Indag. Mathem., N.S., 16 (3–4), 531–551 December 19, 2005

The Abel, Fourier and Radon transforms on symmetric spaces

by Sigurdur Helgason

77 Massachusetts Avenue, Cambridge, MA 02139, USA

Dedicated to Gerrit van Dijk on the occasion of his 65th birthday

Communicated by Prof. J.J. Duistermaat at the meeting of June 20, 2005

1. INTRODUCTION

In this paper we prove some recent results on the three transforms in the title and discuss their relationships to older results. The spaces we deal with are symmetric spaces $X = G/K$ of the noncompact type, G being a connected noncompact semisimple Lie group with finite center and K a maximal compact subgroup.

For the two natural Radon transforms on X we prove a new inversion formula and a sharpening of an old support theorem; for the Abel transform we prove some new identities with some applications and for the Fourier transform a result for integrable functions which has a strong analog of the Riemann–Lebesgue lemma. These latter results are from a collaboration with Rawat, Sengupta and Sitaram.

Notation. Following Schwartz we use the notation $\mathcal{D}(X)$ for $C_c^\infty(X)$, $\mathcal{E}(X)$ for $C^\infty(X)$ and $\mathcal{S}(\mathbf{R}^n)$ for the space of rapidly decreasing functions on $\mathbf{R}^n$.

2. DIFFERENT RADON TRANSFORMS ON THE SYMMETRIC SPACE X

Radon's paper [40] suggested the general problem of determining a function on a manifold on the basis of its integrals over certain submanifolds. A natural case of

E-mail: helgason@mit.edu (S. Helgason).

this problem is the inversion of the X-ray transform on a Riemannian manifold. It is the transform $f \to \hat{f}$ defined by the arc-length integral

$$(2.1) \qquad \hat{f}(\gamma) = \int_{\gamma} f(x)\, dm(x),$$

f being an "arbitrary" function on the Riemannian manifold X and γ any complete geodesic in X.

In general this injectivity problem seems to be unresolved. For a Cartan symmetric space $X \neq \mathbf{S}^n$ the injectivity, however, holds. For a symmetric X of the noncompact the injectivity holds in the stronger form of the

Support theorem [26]. *If* $\hat{f}(\gamma) = 0$ *for all geodesics* γ *disjoint from a ball* $B \subset X$ *then* $f(x) = 0$ *for* $x \notin B$.

This last result requires stronger decay assumption at ∞ than the injectivity result does.

Here we shall prove an explicit inversion formula for the X-ray transform for rank $X > 1$. See Section 5 for the contact with Rouvière's different solution.

Funk [11] and Radon [40] inverted this transform for the sphere $\mathbf{S}^2$ and $\mathbf{R}^2$. Denoting the set of geodesics by Ξ we have the coset space representations

$$\mathbf{S}^2 = \mathbf{O}(3)/\mathbf{O}(2), \qquad \Xi = \mathbf{O}(3)/\mathbf{O}(2)\mathbf{Z}^2,$$
$$\mathbf{R}^2 = \mathbf{M}(2)/\mathbf{O}(2), \qquad \Xi = \mathbf{M}(2)/\mathbf{M}(1)\mathbf{Z}_2,$$

$\mathbf{M}(n)$ denoting the isometry group of $\mathbf{R}^n$.

This suggests the following generalization. Let $X = G/K$ and $\Xi = G/H$ be coset spaces of the same locally compact group G, K and H being closed subgroups. Here it will be convenient to assume all these groups as well as $L = K \cap H$ to be unimodular. We do not assume the elements $\xi \in \Xi$ to be subsets of X but instead use Chern's concept of incidence:

$$x = gK \text{ is incident to } \xi = \gamma H$$

if $gK \cap \gamma H \neq \emptyset$ as subsets of G. Given $x \in X$, $\xi \in \Xi$ define

$$\check{x} = \{\xi \in \Xi \colon\ x, \xi \text{ incident}\},$$
$$\hat{\xi} = \{x \in X \colon\ x, \xi \text{ incident}\}.$$

These are orbits of certain subgroups of G and have natural measures dm, $d\mu$ (up to factors) and we define the abstract Radon transform $f \to \hat{f}$ and its dual $\varphi \to \check{\varphi}$ by

$$(2.2) \qquad \hat{f}(\xi) = \int_{\hat{\xi}} f(x)\, dm(x), \qquad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\, d\mu(\xi).$$

The normalizations of dm and $d\mu$ are unified by taking $x_0 = eK$, $\xi_0 = eH$ and

$$(2.3) \qquad \hat{f}(\gamma H) = \int_{H/L} f(\gamma h \cdot x_0)\, dh_L, \qquad \check{\varphi}(gK) = \int_{K/L} \varphi(gk \cdot \xi_0)\, dk_L$$

the invariant measures dh_L, dk_L being fixed by Haar measures of H, K and L.

