$

if for $x \in G$, $k(x) \in K$ is given by $x \in k(x)AN$.

Now let $A^+ = \exp \mathfrak{a}^+$. Then we have the following "polar coordinate representation" of the symmetric space G/K.

Theorem 7.1. The mapping $(kM, a) \to kaK$ is a diffeomorphism of $K/M \times A^+$ onto an open submanifold of G/K whose complement in G/K has lower dimension.

Without spelling out the proof in detail we remark that it is a fairly direct consequence of Theorems 3.4, 5.2, and 6.3.

CHAPTER 4: FUNCTIONS ON SYMMETRIC SPACES

4-1 Invariant Differential Operators

Let M be a manifold and D a differential operator on M, that is, a linear mapping of $C_c^\infty(M)$ into itself which in an arbitrary coordinate system is expressed by partial derivatives in the coordinates. Let $\phi : M \to M$ be a diffeomorphism, and if f is a function on M put $f^\phi = f \circ \phi^{-1}$ and let D^ϕ denote the operator

$$D^\phi f = (Df^{\phi^{-1}})^\phi$$

Then D^ϕ is another differential operator, and we say D is *invariant under* ϕ if $D^\phi = D$.

Examples

Let us find all differential operators D on $\boldsymbol{R}^n$ which are invariant under all rigid motions. Since D is invariant under all translations it has constant coefficients so $D = P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, where P is a polynomial. But D is also invariant under all rotations around 0 so P is rotation-invariant, and since the rotations are transitive on each sphere $|x| = r$, we find P is constant on each such sphere so $P(x_1, \ldots, x_n)$ is a function of $x_1{}^2 + \cdots + x_n{}^2$, hence a polynomial in $x_1{}^2 + \cdots + x_n{}^2$.

Proposition 1.1. The differential operators on $\boldsymbol{R}^n$ which are invariant under all isometries are the operators $\Sigma a_n \Delta^n$ ($a_n \in \boldsymbol{C}$), where Δ is the Laplacian.

This result holds also if $\boldsymbol{R}^n$ is replaced by a symmetric space of rank 1 (and Δ by the Laplace–Beltrami operator) and also if we replace the isometries of $\boldsymbol{R}^n$ by the inhomogeneous Lorentz group, in which case the Laplacian is replaced (*cf.* [29], p. 271) by the operator

$$\frac{\partial^2}{\partial x_1{}^2} - \frac{\partial^2}{\partial x_2{}^2} - \cdots - \frac{\partial^2}{\partial x_n{}^2}$$

Now if M is a Riemannian manifold the Laplace–Beltrami operator Δ on M is invariant under all isometries of M. The examples above have a high degree of mobility, that is, a large group of isometries, so essentially only Δ is invariant. The following interesting generalization is essentially a combination of results of Harish–Chandra and Chevalley (see [31] p. 432). It expresses in a precise way how higher rank of the space, that is, lower degree of mobility, leads to more invariant operators.

Theorem 1.2. Let G/K be a symmetric space of rank l. Then the algebra of all G-invariant differential operators on G/K is a commutative algebra with l algebraically independent generators.

It will now be convenient to assume that G has finite center so K is compact. As pointed out in §3-3, this is no restriction on the symmetric space G/K. Let $L(g)$ and $R(g)$ denote left and right translations on G by the group element g and let $\boldsymbol{D}(G)$ denote the set of all differential operators on G invariant under all $L(g)$. If $X \in \mathfrak{g}$ the operator

$$\tilde{X} : F(g) \to \{(d/dt)F(g \exp tX)\}_{t=0}$$

belongs to $\boldsymbol{D}(G)$. Let $\boldsymbol{D}_K(G)$ denote the set of elements in $\boldsymbol{D}(G)$ which are invariant under all $R(k)$ ($k \in K$). For $D \in \boldsymbol{D}(G)$ we put

$$D^\natural = \int_K D^{R(k)}\, dk \tag{1}$$

where dk denotes the normalized Haar measure on K. The integral makes sense since all the operators $D^{R(k)}$ $(k \in K)$ belong to a fixed finite-dimensional vector space, so $D^\natural$ is a differential operator on G. Clearly $D^\natural \in \boldsymbol{D}_K(G)$, and we have

$$(D^\natural F)(e) = (DF)(e) \tag{2}$$

for every $F \in C^\infty(G)$ which is bi-invariant under K (that is, $F(k_1 g k_2) = F(g)$, $g \in G$, $k_1, k_2 \in K$). In fact,

$$\begin{aligned}(D^\natural F)(e) &= \int_K (D^{R(k)}F)(e)\, dk = \int_K ((DF^{R(k^{-1})})^{R(k)})(e)\, dk \\ &= \int_K (DF)(k^{-1})\, dk = \int_K (DF)^{L(k)}(e)\, dk \\ &= \int_K (DF)(e)\, dk = (DF)(e)\end{aligned}$$

Let π denote the natural projection $g \to gK$ of G onto G/K; if f is a function on G/K we put $\tilde{f} = f \circ \pi$. Then the mapping $f \to \tilde{f}$ is an isomorphism of $C^\infty(G/K)$ onto the space $C_K^\infty(G)$ of functions $F \in C^\infty(G)$ satisfying $F(gk) \equiv F(g)$. Similarly, we would like to "lift" the operators in $\boldsymbol{D}(G/K)$ to the group G. If $D \in \boldsymbol{D}_K(G)$ let $\pi(D)$ denote the operator on $C^\infty(G/K)$ determined by $(\pi(D)f)^\sim = D\tilde{f}$ $(f \in C^\infty(G/K))$. It is easy to see (*cf.* [31], p. 390) that the map $D \to \pi(D)$ maps $\boldsymbol{D}_K(G)$ onto $\boldsymbol{D}(G/K)$.

As before let $\tau(g)$ denote the diffeomorphism $hK \to ghK$ of G/K onto itself. We shall often denote the symmetric space G/K by X.

4-2 Harmonic Functions on Symmetric Spaces

In view of Prop. 1.1 it is natural to make the following definition.

Definition. A function $u \in C^\infty(G/K)$ is called *harmonic* if $Du = 0$ for all $D \in \boldsymbol{D}(G/K)$ which annihilate the constants (that is, "without constant term").

Godement made this definition in [22] (even for nonsymmetric spaces G/K), where he proved also the mean value theorem below.

Theorem 2.1. A function $u \in C^\infty(G/K)$ is harmonic if and only if

$$\int_K u(gkh \cdot o)\, dk = u(g \cdot o) \qquad \text{for all } g, h \in G \tag{1}$$

This result is most easily interpreted if rank $(G/K) = 1$. Then the orbit $K \cdot (h \cdot o)$ is a sphere and $gK \cdot (h \cdot o)$ is a sphere with center $g \cdot o$. Thus the theorem states in this case that u is harmonic if and only if the mean value of u over an arbitrary sphere is equal to the value of u in the center (*cf.* Gauss' mean value theorem for harmonic functions in $\boldsymbol{R}^n$).

PROOF. Suppose first that u is harmonic and for a fixed $g \in G$ consider the function

$$F : h \to \int_K \tilde{u}(gkh)\, dk \qquad (h \in G)$$

Let D be an operator in $\boldsymbol{D}(G)$ annihilating the constants. Then using (2) in §4-1,

$$(DF)(e) = (D^\natural F)(e) = \left\{(D^\natural)_h\left(\int_K \tilde{u}(gkh)\, dk\right)\right\}_{h=e}$$

which by the left invariance of $D^\natural$ equals

$$\int_K (D^\natural \tilde{u})(gk)\, dk = (D^\natural \tilde{u})(g)$$

(the last relation coming from the right invariance of $D^\natural \tilde{u}$ under K). However, $(D^\natural \tilde{u}) = (\pi(D^\natural)u)^\sim = 0$ since $\pi(D^\natural)$ annihilates the constants. Thus $(DF)(e) = 0$ for all $D \in \boldsymbol{D}(G)$ which annihilate the constants.

Since u satisfies the elliptic equation $\Delta u = 0$ and since Δ has analytic coefficients, it follows from a theorem of Bernstein (John [44], p. 142) that u is also analytic. Hence $\tilde{u}$ and F are also analytic so from Taylor's formula (§2-2) we can conclude that F is constant. But the relation $F(h) = F(e)$ is (1).

On the other hand, suppose (1) holds. Let $D \in \boldsymbol{D}(G/K)$ annihilate the constants. Writing (1) as

$$\int_K u^{\tau(k^{-1}g^{-1})}(x)\, dk = u(g \cdot o) \qquad g \in G,\ x \in X$$

we deduce by applying D to both sides (considered as functions of x),

$$\int_K (Du)(gk \cdot x)\, dk = 0$$

Taking $x = 0$ we conclude $Du \equiv 0$, so u is harmonic.

Now we intend to study bounded harmonic functions u on the symmetric space G/K and prove a Poisson integral representation formula due to Furstenberg [19].* Let Q_u denote the set of all functions $\psi \in L^\infty(G)$ (the space of bounded measurable functions on G) such that the sup norm $\|\psi\|_\infty = \sup_{h \in G} |\psi(h)|$ satisfies $\|\psi\|_\infty \leqslant \|u\|_\infty$ and such that

$$u(g \cdot o) = \int_K \psi(gkh)\, dk \qquad \text{for all } g, h \in G$$

According to Godement's theorem $\tilde{u} \in Q_u$, so Q_u is not empty. In addition

* modify as in notes

it is a convex set and closed in the weak* topology of $L^\infty(G)$ (the weakest topology for which all the maps $\psi \to \int f(g)\psi(g)\,dg$ of $L^\infty(G)$ into $\boldsymbol{C}$ are continuous, f being an integrable function on G and dg being a Haar measure). Since the unit ball in $L^\infty(G)$ is compact in the weak* topology (see, for example, [50]) it follows that Q_u is compact. Now if $\psi \in Q_u$ we have $\psi^{R(g)} \in Q_u$ for all $g \in G$ so G acts as a transformation group of Q_u by right translations. We would like to find a fixed point under the sugbroup MAN, which then would give us a function on the boundary G/MAN.