Main problems.

(i) Injectivity of $f \to \hat{f}$, $\varphi \to \check{\varphi}$.
(ii) Inversion formulas.
(iii) Range and kernel question for these transforms.
(iv) Applications elsewhere.

An easy general result relevant to problem (iii) is the following. For a suitable normalization of the measures $dx = dg_K$, $d\xi = d\gamma_H$ we have

$$(2.4) \qquad \int_X f(x)\check{\varphi}(x)\, dx = \int_\Xi \hat{f}(\xi)\varphi(\xi)\, d\xi,$$

a result which suggests the extension of (2.3) to distributions.

3. d-PLANES IN $\mathbf{R}^n$

Here we consider the space $X = \mathbf{R}^n$ and $\Xi = \mathbf{G}(d,n)$ the set of d-dimensional planes in $\mathbf{R}^n$. These are both homogeneous under the group $G = \mathbf{M}(n)$. Fix $x_0 \in \mathbf{R}^n$, $\xi_0 \in \mathbf{G}(d,n)$ at distance $d(x_0, \xi_0) = p$. Then we have

$$(3.1) \qquad X = \mathbf{R}^n = \mathbf{M}(n)/K_p, \qquad \Xi = \mathbf{G}(d,n) = \mathbf{M}(n)/H_p,$$

where K_p and H_p, respectively, are the stability groups of x_0 and ξ_0. Since various p will be considered the transforms (2.2) will be denoted by $\hat{f}_p$ and $\check{\varphi}_p$. Since the action of $\mathbf{M}(n)$ on X and Ξ is quite rich it turns out that for the coset space representation (3.1)

$$z \in X \text{ is incident to } \eta \in \Xi \quad \Leftrightarrow \quad d(z,\eta) = p.$$

Thus the transform $\hat{f}_p$ and $\check{\varphi}_p$ can be written

$$(3.2) \qquad \hat{f}_p(\xi) = \int_{d(x,\xi)=p} f(x)\, dm(x), \qquad \check{\varphi}_p(x) = \int_{d(x,\xi)=p} \varphi(\xi)\, d\mu(\xi).$$

In particular, $\hat{f}_0$ is the usual d-plane transform $\hat{f}$, but in order to invert it we need $\check{\varphi}_p$ for variable p. One of several versions of the inversion formula is the following (see [25,27]):

$$f(x) = c(d)\left[\left(\frac{d}{d(r^2)}\right)^d \int_r^\infty p\big(p^2 - r^2\big)^{d/2-1}\big(\hat{f}\big)_p^\vee(x)\,dp\right]_{r=0} \tag{3.3}$$

with $c(d)$ and constant. Note that $(\hat{f})_p^\vee(x)$ is the average of the integrals of f over all d-planes at distance p from x.

For $d = 1$ this formula reduces to

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{d}{dp}\big((\hat{f})_p^\vee(x)\big)\frac{dp}{p}, \tag{3.4}$$

which for $n = 2$ coincides with Radon's original formula. Radon's proof is very elegant and is based on an exhaustion of the exterior $|x| > r$ by lines. As far as I know this proof has not been extended to higher dimensions n. Formula (3.4) for $n > 2$ is crucial for the inversion of (2.1) given in Theorem 5.1.

4. d-DIMENSIONAL TOTALLY GEODESIC SUBMANIFOLDS IN HYPERBOLIC SPACE $\mathbf{H}^n$

A similar method works here and the analog of (3.3) is the formula

$$f(x) = C(d)\left[\left(\frac{d}{d(r^2)}\right)^d \int_r^\infty \big(t^2 - r^2\big)^{d/2-1} t^d \big(\hat{f}\big)_{s(t)}^\vee(x)\,dt\right]_{r=1}, \tag{4.1}$$

where $C(d)$ is a constant and $s(p) = \cosh^{-1}(p)$ (see [25,27]). Other versions of the inversion exist (e.g., [28] and [6]). For $d = 1$ this reduces to

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{d}{dp}\big((\hat{f})_p^\vee(x)\big)\frac{dp}{\sinh p} \tag{4.2}$$

a formula which for $n = 2$ is stated without proof in Radon [40].

5. X-RAY INVERSION ON THE SYMMETRIC SPACE $X = G/K$

In communication from 2003, Rouvière proved an extension of formula (4.2) to symmetric spaces X of rank $\ell = 1$. Inspired by his methods, I proved the inversion formula (5.2) for the X-ray transform for X of rank $\ell > 1$. Then Rouvière [43] extended his formula to X of arbitrary rank ℓ. Actually he has several such formulas but they are all different from the formula (5.2) below.

Fix a flat totally geodesic submanifold E of X with $\dim E = \ell > 1$ (ℓ the rank of X) passing through the origin $o = eK$ of X. Let $p > 0$ and $S = S_p(o)$ be the sphere in E with radius p and center o. The geodesics γ in E tangent to

S are permuted transitively by the orthogonal group $\mathbf{O}(E)$. Let du and dk denote the normalized Haar measures on U and K. The spaces $k \cdot E$ as k runs through K constitute all flat totally geodesic subspaces of X through o of dimension ℓ. Thus the images $k \cdot \gamma$ ($k \in K$, γ tangent to S) constitute the set Γ_p of all geodesics γ in X lying in some flat ℓ-dimensional totally geodesic submanifold of X through o and $d(o, \gamma) = p$. The set Γ_p has a natural measure ω_p given by the functional

$$\omega_p : \varphi \to \int_K \left(\int_{\mathbf{O}(E)} \varphi\big(k(u \cdot \gamma)\big)\, du \right) dk. \tag{5.1}$$

Theorem 5.1. *The X-ray transform* (2.1) *on a symmetric space* $X = G/K$ *of rank* $\ell > 1$ *is inverted by the formula*

$$f(o) = -\frac{1}{\pi} \int_0^\infty \left(\frac{d}{dp} \int_{\Gamma_p} \hat{f}(\gamma) d\omega_p(\gamma) \right) \frac{dp}{p}, \quad f \in \mathcal{D}(X). \tag{5.2}$$

Since Γ_p and $d\omega_p$ are K-invariant the formula holds at each point x by replacing f by $f \circ g$ where $g \in G$ is such that $g \cdot o = x$.

Proof. First assume f to be K-invariant and consider the restriction $f|E$. Fix an orthonormal frame $H_0, H \in E_0$, the tangent space to E at o, consider the one parameter subgroups $\exp tH_0$, $\exp tH$ and the geodesic $\gamma_0(t) = \exp tH_0 \cdot o$. Then the geodesic $\gamma(t) = \exp pH \cdot \gamma_o(t)$ lies in E and is tangent to $S_p(o)$. Because of (3.4) we have

$$f(o) = -\frac{1}{\pi} \int_0^\infty \frac{d}{dp} \big(\hat{f}\big)^E_p(o) \frac{dp}{p}, \tag{5.3}$$

where the superscript E stands for the dual transform on geodesics in the space E. Thus

$$\big(\hat{f}\big)^E_p(o) = \int_{\substack{\gamma \subset E \\ d(o,\gamma)=p}} \big(\hat{f}\big)(\gamma)\, d\nu(\gamma) = \int_{\mathbf{O}(E)} \big(\hat{f}\big)(u \cdot \gamma)\, du, \tag{5.4}$$

where ν stands for the average over the set of geodesics tangent to $S_p(o)$.

For $f \in \mathcal{D}(X)$ arbitrary we use (5.2) on the function

$$f^\natural(x) = \int_K f(k \cdot x)\, dk.$$

Taking into account the definition (5.1) the inversion formula (5.2) follows immediately. □

Remark. Note that the measure ω_p in (5.1) is a kind of convolution of the Haar measures dk and du. However it is not a strict convolution since the product ku is not defined.