Definition. A group has the *fixed point property* if whenever it acts continuously on a locally convex topological vector space by linear transformations leaving a compact convex set $Q \neq \varnothing$ invariant it has a fixed point in the set.

Lemma 2.2. Connected solvable Lie groups have the fixed point property (*cf.* [6], p. 115).

PROOF. Let V be a locally convex topological vector space and G any Abelian group of linear transformations of V. For each $g \in G$ let $g_n = (1/n)(I + g + \cdots + g^{n-1})$; let $\tilde{G}$ denote the set of all products $g_{n_1} \dots g_{n_k}$ $(n_i \in \boldsymbol{Z}^+, g \in G)$. All elements of $\tilde{G}$ commute. Let $Q \subset V$ be a nonempty compact convex subset of V. By convexity, $hQ \subset Q$ for $h \in \tilde{G}$. Let $h_1, \dots, h_r \in \tilde{G}$. Then for each i, $1 \leqslant i \leqslant r$,

$$h_1 \dots h_r Q = h_i h_1 \dots h_{i-1} h_{i+1} \dots h_r Q \subset h_i Q$$

whence

$$h_1 \dots h_r Q \subset \bigcap_{i=1}^{r} h_i Q$$

so this intersection is $\neq \varnothing$. By compactness of Q (expressed by the finite intersection property), we have

$$\bigcap_{h \in \tilde{G}} hQ \neq \varnothing$$

Let x an element in this intersection and let $g \in G$. Then $x \in g_n Q$, so for a suitable element $y \in Q$,

$$x = \frac{1}{n}(y + gy + \cdots + g^{n-1}y)$$

so

$$gx - x = \frac{1}{n}(g^n y - y) \subset \frac{1}{n}(Q + (-Q))$$

for each n. Using again the compactness of Q we conclude $g \cdot x = x$.

Now assume G is a connected solvable Lie group of linear transformations of V. Let $\mathfrak{g}$ be its Lie algebra and let

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = \{0\} \qquad \mathfrak{g}_{m-1} \neq \{0\}$$

be the sequence of derived algebras, $\mathfrak{g}_i = \mathfrak{D}^i\mathfrak{g}$. Let $G = G_0 \supset G_1 \supset \cdots \supset G_m = \{e\}$ be the corresponding series of analytic subgroups of G. Suppose now the lemma holds for all connected solvable Lie groups whose series (as defined above) has length $< m$. Let A denote the set of points in Q fixed under all $g \in G_1$. By the induction assumption, A is $\neq \varnothing$ and, of course, A is convex and compact. Let $\gamma \in G$. If $g \in G_1$ then $\gamma g \gamma^{-1} \in G_1$, so if $x \in A$, $\gamma g \gamma^{-1} x = x$ so $g\gamma^{-1}x = \gamma^{-1}x$. Thus $\gamma^{-1}x$ is fixed by all elements in G_1; being in Q, $\gamma^{-1}x$ belongs to A. Thus G maps A into itself. The closed subspace V_A of V generated by A is locally convex and since G_1 acts trivially on it, G acts on V_A as an Abelian group. By the first part of the proof there exists a $v \in A$ fixed under all $g \in G$. Q.E.D.

Lemma 2.3. The group MAN has the fixed point property.

PROOF. Let MAN act on a locally convex space V and let $Q \subset V$ be a compact convex subset $\neq \varnothing$ invariant under MAN. Since AN is solvable and connected there exists a point $q \in Q$ fixed under AN. If dm denotes the normalized Haar measure on the compact group M the integral

$$\int_M m \cdot q \, dm$$

(defined by means of approximating sums) represents, because of the compactness and convexity, a point q^* in Q. Since $m(AN)m^{-1} \subset AN$ we have for $s \in AN$

$$sq^* = \int_M sm \cdot q \, dm = \int_M m(m^{-1}sm)q \, dm = \int_M m \cdot q \, dm$$

so q^* is fixed under MAN.

We recall now that the boundary B of the symmetric space is given by the coset space representations $B = K/M$, $B = G/MAN$. The latter shows that G acts on B; this action will be denoted $(g, b) \to g(b)$ in order to distinguish it from the action $(g, x) \to g \cdot x$ of G on $X = G/K$, which we have already used. Let db denote the unique K-invariant measure on B satisfying

$$\int_B db = 1$$

Theorem 2.4. If u is a bounded harmonic function on X then there exists a bounded measurable function $\hat{u}$ on B such that

$$u(g \cdot o) = \int_B \hat{u}(g(b)) \, db \tag{2}$$

On the other hand, if $\hat{u}$ is a bounded measurable function on B then u as defined by (2) is a bounded harmonic function on X.

PROOF. As shown above (Lemma 2.3) the set Q_u has a fixed point under MAN, say u_1. Define $\hat{u}$ on G/MAN by $\hat{u}(gMAN) = u_1(g)$. Then by the definition of Q_u, we have

$$u(g \cdot o) = \int_K \hat{u}(gkhMAN)\, dk$$

no! almost every where formula *

Take $h = e$ and recall that $gkMAN$ is $g(b)$ if $b = kM$. Then (2) follows because if F is any continuous function on B,

$$\int_B F(b)\, db = \int_K F(kM)\, dk$$

On the other hand, if $\hat{u}$ is a function in $L^\infty(B)$, define u by (2). Then

$$u(gkh \cdot o) = \int_B \hat{u}(gkh(b))\, db \tag{3}$$

Now let $b = k'MAN$; then $gkh(b) = gkhk'MAN = gkk_1MAN$ if $hk' = k_1a_1n_1$ (Theorem 5.1, Ch. 3). Hence,

$$\begin{aligned}\int_K u(gkh \cdot o)\, dk &= \int_K \left(\int_K \hat{u}(gkhk'MAN)\, dk' \right) dk \\ &= \int_K \left(\int_K \hat{u}(gkhk'MAN)\, dk \right) dk' = \int_K \left(\int_K \hat{u}(gkk_1MAN)\, dk \right) dk' \\ &= \int_K \left(\int_K \hat{u}(gkMAN)\, dk \right) dk' = \int_K \hat{u}(gkMAN)\, dk = u(g \cdot o).\end{aligned}$$

By Theorem 2.1, u is harmonic, so the theorem is proved.

Now define the *Poisson kernel* $P(x, b)$ on the product space $X \times B$ by the Jacobian

$$P(g \cdot o, b) = \frac{d(g^{-1}(b))}{db} \tag{4}$$

As we saw in Ch. 1. (11) §1-3 this does indeed give the classical Poisson kernel in the case when G/K is the non-Euclidean disk. We shall give the general formula for (4) later. But at any rate formula (2) can be written

$$u(x) = \int_B P(x, b)\hat{u}(b)\, db \tag{5}$$

giving a Poisson integral representation of an arbitrary bounded harmonic function on X. Furstenberg showed in [19], p. 366, that in the weak topology of measures the values of $\hat{u}$ can be regarded as boundary values of u. We shall now see that this is also the case, when we approach the boundary in a more geometric fashion.

Let $\bar{\mathfrak{n}}$ denote the subalgebra of $\mathfrak{g}$ given by

$$\bar{\mathfrak{n}} = \sum_{\alpha<0} \mathfrak{g}_\alpha$$

where the $\mathfrak{g}_\alpha$ are given by (2) §3-5. Let $\bar{N}$ denote the corresponding analytic subgroup of G. As an immediate consequence of the Bruhat lemma (see Harish–Chandra [26]) we have that the subset $\bar{N}MAN \subset G$ is an open subset whose complement has lower dimension. As a result the mapping $T: \bar{n} \to k(\bar{n})M$ maps $\bar{N}$ onto a subset of K/M whose complement has lower dimension [Here $k(\bar{n})$ is the K-component of $\bar{n}$ according to the decomposition $G = KAN$.] One can also prove that the mapping T is one-to-one.

Lemma 2.5. For a certain positive integrable function ψ on $\bar{N}$, we have

$$\int_{K/M} f(kM)\, dk_M = \int_{\bar{N}} f(k(\bar{n})M)\psi(\bar{n})\, d\bar{n} \qquad f \in C^\infty(K/M)$$

Here dk_M is the normalized K-invariant measure on K/M and $d\bar{n}$ is a Haar measure on $\bar{N}$.

PROOF. Let $dk_M \circ T$ denote the measure on $\bar{N}$ given by

$$(dk_M \circ T)(C) = \int_{T(C)} dk_M \qquad C \text{ compact in } \bar{N}$$

Let $\psi(\bar{n})$ denote the Radon–Nikodym derivative (see, for example, [24], p. 128). Then the lemma follows at once from the properties of T given above.

REMARK. This lemma is given in Harish–Chandra [27], p. 287, with an explicit formula for $\psi(\bar{n})$ which will be derived later (Proposition 2.10).

The mapping T is particularly useful for studying the action of A on the boundary. In fact, if $a \in A$, $\bar{n} \in \bar{N}$ we have

$$a(k(\bar{n})M) = ak(\bar{n})MAN = k(a\bar{n})MAN = k(a\bar{n}a^{-1})MAN$$

that is, $\bar{n} = k(\bar{n})\, a_1 n_1$

$$a(k(\bar{n})M) = k(\bar{n}^a)M \tag{6}$$

the superscript denoting conjugation.

Theorem 2.6. Let F be a continuous function on B and u its Poisson integral

$$u(x) = \int_B P(x, b)F(b)\, db \qquad x \in X$$

Then u has boundary values given by F, that is,

$$\lim_{t \to \infty} u(k \exp tH \cdot o) = F(kM) \tag{7}$$

for each $k \in K$ and each $H \in \mathfrak{a}^+$.

PROOF. We may assume $k = e$. We must prove that if $a_t = \exp tH$ then as $t \to \infty$

$$\int_{K/M} F(a_t(kM))\, dk_M \to F(eM)$$

But by Lemma 2.5 and (6) the integral on the left equals

$$\int_{\bar N} F(a_t(k(\bar n)M))\psi(\bar n)\, d\bar n = \int_{\bar N} F(k(\bar n^{a_t})M)\psi(\bar n)\, d\bar n \tag{8}$$

Now

$$\bar n = \exp\left(\sum_{\alpha < 0} X_\alpha\right)$$

where $X_\alpha \in \mathfrak{g}_\alpha$ and by (7) and (9) in §3-1,

$$\bar n^{\exp H} = \exp H \exp\left(\sum_\alpha X_\alpha\right) \exp(-H) = \exp\left(\mathrm{Ad}(\exp H)\left(\sum_\alpha X_\alpha\right)\right)$$

$$= \exp\left(e^{\mathrm{ad}\, H}\left(\sum_\alpha X_\alpha\right)\right) = \exp\left(\sum_\alpha e^{\alpha(H)} X_\alpha\right)$$

But $\alpha(H) < 0$ whenever $\alpha < 0$ so we see that for each $\bar n \in N$, $\bar n^{\exp tH} \to e$. It follows (using the dominated convergence theorem) that the right-hand side of (8) has a limit

$$\int_{\bar N} F(eM)\psi(\bar n)\, d\bar n = F(eM)$$

as $t \to \infty$. This proves the theorem.

The result above is not new (*cf.* Karpelevič [46], Theorem 18.3.2 and also Moore [53], p. 204). Next we prove that the boundary function $\hat u$ in Theorem 2.4 is unique.

Corollary 2.7. Let $F \in L^\infty(B)$ and

$$u(x) = \int_B P(x, b)F(b)\, db \qquad (x \in X)$$

Then if $u \equiv 0$, we have also $F \equiv 0$.

In fact, let $\phi \in L^1(G)$ be continuous and consider the function

$$F_1(b) = \int_G \phi(g)F(g(b))\, dg \qquad b \in B$$

The function F_1 is continuous (as a convolution of a continuous integrable function with a bounded function) and its Poisson integral u_1 is given by

$$u_1(h \cdot o) = \int_B P(h \cdot o, b) F_1(b)\, db = \int_B F_1(h(b))\, db$$

$$= \int_B \left(\int_G \phi(g) F(gh(b))\, dg \right) db = \int_G \phi(g) u(gh \cdot o)\, dg$$

Now if $u \equiv 0$ we have $u_1 \equiv 0$ so by Theorem 2.6, $F_1 \equiv 0$. But since ϕ is arbitrary, we conclude $F \equiv 0$.

The Topology of $X \cup B$

It is possible to define a topology on the union $X \cup B$ such that the limit relation (7) is convergence in this topology. A vector $Y \in \mathfrak{p}$ is called regular if its centralizer Z_Y in $\mathfrak{p}$ is Abelian. A point $x = (\exp Y)K$ in X is called regular if Y is regular. Now a regular vector $Y \in \mathfrak{p}$ belongs to a unique Weyl chamber b_Y in the maximal Abelian subspace Z_Y. We say that a sequence of points $x_1, x_2, \ldots$ in X converges to a boundary point b if

(i) Each $x_n = (\exp Y_n)K$ (where $Y_n \in \mathfrak{p}$) is regular
(ii) The Weyl chambers b_{Y_n} converge to b (in the topology of B)
(iii) The distance from Y_n to the boundary of b_{Y_n} in Z_{Y_n} tends to ∞

It is not hard to verify that this convergence concept (together with the usual convergence definition on X itself) defines a topology on the union $X \cup B$.

We shall now prove some measure–theoretic results due to Harish–Chandra ([25], p. 239, [27], p. 294) and give an explicit formula for the Poisson kernel $P(x, b)$ as a consequence (*cf.* also Schiffmann [56]).

Lemma 2.8. Let dk, da, and dn be left invariant Haar measures on the groups K, A, and N, respectively. Then for a suitable normalization of the Haar measure dg of G, we have

$$\int_G f(g)\, dg = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)}\, dk\, da\, dn$$

for all $f \in C_c^\infty(G)$. This ρ is defined in §3-5 and log denotes the inverse of the mapping $\exp : \mathfrak{a} \to A$.

PROOF. Since the mapping $(k, a, n) \to kan$ is a diffeomorphism of $K \times A \times N$ onto G (§3-5) there exists a function $D(k, a, n)$ on $K \times A \times N$ such that

$$\int_G f(g)\, dg = \int_{K \times A \times N} f(kan) D(k, a, n)\, dk\, da\, dn \qquad (9)$$

for all $f \in C_c^\infty(G)$. The groups G, K, A, N are all unimodular, that is, the left invariant Haar measures are all right invariant. Thus the left-hand side of (9) does not change if we replace $f(g)$ by $f(k_1 g n_1)$, $k_1 \in K$, $n_1 \in N$. It follows that $D(k_1^{-1}k, a, nn_1^{-1}) \equiv D(k, a, n)$ so $D(k, a, n)$ is a function $\delta(a)$ of a alone. Let $a_1 \in A$. Then

$$\begin{aligned}\int_G f(g)\,dg = \int_G f(ga_1)\,dg &= \int_{KAN} f(kana_1)\delta(a)\,dk\,da\,dn \\ &= \int_{KAN} f(kaa_1(a_1^{-1}na_1))\delta(a)\,dk\,da\,dn \\ &= \int_{KAN} f(ka(a_1^{-1}na_1))\delta(aa_1^{-1})\,dk\,da\,dn \\ &= \int_{KAN} f(kan)\delta(aa_1^{-1})J(a_1, n)\,dk\,da\,dn\end{aligned}$$

where $J(a_1, b)$ denotes the Jacobian determinant of the mapping $n \to a_1 n a_1^{-1}$ of N onto N. The computation in the proof of Theorem 2.6 shows that

$$J(a_1, n) = e^{2\rho(\log a_1)}$$

Thus

$$\delta(a) = \delta(aa_1^{-1})e^{2\rho(\log a_1)}$$

and the lemma follows.

Given $g \in G$, let $k(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$ be determined by $g = k(g) \exp H(g) n(g)$.

Corollary 2.9. The Poisson kernel on $G/K \times K/M$ is given by

$$P(gK, kM) = e^{-2\rho(H(g^{-1}k))}$$

PROOF. The mapping $k \to k(gk)$ is a diffeomorphism of K onto itself. Now fix $h \in G$. Then for $f \in C_c^\infty(G)$,

$$\int f(kan)e^{2\rho(\log a)}\,dk\,da\,dn = \int f(g)\,dg = \int f(hg)\,dg \qquad (10)$$

Now if $g = kan$, then

$$hg = hkan = k(hk)\exp H(hk)n(hk)an = k(hk)\exp H(hk)a(a^{-1}n(hk)an)$$

which we write as $k_1 a_1 n_1$. Then our integral on the right-hand side of (10) equals

$$\int f(k_1 a_1 n_1)e^{2\rho(\log a)}\,dk\,da\,dn. \qquad (11)$$

But the map $a \to \exp H(hk)a$ preserves the measure da and the map $n \to (a^{-1}n(hk)a)n$ preserves the measure dn. The integral (11) therefore equals

$$\int f(k(hk)a_1 n)e^{2\rho(\log a_1)}e^{-2\rho(H(hk))}dk\, da_1\, dn$$

so comparing with the left-hand side of (10), we find

$$\int_K F(k)\, dk = \int_K F(k(hk))e^{-2\rho(H(hk))}\, dk \qquad (F \in C^\infty(K)) \tag{12}$$

In particular, let us use this for $F(k) = \phi(kM)$, ϕ being an arbitrary C^∞ function on the boundary. Since

$$\int_K F(k)\, dk = \int_{K/M} \phi(kM)\, dk_M$$

$$\int_K F(k(hk))e^{-2\rho(H(hk))}\, dk = \int_{K/M} \phi(k(hk)M)e^{-2\rho(H(hk))}\, dk_M$$

and since $k(hk)M = h(kM)$ the corollary follows from (12).

As another application let us compute the function $\bar{n} \to \psi(\bar{n})$ in Lemma 2.5.