6. THE HOROCYCLE TRANSFORM IN $X = G/K$

Consider the usual Iwasawa decomposition of G, $G = NAK$ where N and A are nilpotent and abelian, respectively. A *horocycle* is by definition [12] an orbit in X of a conjugate gNg^{-1} of N. The group G permutes the horocycles transitively and the space Ξ of horocycles can be written $\Xi = G/MN$ where M is the centralizer of A in K. In the double fibration

$$\begin{array}{ccc} & G/M & \\ \swarrow & & \searrow \\ X = G/K & & G/MN = \Xi \end{array}$$

it turns out that $x = gK$ is incident to $\xi = \gamma MN$ if and only if $x \in \xi$. The transforms (2.2) become

$$(6.1) \qquad \hat{f}(\gamma MN) = \int_N f(\gamma n \cdot o)\, dn, \qquad \check{\varphi}(gK) = \int_K \varphi(gk \cdot \xi_0)\, dk,$$

where $\xi_o = N \cdot o$. While the map $\varphi \to \check{\varphi}$ has a big kernel, the horocycle transform $f \to \hat{f}$ is injective (see [17] or [13]). The following result from [21] is considerably stronger.

Theorem 6.1 (Support theorem). *Let B be a closed ball in X. Then*

$$\hat{f}(\xi) = 0 \quad \text{for } \xi \cap B = \emptyset \quad \text{implies}$$
$$f(x) = 0 \quad \text{for } x \notin B.$$

Here one requires stronger decay conditions on f than for the injectivity. A different proof was given in [14]. We have also the following inversion and Plancherel formula for the Radon transform [18,19]. The pseudodifferential operator Λ and the differential operator $\square$ below are constructed by means of the Harish-Chandra c-function, and w denotes the order of the Weyl group. For G complex a result similar to (6.2) appears in [12].

Theorem 6.2. *For $f \in \mathcal{D}(X)$ or sufficiently rapidly decreasing we have the inversion formula*

$$(6.2) \qquad f = \frac{1}{w}\left(\Lambda \bar{\Lambda} \hat{f}\right)^{\vee}.$$

If all Cartan subgroups of G are conjugate the formula has the improved version

$$f = \frac{1}{w}\square((\hat{f})^{\vee}).$$

For G arbitrary

$$(6.3) \qquad w\int_X |f(x)|^2\,dx = \int_{\Xi} |\Lambda\hat{f}|^2(\xi)\,d\xi,$$

with a suitable normalization of the invariant measures dx and $d\xi$.

The range question (iii) for $f \to \hat{f}$ is more complicated. Consider first the hyperbolic plane $\mathbf{H}^2$ in the Poincaré unit disk model D. Here the horocycles are the circles in the disk tangential to the boundary $\{e^{i\theta}\colon\ \theta \in \mathbf{R}\}$. Let $\xi_{t,\theta}$ denote the horocycle through $e^{i\theta}$ with distance t (with sign) from the origin. Then we have the following result from [23].

Theorem 6.3. *The range $\mathcal{D}(D)\widehat{\ }$ consists of the functions $\psi \in \mathcal{D}(\Xi)$*

$$\psi(\xi_{t,\theta}) = \sum_n \psi_n(t)e^{in\theta}$$

where

$$(6.4) \qquad \psi_n(t) = e^{-t}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - 2|n| + 1\right)\varphi_n(t)$$

where $\varphi_n \in \mathcal{D}(D)$ is even.

This implies a relationship between $\psi(\xi_{t,\theta})$ and $\psi(\xi_{-t,\theta})$. More specifically, if $f'(t) = f(-t)$, $\Psi_n = e^t\psi_n$ then $*$ denoting convolution on $\mathbf{R}$

$$\Psi_n' = S_n * \Psi_n$$

where the distribution S_n on $\mathbf{R}$ has Fourier transform

$$\hat{S}_n = \frac{(i\lambda+1)\cdots(i\lambda+2|n|-1)}{(i\lambda-1)\cdots(i\lambda-2|n|+1)}, \qquad \lambda \in \mathbf{R}.$$

This relationship between $\psi(\xi_{-t,\theta})$ and $\psi(\xi_{t,\theta})$ implies that in (6.3) $f \to \Lambda\hat{f}$ does not map $L^2(X)$ onto $L^2(\Xi)$.

For the generalization of (6.4) to $X = G/K$ we need some additional notation. Let $\hat{K}$ be the unitary dual of K and $d(\delta)$ the degree of a $\delta \in \hat{K}$. Given δ acting on V_δ let

$$V_\delta^M = \{v \in V_\delta\colon\ \delta(m)v = v \text{ for } m \in M\}$$

and put $\ell(\delta) = \dim V_\delta^M$. Let

$$\hat{K}_M = \{\delta \in \hat{K} : \ell(\delta) > 0\}.$$

In the following theorem from [25] the expansion (6.5) is a generalization of (6.4). Put $\rho(H) = \frac{1}{2}\,\mathrm{Trace}(\mathrm{ad}\, H|\mathfrak{n})$.

Theorem 6.4. *The range $\mathcal{D}(X)\widehat{\ }$ consists of the functions $\psi \in \mathcal{D}(\Xi)$*

$$\psi(ka \cdot \xi_0) = \sum_{\delta \in \hat{K}_M} d(\delta)\,\mathrm{Tr}\big(\delta(k)\Psi_\delta(a)\big) \tag{6.5}$$

($\mathrm{Tr} = \mathrm{Trace}$) *where Ψ_δ is a function on A with values in* $\mathrm{Hom}(V_\delta, V_\delta^M)$*, i.e.,* $\Psi_\delta \in \mathcal{D}(A, \mathrm{Hom}(V_\delta, V_\delta^M))$*, given by*

$$\Psi_\delta(a) = e^{-\rho(\log a)} Q^\delta(D)_a\big(\Phi_\delta(a)\big), \quad a \in A, \tag{6.6}$$

where

$$\Phi_\delta \in \mathcal{D}\big(A, \mathrm{Hom}\big(V_\delta, V_\delta^M\big)\big) \tag{6.7}$$

is W-invariant and $Q^\delta(D)$ is a certain $\ell(\delta) \times \ell(\delta)$ matrix of constant coefficient differential operators on A.

From this result we can derive the following (unpublished) refinement of the support theorem above. Let A^+ be the Weyl chamber corresponding to the choice of the group N.

Theorem 6.5. *Suppose $f \in \mathcal{D}(X)$ satisfies*

$$\hat{f}(ka \cdot \xi_0) = 0 \quad \textit{for } k \in K,\ a \in A^+,\ |\log a| > R.$$

Then

$$\hat{f}(ka \cdot \xi_0) = 0 \quad \textit{for } k \in K,\ |\log a| > R,\ a \in A$$

so by Theorem 6.1

$$f(x) = 0 \quad \textit{for } d(0, x) > R.$$

Proof. Let $Q_c(D)$ be the matrix of cofactors of $Q^\delta(D)$ so that

$$Q_c(D)Q^\delta(D) = \det Q^\delta(D)I. \tag{6.8}$$

Then (6.6) implies

$$Q_c(D)(e^\rho \Psi_\delta) = \det Q^\delta(D)\Phi_\delta. \tag{6.9}$$

Now it is known [34], [25, pp. 267, 348] that $\det Q^{\delta}(D)$ is a product of linear factors $\delta(H_i) + c$ where $H_i \in \mathfrak{a}$ and $\partial(H_i)$ the corresponding directional derivative.

Suppose the function $\psi = \hat{f}$ satisfies

$$\psi(ka \cdot \xi_0) = 0 \quad \text{for } k \in K,\ a \in A^+,\ |\log a| > R.$$

Since

$$\Psi_\delta(a) = \int_K \psi(ka \cdot \xi_0)\delta(k^{-1})\, dk$$

we deduce from (6.8) and (6.9) that

$$\det Q^{\delta}(D)\Phi_\delta(a) = 0 \quad \text{for } a \in A^+,\ |\log a| > R.$$

Consider this equation on a ray in A^+ starting at e. Because of the mentioned factorization of $\det Q^{\delta}(D)$ we deduce that on this ray Φ_δ satisfies an ordinary differential equation on the interval (R, ∞). Having compact support we deduce that $\Phi_\delta(a) = 0$ for $a \in A^+$, $|\log a| > R$. By its Weyl group invariance it vanishes for all $a \in A$, $|\log a| > R$ which by (6.5) proves the theorem. □

Consider the case rank $X = 1$. Let $B_R(o)$ be a ball in X with radius R and center 0. Fix a unit vector H in the Lie algebra of A such that $\exp H \in A^+$. Put $a_t = \exp tH$. The *interior* of the horocycle $kNa_t \cdot o$ is the union $\bigcup_{\tau > t} kNa_\tau \cdot o$. A horocycle ξ is said to be *external* to $B_R(o)$ if its interior is disjoint from $B_R(o)$; ξ is said to *enclose* $B_R(o)$ if its interior contains $B_R(o)$.