Proposition 2.10. For a suitable Haar measure $d\bar{n}$ on $\bar{N}$ we have

$$\int_{K/M} f(k_M)\, dk_M = \int_{\bar{N}} f(k(\bar{n})M)e^{-2\rho(H(\bar{n}))}\, d\bar{n} \qquad \in C^\infty(K/M).$$

PROOF. Fix an element $\bar{n}_o \in \bar{N}$ and consider the function $f^{\bar{n}_o} : kM \to f(\bar{n}_0(kM))$ on K/M. Since $\bar{n}_0(k(\bar{n})M) = k(\bar{n}_0\bar{n})M$ we conclude from Lemma 2.5,

$$\int_{K/M} f(\bar{n}_o(kM))\, dk_M = \int_{\bar{N}} f(k(\bar{n}_o\bar{n})M)\psi(\bar{n})\, d\bar{n} = \int_{\bar{N}} f(k(\bar{n})M)\psi(\bar{n}_o^{-1}\bar{n})\, d\bar{n},$$

and from the definition of the Poisson kernel,

$$\int_{K/M} f(\bar{n}_o(kM))\, dk_M = \int_{K/M} f(kM)P(\bar{n}_o \cdot o, kM)\, dk_M$$

$$= \int_{\bar{N}} f(k(\bar{n})M)P(\bar{n}_o \cdot o, k(\bar{n})M)\psi(\bar{n})\, d\bar{n}$$

Comparing the formulas we conclude,

$$\psi(\bar{n}_o^{-1}\bar{n}) = P(\bar{n}_o \cdot o, k(\bar{n})M)\psi(\bar{n})$$

so putting $\bar{n} = e$ the proposition follows from Cor. 2.9.

To conclude this section we state two theorems without proof. Let Δ denote the Laplace–Beltrami operator on X.

Theorem 2.11. Let u be a bounded solution of the equation $\Delta u = 0$ on X. Then u is harmonic.

A probabilistic proof of this theorem is given in Furstenberg [19] (*cf.* also Berezin [2] and Karpelevič [46]).

Using this result, A. Korányi and the author ([38]) have proved the following theorem which generalizes the classical Fatou theorem for the unit disk.

Theorem 2.12. Let u be a bounded solution of the equation $\Delta u = 0$ on X. Then for almost all geodesics $t \to \gamma(t)$ in X starting at the origin o the limit

$$\lim_{t \to \infty} u(\gamma(t))$$

exists.

4-3 Spherical Functions on Symmetric Spaces

Let $X = G/K$ be a symmetric space of the noncompact type as in the last section. A *spherical function* on G/K is by definition a K-invariant eigenfunction ϕ of all the operators $D \in \mathbf{D}(G/K)$ satisfying $\phi(o) = 1$. According to a theorem of Harish–Chandra the spherical functions are precisely the functions on G/K given by

$$\phi_\lambda(gK) = \int_K e^{(i\lambda - \rho)(H(gk))}\, dk \tag{1}$$

where λ is an arbitrary complex-valued linear function on $\mathfrak{a}$.

In the simplest case when X is the non-Euclidean disk D from Ch. 1 the spherical functions are the Legendre functions P_ν and their integral formula

$$P_\nu(\cosh r) = \frac{1}{2\pi}\int_0^{2\pi} (\cosh r + \sinh r \cos \theta)^\nu \, d\theta$$

is the simplest example of (1) (see, for example, [31], p. 406).

We shall now state Harish–Chandra's result ([27], p. 612, [28], p. 48) which describes how an arbitrary K-invariant function $f \in C_c^\infty(X)$ can be decomposed into spherical functions. In view of Theorem 7.1, Ch. 3 such a function f is completely determined by the values $f(a \cdot o)$, $(a \in A^+)$ and we define the transform (spherical Fourier transform) $\tilde{f}(\lambda)$ by

$$\tilde{f}(\lambda) = \int_{A^+} f(a \cdot o)\overline{\phi_\lambda(a)} D(a)\, da \qquad (\lambda \in \mathfrak{a}^*) \tag{2}$$

Here $\mathfrak{a}^*$ is the dual of the vector space $\mathfrak{a}$ and the function $D(a)$ is the density for the volume element dx on X in polar coordinates (Theorem 7.1, Ch. 3). More precisely, if $x = ka \cdot o$ then $dx = D(a)\, dk_M\, da$.

The problem is now to invert formula (2). Motivated by the spectral theory of singular ordinary differential operators, Harish–Chandra expands the function ϕ_λ (exp H) in a series of the form

$$\phi_\lambda(\exp H) = \sum_\mu \left(\sum_{s \in W} \gamma_\mu(s\lambda) e^{is\lambda(H)} \right) e^{-\mu(H)} \qquad (H \in \mathfrak{a}^+) \tag{3}$$

Here μ runs through certain subset of $\mathfrak{a}^*$, the γ_μ are certain functions on $\mathfrak{a}^*$ and W denotes the Weyl group (which acts on $\mathfrak{a}^*$ by duality). The dominating term in this series has the form

$$e^{-\rho(H)} \sum_{s \in W} c(s\lambda) e^{is\lambda(H)} \tag{4}$$

where $1/c(\lambda)$ is a certain analytic function on $\mathfrak{a}^*$. From (1) above and Prop. 2.10, Harish–Chandra derives the integral formula

$$c(\lambda) = \int_{\bar{N}} e^{(-i\lambda-\rho)(H(\bar{n}))}\, d\bar{n} \tag{5}$$

whenever the integral converges absolutely.

Theorem 3.1. The inverse of the spherical Fourier transform $f \to \tilde{f}$ in (2) is given by

$$f(a \cdot o) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda) \phi_\lambda(a) \, |c(\lambda)|^{-2} \, d\lambda \tag{6}$$

where $d\lambda$ is a constant multiple of the Euclidean measure on $\mathfrak{a}^*$.

The simplest case of this theorem is the inversion formula for the Mehler transform stated in Ch. 1.

We shall now attempt to describe some of the main steps in the proof of this theorem. For a restricted root $\alpha > 0$ let $m_\alpha = \dim(\mathfrak{g}_\alpha)$, where $\mathfrak{g}_\alpha$ is as defined in §3.5. Let (,) denote the inner product on $\mathfrak{a}^*$ induced by the Killing form B of $\mathfrak{g}$, restricted to $\mathfrak{a}$.

(i) The function $c(\lambda)$ is given by $c(\lambda) = I(i\lambda)/I(\rho)$, where

$$I(\nu) = \prod_{\alpha > 0} B\left(\frac{1}{2} m_\alpha, \frac{1}{4} m_{\alpha/2} + \frac{(\nu, \alpha)}{(\alpha, \alpha)}\right) \qquad (\nu \in \mathfrak{a}^*) \tag{7}$$

and B denotes the Beta function,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

Let us first consider the case rank $(G/K) = 1$. Then $\phi_\lambda(a)$ is a function of one real variable and is characterized by a single second-order ordinary differential equation (which comes from ϕ_λ being an eigenfunction of Δ). One finds then that ϕ_λ is given by a hypergeometric function. If one now compares the series expansion for the hypergeometric function with the expansion (3), formula (7) follows. For the details see Harish–Chandra [27], p. 301.

Bhanu–Murthy [4, 5] extended (7) to several other special cases whereupon Gindikin and Karpelevič [21] proved (7) in general along the following lines. Let $\alpha > 0$ be a restricted root which is not a positive integral multiple of other restricted roots. Let $\mathfrak{g}^\alpha$ denote the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Then $\mathfrak{g}^\alpha$ is semisimple and has a Cartan decomposition

$$\mathfrak{g}^\alpha = \mathfrak{k}^\alpha + \mathfrak{p}^\alpha \qquad \mathfrak{k}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{k} \qquad \mathfrak{p}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{p} \tag{8}$$

Let G^α and K^α denote the analytic subgroups of G corresponding to $\mathfrak{g}^\alpha$ and $\mathfrak{k}^\alpha$, respectively. The symmetric space G^α/K^α (which can be identified with the orbit $G^\alpha \cdot o$ and is a totally geodesic submanifold of G/K) has rank one. In fact if $\mathfrak{a}^\alpha$ denotes the orthogonal complement in $\mathfrak{a}$ of the hyperplane $\alpha(H) = 0$ then $\mathfrak{a}^\alpha$ is maximal Abelian in $\mathfrak{p}^\alpha$. Now G^α has an Iwasawa decomposition $G^\alpha = K^\alpha A^\alpha N^\alpha$, and the $\boldsymbol{c}$-function for G^α/K^α, denoted $\boldsymbol{c}^\alpha$, is given by an integral of the form (5) over the group $\overline{N}^\alpha$. Now Gindikin and Karpelevič prove that the product of these integrals (for the various α) is equal to the integral (5) over $\overline{N}$; more precisely,

$$\boldsymbol{c}(\lambda) = \prod_{\alpha} \boldsymbol{c}^\alpha(\lambda^\alpha) \tag{9}$$

where λ^α denotes the restriction of λ to $\mathfrak{a}^\alpha$ and α runs through the restricted roots specified above. Now (7) follows from the rank-one case.

Now let $\mathscr{S}(\mathfrak{a}^*)$ denote the set of rapidly decreasing functions on $\mathfrak{a}^*$ in the sense of Schwartz [57] and let $\mathscr{I}(\mathfrak{a}^*)$ denote the set of W-invariant functions in $\mathscr{S}(\mathfrak{a}^*)$. (Here W is the Weyl group.)

(ii) Let $\mu \in \mathfrak{a}^*$. Then the mapping

$$S_\mu : b \to \int_{A^+} \overline{\phi_\mu(a)} \left(\int_{\mathfrak{a}^*} b(\lambda)\phi_\lambda(a)\, |\boldsymbol{c}(\lambda)|^{-2}\, d\lambda \right) D(a)\, da$$

$[b \in \mathscr{I}(\mathfrak{a}^*)]$ is a tempered distribution on $\mathfrak{a}^*$.

It is easy to see from (7) that the integral over λ is absolutely convergent. On the other hand to show that the integral with respect to a is absolutely convergent and makes S_μ a distribution requires very detailed study of the behavior of $\phi_\lambda(a)$ for large a (see Harish–Chandra [27], p. 588).

Eventually one wants to prove that for a suitable normalization of $d\lambda$, $S_\mu(b) = b(\mu)$. But first one proves

(iii) If p is a Weyl group invariant polynomial on $\mathfrak{a}^*$ then

$$pS_\mu = p(\mu)S_\mu$$

To see this select a differential operator $D \in \boldsymbol{D}(G/K)$ such that $D\phi_\lambda = p(\lambda)\phi_\lambda$ (see [27], p. 591 or [31], p. 432). Then

$$pS_\mu(b) = S_\mu(pb) = \int_X \overline{\phi_\mu(x)} \left(\int_{\mathfrak{a}^*} p(\lambda) b(\lambda) \phi_\lambda(x) \, |c(\lambda)|^{-2} \, d\lambda \right) dx$$

Here we replace $p(\lambda)\phi_\lambda(x)$ by $(D\phi_\lambda)(x)$ and carry D over on $\bar{\phi}_\mu$ by replacing it with its adjoint; the result is $p(\mu)S_\mu(b)$ as desired.

As a fairly easy consequence of (iii) we obtain (*cf.* [27], p. 591).

(iv) There exists a function γ on $\mathfrak{a}^*$ such that

$$S_\mu(b) = \gamma(\mu)b(\mu) \qquad b \in \mathscr{I}(\mathfrak{a}^*).$$

Now we must prove that γ is a constant. Consider for f as in (2) the function F_f defined by

$$F_f(a) = e^{\rho(\log a)} \int_N f(\bar{n}a \cdot o) \, d\bar{n} \qquad a \in A$$

Then we have as a simple consequence of (1) and Lemma 2.8 that

$$\tilde{f}(\lambda) = \int_X f(x) \overline{\phi_\lambda(x)} \, dx = \int_A F_f(a) e^{-i\lambda(\log a)} \, da \tag{10}$$

If $b \in \mathscr{I}(\mathfrak{a}^*)$ consider the function

$$\phi_b(x) = \int_{\mathfrak{a}^*} b(\lambda)\phi_\lambda(x) \, |c(\lambda)|^{-2} \, d\lambda$$

The integral for F_{ϕ_b} can be shown to converge and by the inversion formula for the Fourier transform on A and $\mathfrak{a}^*$ we obtain

$$\begin{aligned} F_{\phi_b}(a) &= e^{\rho(\log a)} \int_N \phi_b(\bar{n}a \cdot o) \, d\bar{n} = \int_{\mathfrak{a}^*} \tilde{\phi}_b(\lambda) e^{i\lambda(\log a)} \, d\lambda \\ &= \int_{\mathfrak{a}^*} S_\lambda(b) e^{i\lambda(\log a)} \, d\lambda = \int_{\mathfrak{a}^*} \gamma(\lambda) b(\lambda) e^{i\lambda(\log a)} \, d\lambda \\ &= \frac{1}{w} \int_{\mathfrak{a}^*} \gamma(\lambda) b(\lambda) \sum_{s \in W} e^{is\lambda(\log a)} \, d\lambda \end{aligned}$$

where w denotes the order of W. The relation $\gamma \equiv w$ would therefore result from the following statement.

(v) The relation

$$|c(\lambda)|^{-2} e^{\rho(\log a)} \int_N \phi_\lambda(\bar{n}a \cdot o) \, d\bar{n} = \sum_{s \in W} e^{is\lambda(\log a)} \tag{11}$$

holds in the weak sense in λ, that is, it gives the right result when integrated against any $b \in \mathscr{I}(\mathfrak{a}^*)$.

This is carried out by means of a beautiful analysis in §15, p. 597, of Harish–Chandra [27]. Here we have to settle for a vague plausibility argument. Writing $\bar{n}a = k_1 a' k_2$ $(k_1, k_2 \in K,\ a' \in \overline{A^+})$ we have (*loc. cit.* p. 604)

$$\log a' \sim \log a + H(\bar{n})$$

as $a \to \infty$ in A^+. Since (4) is the dominating term in the expansion for $\phi_\lambda(\exp H)$ let us replace $\phi_\lambda(\bar{n}a \cdot o) = \phi_\lambda(a' \cdot o)$ by

$$e^{-\rho(\log a + H(\bar{n}))} \sum_{s \in W} c(s\lambda) e^{is\lambda(\log a + H(\bar{n}))}$$

When this expression is integrated over $\bar{N}$ we obtain from (5) the expression

$$e^{-\rho(\log a)} \sum_{s \in W} c(s\lambda) c(-s\lambda) e^{is\lambda(\log a)}$$

which equals $e^{-\rho(\log a)} |c(\lambda)|^2 \sum_{s \in W} e^{is\lambda(\log a)}$ in accordance with (11).

In order to deduce Theorem 3.1 from the relation $S_\mu(b) = (\text{const}) b(\mu)$ $(b \in \mathscr{I}(\mathfrak{a}^*))$ we still have to prove the following statement.

(vi) Each K-invariant function $f \in C_c^\infty(X)$ can be written in the form

$$f(x) = \int_{\mathfrak{a}^*} b(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2}\, d\lambda \qquad b \in \mathscr{I}(\mathfrak{a}^*)$$

This was stated as a conjecture in Harish–Chandra [27], p. 612, and was finally proved by him in [28], p. 48. Since this proof involves so much work on the general Plancherel formula for G (in particular, the discrete series) it would not be feasible to describe it here. Instead let me outline a different effort [37] at proving (vi).

Let F be a W-invariant function in $C_c^\infty(A)$ and F^* its Fourier transform

$$F^*(\lambda) = \int_A F(a) e^{-i\lambda(\log a)}\, da$$

Writing the expansion (3) as

$$\phi_\lambda(\exp H) = \sum_\mu \psi_\mu(\lambda, H) \qquad (H \in \mathfrak{a}^+) \tag{12}$$

we assume that the term-by-term integration

$$\int_{\mathfrak{a}^*} F^*(\lambda) \phi_\lambda(\exp H) |c(\lambda)|^{-2}\, d\lambda = \sum_\mu \int_{\mathfrak{a}^*} F^*(\lambda) \psi_\mu(\lambda, H) |c(\lambda)|^{-2}\, d\lambda \tag{13}$$

is permissible. Then we have (*loc. cit.* p. 302).

(vii) For $H \in \mathfrak{a}$ let $|H| = B(H, H)^{1/2}$. Suppose $R > 0$ such that $F(\exp H) = 0$ for $|H| > R$. Then

$$\int_{\mathfrak{a}^*} F^*(\lambda) \psi_\mu(\lambda, H) |c(\lambda)|^{-2}\, d\lambda = 0 \qquad \text{for } |H| > R \tag{14}$$

This is proved by translating the integration into the complexification $\mathfrak{a}^* + i\mathfrak{a}^*$ by use of Cauchy's theorem. Because of the formula (7) the function $c(\lambda)^{-1}$ can be extended to a function on $\mathfrak{a}^* + i\mathfrak{a}^*$ with singularities, whose location can be determined. The functions $\psi_\mu(\lambda, H)$ are determined by certain recursion formulas which result from ϕ_λ being an eigenfunction of each $D \in \boldsymbol{D}(G/K)$. It is therefore possible to describe the sets of singularities of the functions $\psi_\mu(\lambda, H)$ and the integration in $\mathfrak{a}^*$ can by Cauchy's theorem be translated away from these sets. This leads to estimates of the integral, which prove (14).

In order to prove (vi) let $f \in C_c^\infty(X)$ be K-invariant and let us use (14) on the function $F(a) = F_f(a)$, $(a \in A)$. We put

$$g(x) = \int_{\mathfrak{a}^*} F^*(\lambda)\phi_\lambda(x)\,|c(\lambda)|^{-2}\,d\lambda \tag{15}$$

and by (13) and (14) we have $g \in C_c^\infty(X)$ and K-invariant. On the other hand, we have by (10) and the result $S_\mu(b) = b(\mu)$ (with $d\lambda$ suitably normalized),

$$\tilde{g}(\lambda) = F_g{}^*(\lambda) = F^*(\lambda) \tag{16}$$

The Euclidean Fourier transform $F \to F^*$ is one-to-one so the last relation implies

$$F_g(a) = F(a) = F_f(a)$$

Thus, in view of (10), the function $h = f - g$ is a K-invariant function in $C_c^\infty(X)$ satisfying

$$\int_X h(x)\phi_\lambda(x)\,dx = 0$$

for all complex-valued linear forms λ on $\mathfrak{a}^*$. It is well-known (see, for example, [31], p. 409, 453) that this implies $h = 0$, so

$$f(x) = \int_{\mathfrak{a}^*} F^*(\lambda)\phi_\lambda(x)\,|c(\lambda)|^{-2}\,d\lambda$$

which gives (vi).

What is lacking in this proof of (vi) is a justification of the term-by-term integration (13). In the quoted paper this justification is given for the case rank $(G/K) = 1$; in this case the proof also gives a Paley–Wiener type of theorem for the transform $f \to \tilde{f}$, that is, an intrinsic characterization of the functions $\tilde{f}(\lambda)$ as f runs through the K-invariant functions in $C_c^\infty(X)$.

We conclude this section with a simple remark on the formulas

$$\tilde{f}(\lambda) = \int_{A^+} f(a \cdot o)\overline{\phi_\lambda(a)}D(a)\,da$$

$$f(a \cdot o) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\phi_\lambda(a)\delta(\lambda)\,d\lambda \qquad \delta(\lambda) = |c(\lambda)|^{-2}$$

In analogy with the product formula (9)

$$\delta(\lambda) = \prod_{\alpha} \delta_\alpha(\lambda^\alpha) \tag{17}$$

one can prove (and this is an elementary result) that

$$D(\exp H) = \prod_{\alpha} D_\alpha(\exp H^\alpha) \tag{18}$$

where D_α is the D function for the space G^α/K^α, and H^α is the projection of H on $\mathfrak{a}^\alpha$. It seems conceivable that a fuller understanding of the reason for the product formulas (17) and (18) might lead to a reduction of Theorem 3.1 to the rank-one case.