Corollary 6.6. *Let X have rank one and $B_R(o)$ as above. Let $f \in \mathcal{D}(X)$. Then the following are equivalent*:

(i) $\hat{f}(\xi) = 0$ *whenever ξ is external to $B_R(o)$.*
(ii) $\hat{f}(\xi) = 0$ *whenever ξ encloses $B_R(o)$.*
(iii) $f \equiv 0$ *outside $B_R(o)$.*

For hyperbolic space this is clear from Theorem 6.3 and was proved in a different way by Lax and Phillips [35].

Problem (iii) for the dual transform $\varphi \to \check{\varphi}$ has a satisfactory answer (see [25, IV §§2 and 4]). The kernel can be described in the spirit of Theorem 6.5 and for the range one has the surjectivity

$$(6.10) \qquad \mathcal{E}(\Xi)^\vee = \mathcal{E}(X).$$

Let $\mathcal{D}_K(X)$ denote the space of K-invariant functions in $\mathcal{D}(X)$. The *Abel transform* $f \to \mathcal{A}f$ is defined by

$$(7.1) \qquad (\mathcal{A}f)(a) = e^{\rho(\log a)} \int_N f(an \cdot o)\, dn, \quad a \in A,\ f \in \mathcal{D}_K(X).$$

Except for the factor e^{ρ} it is the restriction of the Radon transform to K-invariant functions

$$(7.2) \qquad \mathcal{A}f = e^{\rho} \hat{f}.$$

Some of its properties are best analyzed by means of the spherical functions

$$(7.3) \qquad \varphi_\lambda(g) = \int_K e^{(i\lambda - \rho)(H(gk))}\, dk, \quad g \in G,\ \lambda \in \mathfrak{a}_c^*,$$

where $H(g) \in \mathfrak{a}$ is determined by $g \in k \exp H(g) N$ and $\mathfrak{a}_c^*$ is the complex dual of $\mathfrak{a}$. The *spherical transform*

$$(7.4) \qquad \tilde{f}(\lambda) = \int_X f(x) \varphi_{-\lambda}(x)\, dx, \quad f \in \mathcal{D}_K(X)$$

(where $\varphi_\lambda(g \cdot o) = \varphi_\lambda(g)$) is a homomorphism relative to convolution $\times$ on X:

$$(7.5) \qquad (f_1 \times f_2)^{\sim}(\lambda) = \tilde{f}_1(\lambda) \tilde{f}_2(\lambda).$$

As proved in [15], $\mathcal{A}$ intertwines the spherical transform and the Euclidean Fourier transform $F \to F^*$ on A so

$$(7.6) \qquad \int_A (\mathcal{A}f)(a) e^{-i\lambda(\log a)}\, da = \int_X \varphi_{-\lambda}(x) f(x)\, dx, \quad (\mathcal{A}f)^* = \tilde{f}.$$

Thus $\mathcal{A}f$ is W-invariant and by (7.5)

$$(7.7) \qquad \mathcal{A}(f_1 \times f_2) = \mathcal{A}f_1 * \mathcal{A}f_2,$$

where $*$ is convolution on A. Let $\mathbf{D}(X)$ denote the algebra of G-invariant differential operators on X and $\Gamma : \mathbf{D}(X) \to \mathbf{D}_W(A)$ the isomorphism onto the W-invariant constant coefficient differential operators on A.

The Abel transform is a simultaneous transmutation operator between $\mathbf{D}(X)$ and $\mathbf{D}_W(A)$, i.e.,

$$(7.8) \qquad \mathcal{A}Df = \Gamma(D)\mathcal{A}f, \quad D \in \mathbf{D}(X),\ f \in \mathcal{D}_K(X)$$

as shown in [29] which for example can be used to prove that each D has a fundamental solution. By the Paley–Wiener theorem for (7.3) one has that

$\mathcal{A}: \mathcal{D}_K(X) \to \mathcal{D}_W(A)$ is a bijective homeomorphism. (Here the subscript W means W-invariance.) Hence we have a bijection

$$(7.9) \qquad \mathcal{A}^* : \mathcal{D}'_W(A) \to \mathcal{D}'_K(X)$$

between the corresponding distribution spaces. Also if $\varphi \in \mathcal{E}_W(A)$ we have easily (see [5] or [25, IV, §4])

$$(7.10) \qquad (\mathcal{A}^*\varphi)(gK) = \int_{K/M} \varphi\big(\exp H(gk)\big) e^{-\rho(H(gk))}\, dk.$$

The Radon transform has the advantage over $\mathcal{A}$ that it commutes with the action of G. Thus we can deduce from (6.10) and (7.2) that as in [25]

$$\mathcal{A}^*\mathcal{E}_W(A) = \mathcal{E}_K(X).$$

We now add a few new results about $\mathcal{A}$ and $\mathcal{A}^*$ which will be useful later. Some are closely related to rank-one results in Bagchi and Sitaram in [3].