4-4 Fourier Transform on Symmetric Spaces

As before let X denote the symmetric space G/K. Now we would like to define a Fourier transform for arbitrary functions $f \in C_c^\infty(X)$, not just for the K-invariant ones. We motivate this by means of the definition given in §1-3 for the non-Euclidean disk D. In this case the group G equals $\boldsymbol{SU}(1, 1)$ and as calculated in §3-5 the group N consists of the group of matrices

$$\begin{pmatrix} 1 + in & -in \\ in & 1 - in \end{pmatrix} \quad n \in \boldsymbol{R}$$

The orbit $N \cdot O$ consists of the points $in/(in - 1)$, which clearly form a horocycle and it is a simple matter to verify that the horocycles in D are the orbits in D of all groups of the form gNg^{-1}.

Hence, we define for the general symmetric space $X = G/K$ a *horocycle* to be an orbit in X of a subgroup of G of the form gNg^{-1}, g being an arbitrary element in G.

Lemma 4.1. The group G permutes the horocycles transitively.

PROOF. The most general horocycle ξ is of the form $\xi = gNg^{-1}h \cdot o$, g and h being fixed elements in G. By the Iwasawa decomposition we can write $h^{-1}g = kan$ and deduce (since $aNa^{-1} \subset N$) that $gNg^{-1}h \cdot o = hkN \cdot o$. In other words, the element $hk \in G$ maps the horocycle $\xi_o = N \cdot o$ onto ξ, so the lemma is proved.

In particular, all the horocycles are submanifolds of X of the same dimension and since $N \cap K = \{e\}$ the mapping $n \to n \cdot o$ is a diffeomorphism of N onto ξ_o.

Lemma 4.2. Each horocycle ξ can be written

$$\xi = ka \cdot \xi_o \tag{1}$$

where $a \in A$ is unique and the coset $kM \in K/M$ is unique.

Although the proof of this lemma is not difficult we shall not stop to prove it here. For the case $X = D$ the lemma is quite obvious.

Definition. The Weyl chamber kM in (1) is called the *normal* to the horocycle ξ; the element $a \in A$ in (1) is called the *complex distance* from o to ξ.

Considering the example $X = D$ the term "normal" is quite reasonable; so is the term "complex distance" because the point $ka \cdot o$ is the unique point in ξ at minimum distance from o. (If $a = \exp H$, $H \in \mathfrak{a}$, the distance is $B(H, H)^{1/2}$, *cf.* [37], p. 306.)

We recall now that given the maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, the group N is determined following a choice of a Weyl chamber $\mathfrak{n}^+ \subset \mathfrak{a}$.

Lemma 4.3. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_w$ denote the various Weyl chambers in $\mathfrak{a}$ and $N_1, \ldots, N_w$ the corresponding Iwasawa groups. Then the horocycles $N_1 \cdot o, \ldots, N_w \cdot o$ all have the same tangent space at the point o.

PROOF. The projection $\pi : G \to G/K$ given by $\pi(g) = g \cdot o$ maps N onto ξ_o and the differential $d\pi : \mathfrak{g} \to (G/K)_o$ maps $\mathfrak{n}$ onto $(\xi_o)_o$. But the map $d\pi : \mathfrak{p} \to (G/K)_o$ is an isomorphism so let $\mathfrak{q} \subset \mathfrak{p}$ be the subspace which $d\pi$ maps onto $(\xi_o)_o$. We shall prove that the manifolds $N \cdot o$ and $A \cdot o$ are orthogonal at o and since

$$(\xi_o)_o = d\pi(\mathfrak{q}) \qquad (A \cdot o)_o = d\pi(\mathfrak{a})$$

it suffices, because of the choice of metric on G/K (§3-3), to prove $B(\mathfrak{q}, \mathfrak{a}) = 0$, that is, $\mathfrak{q}$ and $\mathfrak{a}$ are orthogonal with respect to B. But if $H \in \mathfrak{a}$, $X \in \mathfrak{q}$ then there exists an $X_1 \in \mathfrak{n}$ such that $d\pi(X) = d\pi(X_1)$. Thus $X - X_1 \in \mathfrak{k}$ so since $B(\mathfrak{a}, \mathfrak{k}) = 0$ and $B(\mathfrak{a}, \mathfrak{n}) = 0$, we obtain

$$B(X, H) = B(X_1, H) = 0$$

Thus each of the tangent spaces $(N_i \cdot o)_o$ is perpendicular to the tangent space $(A \cdot o)_o$ and since $\dim N \cdot o + \dim A \cdot o = \dim G/K$, the lemma follows.

Lemma 4.4. Given $x \in X$, $b \in B$, there exists exactly one horocycle passing through x with normal b.

PROOF. Let $b = kM$. We must find a unique $a \in A$ such that x lies on the horocycle $\xi = ka \cdot \xi_o$. But $x \in \xi$ means $x = kan \cdot o$ for some $n \in N$ so $an \cdot o = k^{-1} \cdot x$. Thus, by the Iwasawa decomposition, a is uniquely determined by k and x.

We denote the horocycle determined by this lemma by $\xi(x, b)$ and write $\exp A(x, b)$ $(A(x, b) \in \mathfrak{a})$ for the complex distance from o to $\xi(x, b)$. We can now write down the analogs of the functions $e^{\mu\langle z, b\rangle}$ in §1-3.

For $b \in B$ and λ a complex-valued linear function on $\mathfrak{a}$, define the function $e_{\lambda, b}$ by

$$e_{\lambda,b} : x \to e^{\lambda(A(x, b))} \qquad x \in X$$

We state without proof two properties of $e_{\lambda, b}$, the second of which is trivial.

(i) $e_{\lambda, b}$ is an eigenfunction of each operator $D \in \boldsymbol{D}(G/K)$

(ii) $e_{\lambda, b}$ is constant on each horocycle with normal b. A function on X with this property will be called a *plane wave with normal b*.

One can also prove that these two properties characterize the functions $e_{\lambda, b}$ (if certain singular eigenvalue systems are excluded). In accordance with the definition in §1-3 we define *Fourier analysis on the symmetric space X* to be a decomposition of "arbitrary" functions on X into functions of the form $e_{\lambda, b}$.

As before let dx denote the volume element on X and

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \dim (\mathfrak{g}_\alpha)\alpha$$

Let $\mathfrak{a}^*$ denote the dual of $\mathfrak{a}$, that is, the set of real linear functions on $\mathfrak{a}$. Then the following theorem holds (*cf.* [35]).

Theorem 4.5. For $f \in C_c^\infty(X)$ define the Fourier transform $\tilde{f}$ on $\mathfrak{a}^* \times B$ by

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)(A(x, b))}\, dx \qquad \lambda \in \mathfrak{a}^*, b \in B$$

Then

$$f(x) = \int_{\mathfrak{a}^*} \int_B \tilde{f}(\lambda, b) e^{(i\lambda + \rho)(A(x, b))} |c(\lambda)|^{-2}\, d\lambda\, db$$

if the Euclidean measure $d\lambda$ on $\mathfrak{a}^*$ is suitably normalized.

This theorem is proved by reducing it to Theorem 3.1 in a way which is similar to the reduction of Theorem 3.1, Ch. 1, to the inversion formula for the Mehler transform. That reduction made use of the geometric identity

$$\langle \tau \cdot z, \tau \cdot b \rangle = \langle z, b \rangle + \langle \tau \cdot o, \tau \cdot b \rangle \tag{2}$$

and the formula

$$\left| \frac{d(\tau \cdot b)}{db} \right| = e^{2\langle \tau^{-1} \cdot o, b \rangle} \tag{3}$$

valid for an arbitrary isometry τ of the non-Euclidean disk D.

The generalization of the formula (2) to the symmetric space X is

$$A(g \cdot x, g(b)) = A(x, b) + A(g \cdot o, g(b)) \tag{4}$$

for $g \in G$, $x \in X$ and $b \in B$. (Here the action of G on X and on B is denoted as in §2.) In order to prove (4) let $x = hK$, $b = kM$. Then

$$h \cdot o \in k \exp A(x, b) N \cdot o$$

so for some $n_1 \in N$, $k_1 \in K$

$$gh = gk \exp A(x, b) n_1 k_1$$

which by the Iwasawa decomposition can be written

$$gh = k(gk) \exp H(gk) n(gk) \exp A(x, b) n_1 k_1$$

Since $aNa^{-1} \subset N$ $(a \in A)$, this relation implies

$$g \cdot x \in k(gk) \exp (H(gk) + A(x, b)) N \cdot o$$

and since $k(gk)M = g(kM)$, we conclude

$$A(g \cdot x, g(b)) = H(gk) + A(x, b) \tag{5}$$

On the other hand, we have by the definition of $A(g \cdot o, kM)$ that for some $n_2 \in N$, $k_2 \in K$,

$$g = k \exp A(g \cdot o, kM) n_2 k_2$$

so

$$H(g^{-1}k) = -A(g \cdot o, kM) \tag{6}$$

Hence, (5) becomes

$$A(g \cdot x, g(b)) = -A(g^{-1} \cdot o, b) + A(x, b)$$

In particular, putting $x = o$, we get $A(g \cdot o, g(b)) = -A(g^{-1} \cdot o, b)$, so the desired formula (4) follows. The generalization of (3) to the space X is given by

$$\left| \frac{d(g(b))}{db} \right| = e^{2\rho(A(g^{-1} \cdot o, b))} \tag{7}$$

and this of course is a direct consequence of Cor. 2.9 and (6). Now the proof of Theorem 4.5 proceeds essentially as the proof of Theorem 3.1 in Ch. 1.