Because of the convolution property (7.7) one can ask how $\mathcal{A}^*$ behaves relative to convolution. Let L be the operator on $\mathcal{S}(A)$ given by

$$(7.11) \qquad (L\varphi)^*(\lambda) = \big|c(\lambda)\big|^{-2} \varphi^*(\lambda), \quad \lambda \in \mathfrak{a}^*,$$

where $c(\lambda)$ is Harish-Chandra's c-function.

Theorem 7.1. *Let $\varphi \in \mathcal{D}_W(A)$, $\psi \in \mathcal{E}_W(A)$. Then*

$$\mathcal{A}^*(L\varphi) = w\mathcal{A}^{-1}(\varphi) \quad (w = \textit{order of } W)$$

and

$$(7.12) \qquad \mathcal{A}^*(\varphi * \psi) = \frac{1}{w} \mathcal{A}^*(L\varphi) \times \mathcal{A}^*\psi.$$

Proof. Using the inversion formula for the spherical transform we have

$$\begin{aligned}
\mathcal{A}^*(L\varphi)(gK) &= \int_K (L\varphi)\big(\exp H(gk)\big)^{-\rho(H(gk))}\, dk \\
&= \int_K \left(\int_{\mathfrak{a}^*} (L\varphi)^*(\lambda) e^{i\lambda(H(gk))}\, d\lambda \right) e^{-\rho(H(gk))}\, dk \\
&= \int_{\mathfrak{a}^*} \big|c(\lambda)\big|^{-2} \varphi^*_\lambda(g)\, d\lambda = F(gK),
\end{aligned}$$

where $\tilde{F}(\lambda) = \varphi^*(\lambda) w$. But $\tilde{F} = (\mathcal{A}F)^* = \varphi^* w$ so

$$\varphi = \frac{1}{w} \mathcal{A}F, \quad F = w\mathcal{A}^{-1}(\varphi).$$

Thus $\mathcal{A}^*(L\varphi) = w\mathcal{A}^{-1}(\varphi)$. Consider now the average

$$\psi^\lambda(a) = \frac{1}{w}\sum_{s\in W} e^{is\lambda(\log a)}.$$

Then $\mathcal{A}^*\psi^\lambda = \varphi_\lambda$ and $\varphi * \psi^\lambda = \varphi^*(\lambda)\psi^\lambda$. But $\mathcal{A}^{-1}\varphi = \frac{1}{w}F \in \mathcal{D}_K(X)$ and

$$\mathcal{A}^{-1}\varphi \times \varphi_\lambda = \frac{1}{w}\tilde{F}(\lambda)\varphi_\lambda = \varphi^*(\lambda)\varphi_\lambda.$$

Combining these formulas we have

$$\mathcal{A}^*(\varphi * \psi^\lambda) = \mathcal{A}^{-1}\varphi \times \mathcal{A}^*(\psi^\lambda). \tag{7.13}$$

Now $\psi \in \mathcal{D}_W(A)$ is a superposition

$$\psi(a) = \int_{\mathfrak{a}^*} \psi^*(\lambda)\psi^\lambda(a)\,d\lambda$$

so the identity (7.12) follows from (7.13) for such ψ. For $\psi \in \mathcal{E}_W(A)$ the identity follows by an approximation because φ and $\mathcal{A}^*(L\varphi)$ have compact support and $\mathcal{A}^*$ is continuous on $\mathcal{E}_W(A)$. □

Theorem 7.1 implies the following inversion formula which in reality is a special case of (6.2). It appears also in [5].

Corollary 7.2. *The transform $f \to \mathcal{A}f$ has inversion*

$$f = \frac{1}{w}\mathcal{A}^*(L\mathcal{A}f), \qquad f \in \mathcal{D}_K(X).$$

The above results suggest various ways of defining $\mathcal{A}$ on the space $\mathcal{E}'_K(X)$ of K-invariant compactly supported distributions on X although formula (7.1) does not work.