Finally we observe that the Poisson integral representation of bounded harmonic functions on X (*cf.* (5) in §2) can be written

$$u(x) = \int_B e^{2\rho(A(x, b))} \hat{u}(b) \, db$$

and is, therefore, according to our definition, to be regarded as a formula in Fourier analysis on X.

4-5 Interpretation by Representation Theory; Eigenfunctions of the Invariant Differential Operators

Since the group G leaves the volume element dx on X invariant we get a unitary representation T_X of G on $L^2(X)$ by associating to each $g \in G$ the operator $f \to f^{\tau(g)}$ on $L^2(X)$. (Here $f^{\tau(g)}$ denotes the function $x \to f(g^{-1} \cdot x)$.) We shall now indicate how Theorem 4.5 gives a decomposition of this representation into irreducible ones.

For $\lambda \in \mathfrak{a}^*$ let $\mathfrak{H}_\lambda$ denote the vector space

$$\mathfrak{H}_\lambda = \left\{ h_\lambda(x) = \int_B e^{(i\lambda+\rho)(A(x,b))} h(b)\, db \mid h \in L^2(B) \right\}$$

of functions on X. If λ is regular, that is, $s\lambda \neq \lambda$ for all $s \neq e$ in the Weyl group W, one can use an irreducibility criterion of Bruhat [7], p. 193, to prove that the function $h \in L^2(B)$ above is uniquely determined by h_λ. If we define a Hilbert space norm on $\mathfrak{H}_\lambda$ by

$$\|h_\lambda\| = \left\{ \int_B |h(b)|^2\, db \right\}^{1/2}$$

then the mapping which assigns the operator $h_\lambda(x) \to h_\lambda(g^{-1} \cdot x)$ to each $g \in G$ is by (4) and (7) seen to be a unitary representation T_λ of G on $\mathfrak{H}_\lambda$. Using the irreducibility criterion cited, one can show this representation to be irreducible. Now with the notation of Theorem 4.5 there is a Plancherel formula, namely,

$$\int_X |f(x)|^2\, dx = \int_{\mathfrak{a}^*} \int_B |\tilde{f}(\lambda, b)|^2\, |c(\lambda)|^{-2}\, d\lambda\, db$$

In terms of direct integrals of representations (see, for example, Dixmier [15]), Theorem 4.5 can therefore be written:

$$L^2(X) = \int \mathfrak{H}_\lambda |c(\lambda)|^{-2}\, d\lambda \qquad T_X = \int T_\lambda |c(\lambda)|^{-2}\, d\lambda$$

λ running through $\mathfrak{a}^*$ (mod W).

The functions in $\mathfrak{H}_\lambda$ are eigenfunctions of each $D \in \boldsymbol{D}(G/K)$. More generally, if T is an analytic functional on B and $\mu \in \boldsymbol{C}$ the function

$$f(x) = \int_B e^{\mu(A(x,b))}\, dT(b)$$

is an eigenfunction of each $D \in \boldsymbol{D}(G/K)$; it appears likely that for sufficiently general functionals T these functions constitute all the simultaneous eigenfunctions of the operators $\boldsymbol{D}(G/K)$ (*cf.* Theorem 5.1, Ch. 1).

4-6 Invariant Differential Equations on Symmetric Spaces

We shall now discuss general existence theorems for invariant differential equations on the symmetric space G/K. In order to motivate the method followed we first describe a well-known geometric method for solving differential equations in $\boldsymbol{R}^n$ with constant coefficients (Courant–Lax [14], Gelfand–Shapiro [20], John [44]). The basis of the method is a formula of Radon–John which in an explicit manner describes a function on $\boldsymbol{R}^n$ by means of its integrals over the various hyperplanes in $\boldsymbol{R}^n$.

For $f \in C_c^\infty(\boldsymbol{R}^n)$ let $\hat{f}(\omega, p)$ denote the integral of f over the hyperplane $(x, \omega) = p$ (here ω is a unit vector and $p \in \boldsymbol{R}$ and $(\ ,\ )$ the scalar product). The function $\hat{f}$ is called the *Radon transform* of f.

Theorem 6.1. For the Radon transform $f \to \hat{f}$ the following inversion formula holds:

$$f(x) = c(\Delta_x)^{[(n-1)/2]}\left(\int_{S^{n-1}} \hat{f}(\omega, (x, \omega))\, d\omega\right) \tag{1}$$

for $f \in C_c^\infty(\boldsymbol{R}^n)$. Here Δ denotes the Laplacian, $d\omega$ is the surface element on the unit sphere $\boldsymbol{S}^{n-1}$, and c is a constant.

For the proof see [44]. There the cases $n =$ odd and $n =$ even are presented in different forms; the unified version can be found in [34], p. 163.

Formula (1) states that when for $x \in \boldsymbol{R}^n$ we form the integral of f over each hyperplane through x, then take the average of these integrals, and finally apply the operator $\Delta^{(n-1)/2}$, we recover the function f. However, for the applications indicated, the important feature of (1) is an explicit decomposition of f into plane waves. (A plane wave is a function which is constant on each hyperplane with a given normal vector; this normal vector is then called the normal to the plane wave.) In fact, for any fixed $\omega \in S^{n-1}$ the function $f_\omega : x \to \hat{f}(\omega, (x, \omega))$ is a plane wave with normal ω.

We shall now apply formula (1) to differential equations. Let D be a differential operator on $\boldsymbol{R}^n$ with constant coefficients and consider a differential equation

$$Du = f \tag{2}$$

where $f \in C_c^\infty(\boldsymbol{R}^n)$ is a given function. We begin by considering the differential equation

$$Dv = f_\omega \tag{3}$$

where f_ω is as above and we look for a solution v which is a plane wave with normal ω. But a plane wave with normal ω is just a function of one variable; furthermore if v is a plane wave with normal ω then so is the function Dv. Our problem of finding v of the specified type satisfying (3) is therefore just an ordinary differential equation with constant coefficients. Pick a solution u_ω and assume that this choice can be made smoothly in ω. Then the function

$$u = c\,\Delta^{(n-1)/2} \int_{S^{n-1}} u_\omega\, d\omega \tag{4}$$

is a solution of the equation (1). In fact, since differential operators with constant coefficients commute we have (at least for n odd)

$$Du = c\,\Delta^{(n-1)/2} \int_{S^{n-1}} Du_\omega\, d\omega = c\Delta^{(n-1)/2} \int_{S^{n-1}} f_\omega\, d\omega = f$$

This proof actually works also for n even. The weakness of the method lies in the assumption that u_ω can be chosen so as to vary smoothly in ω. In fact the example $D = \partial^2/\partial x_1 \partial x_2$, $\omega = (1, 0)$ shows that u_ω may not exist for all ω.

For a symmetric space $X = G/K$ the inversion formula for the Fourier transform (Theorem 4.5) does give a decomposition of an arbitrary function $f \in C_c^\infty(X)$ into plane waves. In fact let as before

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)(A(x, b))}\, dx \qquad \lambda \in \mathfrak{a}^*, b \in B$$

and put

$$f_b(x) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda, b) e^{(i\lambda + \rho)(A(x, b))} |c(\lambda)|^{-2}\, d\lambda \tag{5}$$

Then $f_b(x)$ is a plane wave with normal b so the formula

$$f(x) = \int_B f_b(x)\, db \tag{6}$$

does indeed give a decomposition of f into plane waves. We shall now apply this formula to the problem of solving a differential equation

$$Du = f \tag{7}$$

where D is a given differential operator in $\boldsymbol{D}(G/K)$ and $f \in C_c^\infty(X)$ is a given function. First we need a simple lemma concerning the action of invariant differential operators on plane waves (*cf.* [27], p. 247, or [45]).

Lemma 6.2. Let $D \in \boldsymbol{D}(G/K)$. Then there exists a unique differential operator $\delta(D)$ on the submanifold $A \cdot o \subset X$ such that if bar denotes restriction to $A \cdot o$,

$$\overline{DF} = \delta(D)\bar{F}$$

for every $F \in C^\infty(X)$ which is N-invariant (that is, a plane wave with normal $\mathfrak{a}^+$). This differential operator $\delta(D)$ is invariant under A.

PROOF. Since the mapping $(n, a \cdot o) \to na \cdot o$ is a diffeomorphism of $N \times (A \cdot o)$ onto X the existence and uniqueness of $\delta(D)$ is obvious. Hence we just have to prove its invariance under A. Let $a \in A$ and, as before, if $F \in C^\infty(X)$ let $F^{\tau(a)}$ denote the function $x \to F(a^{-1} \cdot x)$ on X. If F is invariant under N then the function $F^{\tau(a)}$ is too; in fact,

$$F^{\tau(a)}(n \cdot x) = F(a^{-1} n \cdot x) = F(n_1 a^{-1} \cdot x)$$

for some $n_1 \in N$. Thus $F^{\tau(a)}(n \cdot x) = F^{\tau(a)}(x)$, and of course $\overline{F^{\tau(a)}} = (\bar{F})^{\tau(a)}$.

Thus,

$$(\delta(D)^{\tau(a)}\bar{F}) = (\delta(D)(\bar{F})^{\tau(a^{-1})})^{\tau(a)} = (\delta(D)\overline{F^{\tau(a^{-1})}})^{\tau(a)}$$
$$= \overline{((DF^{\tau(a^{-1})})^{\tau(a)}} = ((DF^{\tau(a^{-1})})^{\tau(a)})^{-} = \overline{DF} = \delta(D)\bar{F}$$

This proves the lemma because each function in $C^\infty(A \cdot o)$ can be extended to an N-invariant function in $C^\infty(X)$.

In order to solve the differential equation (7) we begin by considering the differential equation

$$Dv = f_b \tag{8}$$

for an arbitrary $b \in B$. We look for a solution $v = v^b$ which like the function f_b [*cf.* (5)] is a plane wave with normal b. For example, consider the case $b = \mathfrak{a}^+$. Then the function f_b is invariant under N and so is the required function v^b. According to Lemma 6.2, the differential equation $Dv^b = f_b$ on X amounts to the differential equation

$$\delta(D)\overline{v^b} = \overline{f_b} \tag{9}$$

which is by the A-invariance of $\delta(D)$ a differential equation with constant coefficients on the Euclidean space $A \cdot o$. But by a result of Ehrenpreis [16] and Malgrange [52], a differential operator on $\boldsymbol{R}^n$ with constant coefficients maps the space $C^\infty(\boldsymbol{R}^n)$ onto itself. Hence a solution $v = v^b$ exists. Now we assume that v^b can be chosen so that it depends smoothly on b. Then we put

$$u(x) = \int_B v^b(x)\, db \qquad x \in X$$

and have

$$Du = \int_B Dv^b\, db = \int_B f_b\, db = f$$

This is not an existence proof for the differential equation (7) because of the smoothness assumption about v^b (see, however, Trèves [59], p. 131). Nevertheless, we have the following general theorem (Helgason [33], p. 577–578).

Theorem 6.3. Let $D \neq 0$ be an arbitrary G-invariant differential operator on the symmetric space G/K. For each $f \in C_c^\infty(G/K)$ the differential equation $Du = f$ has a solution $u \in C^\infty(G/K)$.

It suffices to find a distribution T on X satisfying the differential equation $DT = \delta$, where δ is the delta-distribution at the origin o. In fact, the desired solution is then $u = f \times T$, where $\times$ is the operation on distributions on X which is induced by the convolution product of distributions on G. Since D and δ are K-invariant we look for a K-invariant T. For this we use the transform $f \to F_f$ discussed in §3. As proved in Harish-Chandra [28],

p. 46, this transform is one-to-one on the space $I(X)$ of K-invariant, square-integrable functions on X which are rapidly decreasing on X in a certain technical sense, and the transform maps $I(X)$ into the space $I(A)$ of Weyl group invariant functions on A which are rapidly decreasing on A (considered as a Euclidean space). On the other hand, it is proved in Helgason [33] that the range of the mapping $f \to F_f$ $(f \in I(X))$ is precisely $I(A)$ and furthermore, $F_{Df} = \gamma(D)F_f$, where $\gamma(D)$ is a certain constant-coefficient differential operator on A. The isomorphism $f \to F_f$ of $I(X)$ onto $I(A)$ has a transpose, mapping the dual $I'(A)$ of $I(A)$ onto the dual $I'(X)$ of $I(X)$. Under this isomorphism the differential equation $DT = \delta$ on X is transformed into a differential equation for tempered distribution on A, and this last differential equation has constant coefficients since $\gamma(D)$ does. But by a theorem of Hörmander [40] and Lojasiewicz [49] any differential operator on $\boldsymbol{R}^n$ with constant coefficients maps the space of tempered distributions on $\boldsymbol{R}^n$ onto itself. This leads to the desired distribution T on X, proving the theorem.

4-7 The Wave Equation on Symmetric Spaces

We shall now discuss a different method for solving differential equations on the symmetric space X. It uses the Radon transform on X which we now define. Let Ξ denote the set of all horocycles in X. For $f \in C_c^\infty(X)$ we define the function $\hat{f}$ on Ξ by

$$\hat{f}(\xi) = \int_\xi f(x)\, d\sigma(x) \qquad (\xi \in \Xi) \tag{1}$$

where $d\sigma$ is the volume element on ξ. (The Riemannian structure on X induces in an obvious way a Riemannian structure on the submanifold ξ.) The function $\hat{f}$ is called the *Radon transform* of f.

If $x \in X$ the (compact) subgroup K_x of G which keeps x fixed permutes the horocycles through x transitively. For $x = o$ this is obvious from Lemma 4.2 and in general it follows by the homogeneity of X. The set of horocycles passing through x has a unique normalized measure, say ν, invariant under K_x.

If ϕ is a function on Ξ the function $\check{\phi}$ on X is defined by

$$\check{\phi}(x) = \int_{x \in \xi} \phi(\xi)\, d\nu(\xi) \tag{2}$$

***Theorem** 7.1.* Suppose all Cartan subgroups of G are conjugate. Then for a certain fixed differential operator $\square \in \boldsymbol{D}(G/K)$

$$f = \square((\hat{f})^\vee) \qquad \in C_c^\infty(G/K) \tag{3}$$

This formula is analogous to the inversion formula of Radon–John (Theorem 6.1) for the case of an odd-dimensional Euclidean space. The even-dimensional Euclidean case corresponds here to the existence of non-conjugate Cartan subgroups and in this case (3) still holds in a slightly modified form (*cf.* [35], p. 759). We emphasize that the differential operator $\square$ can be written down quite explicitly.

By means of (3) one can write down a solution of the wave equation on X,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \tag{4}$$

with initial data

$$u(x, 0) = 0 \qquad \left\{\frac{\partial}{\partial t} u(x, t)\right\}_{t=0} = f(x) \tag{5}$$

Here Δ denotes the Laplace–Beltrami operator on X and f is an arbitrary given function in $C_c^\infty(X)$.

In the notation of §1, let $\tilde{\square} \in \boldsymbol{D}_K(G)$ be an operator satisfying $(\square f)^\sim = \tilde{\square}\tilde{f}$ for all $f \in C^\infty(G/K)$. Let $|\rho|$ denote the norm of the linear form ρ, and let dn be a Haar measure on N which corresponds to the volume element $d\sigma$ on $\xi_o = N \cdot o$ under the diffeomorphism $n \to n \cdot o$. Let Δ_A denote the Laplacian on the Euclidean space A.

***Theorem** 7.2.* The solution to the wave equation (4) with initial data (5) is given by

$$u(g \cdot o, t) = \tilde{\square}_g\left(\int_K V_{k,g}(e, t)\, dk\right) \tag{6}$$

where $V_{k,g}$ is the solution to the equation for damped waves on $A \times \boldsymbol{R}$,

$$(\Delta_A - |\rho|^2)V_{k,g} = \frac{\partial^2}{\partial t^2} V_{k,g} \tag{7}$$

$$V_{k,g}(a, 0) \equiv 0, \qquad \left\{\frac{\partial}{\partial t} V_{k,g}(a, t)\right\}_{t=0} = e^{\rho(\log a)} F_{k,g}(a)$$

where

$$F_{k,g}(a) = \int_N f(gkan \cdot o)\, dn$$

Although the verification of this theorem is not long (*cf.* [32], p. 688) we omit it here because it requires some further preparation. The function $V_{k,g}$ is given as a convolution of a certain Bessel function with $F_{k,g}$ so the solution (6) is explicitly given in terms of the initial data $f(x)$.

Huygens' Principle

Let M be an analytic pseudo-Riemannian manifold with Lorentzian signature, in short, a Lorentzian manifold. Since our considerations will be local we assume that M is convex, that is, any two points in M can be joined by a unique geodesic. The geodesics of zero length through a point $p \in M$ generate the light cone C_p in M with vertex p. A submanifold $S \subset M$ is called spacelike if each tangent vector to S is spacelike. Let Δ denote the (hyperbolic) Laplace–Beltrami operator on M, and suppose now that a Cauchy problem is posed for the wave equation $\Delta u = 0$ with initial data on a spacelike hypersurface $S \subset M$. Hadamard proved that the value $u(p)$ of the solution at a point $p \in M$ only depends on the initial data on the piece $S^* \subset S$ which lies inside the light cone C_p. Huygens' principle (in the strong sense) is said to hold for $\Delta u = 0$ if the value $u(p)$ only depends on the initial data in an arbitrary small neighborhood of the edge s of S^*, $s = C_p \cap S$. It is known that this is a property of the space M and does not depend on the particular choice of $S \subset M$. The wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1{}^2} - \cdots - \frac{\partial^2 u}{\partial x_{n-1}^2} = 0$$

for an odd-dimensional $\boldsymbol{R}^{n-1}$ satisfies Huygens' principle. A conjecture, attributed to Hadamard, was that these were essentially the only second-order hyperbolic equations satisfying Huygens' principle. A counter-example of the form $\Delta u + cu = 0$ ($n = 6$) was given by Stellmacher [58] in 1953, and in 1965, P. Günther [23] gave a whole series of counter-examples for the pure equation $\Delta u = 0$ ($n = 4$). These are based on Hadamard's criterion that Huygens' principle holds if and only if n is even and $\geqslant 4$ and the logarithmic part of the fundamental solution (in Hadamard's sense) vanishes.

If M is symmetric the evidence available seems to indicate that "Hadamard's conjecture" might hold for the pure equation $\Delta u = 0$. For M of constant curvature (a "de Sitter space" or an "anti de Sitter space") this is indeed so (*cf.* [29], p. 296; see also [13].) The answer is also affirmative if M has the form $M = M_o \times \boldsymbol{R}$, where M_o has dimension 3 and constant curvature (Hölder [39]). Finally the answer is affirmative if $M = X \times \boldsymbol{R}$, where X is a symmetric space whose group of isometries is a complex semisimple Lie group (Helgason [33], p. 582).

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