Spherical transform method. If $T \in \mathcal{E}'_K(X)$, the spherical transform

$$\tilde{T}(\lambda) = \int_X \varphi_{-\lambda}(x)\,dT(x)$$

is a W-invariant entire function of exponential type on $\mathfrak{a}^*_c$ and of polynomial growth. (See [9] or [21, Theorem 8.5].) By the Euclidean Paley–Wiener theorem there exists an $S \in \mathcal{E}'_W(A)$ such that $\tilde{T} = S^*$. Thus in accordance with (7.6) we put

$$\mathcal{A}T = S. \tag{7.14}$$

Radon transform method. Because of (2.4) the Radon transform of a distribution $T \in \mathcal{E}'(X)$ is defined by

$$\hat{T}(\varphi) = T(\check{\varphi}), \quad \varphi \in \mathcal{E}(\Xi).$$

If T is K-invariant then so is $\hat{T}$ and since $\Xi = K/M \times A$ under the bijection $(kM, a) \to ka \cdot \xi_o$ we see that $\hat{T}$ has the form $\hat{T} = 1 \otimes \sigma$ where $\sigma \in \mathcal{E}'(A)$. Because of (7.2) we put

(7.15) $\quad \mathcal{A}T = e^{\rho}\sigma.$

Functional analysis method. As remarked $\mathcal{A}^*$ is a bijection of $\mathcal{D}'_W(A)$ onto $\mathcal{D}'_K(X)$. The restriction of $\mathcal{A}^*$ to $\mathcal{E}_W(A)$ is a continuous bijection onto $\mathcal{E}_K(X)$ and in fact a homeomorphism since both spaces are Fréchet. Thus we have $(\mathcal{A}^*)^* : \mathcal{E}'_K(X) \to \mathcal{E}'_W(A)$ bijectively so we can define

(7.16) $\quad \mathcal{A}T = (\mathcal{A}^*)^*(T).$

Proposition 7.3. *All the definitions* (7.14)–(7.16) *coincide.*

The convolution property in Theorem 7.1 extends readily to distributions so

$$\mathcal{A}^*(\mathcal{E}'_W(A) * \psi) = \mathcal{A}^{-1}\big(\mathcal{E}'_W(A)\big) \times \mathcal{A}^*\psi = \mathcal{E}'_K(X) \times \mathcal{A}^*\psi.$$

Thus putting

$$V_\psi = \mathcal{E}'_W(A) * \varphi, \quad \psi \in \mathcal{E}_W(A);$$
$$W_f = \mathcal{E}'_K(X) \times f, \quad f \in \mathcal{E}_K(X),$$

we conclude that

(7.17) $\quad \mathcal{A}^*(V_\psi) = W_{\mathcal{A}^*\psi}.$

Theorem 7.4 (Bagchi–Sitaram). *If X has rank one and $f \in \mathcal{E}_K(X)$ then the closure of the space $W_f = \mathcal{E}'_K(X) \times f$ contains a spherical function.*

The authors use (7.17) to reduce the question to the analogous one for the one-dimensional space $A \sim \mathbf{R}$ where by Schwartz's theorem stated in Section 9 below some exponentials $e^{i\mu}$ and $e^{-i\mu}$ belong to the closure and $\mathcal{A}^*(e^{i\mu} + e^{-i\mu}) = 2\varphi_\mu$.

8. THE FOURIER TRANSFORM ON $X = G/K$

We now go to the notation of Section 6 with the Iwasawa decomposition $G = NAK$, and $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ for the corresponding Lie algebras. For $g \in G$ let $A(g) \in \mathfrak{n}$ be determined by $g = n \exp A(g) k$ $(n \in N, k \in K)$. Given $x = gK$ in X, $b = kM$ in $B = K/M$ we put

$$A(x, b) = A(k^{-1}g)$$

and as usual we put $\rho(H) = \frac{1}{2}\,\mathrm{Trace}(\mathrm{ad}\,H|\mathfrak{n})$. Let $\mathfrak{a}_c^*$ denote the space of complex-valued linear forms on $\mathfrak{a}$.

Given a function f on X we define its *Fourier transform* by

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda+\rho)(A(x,b))}\, dx \tag{8.1}$$

for those $(\lambda, b) \in \mathfrak{a}_c^* \times B$ for which the integral is defined. Many of the principal theorems for Fourier transforms on $\mathbf{R}^n$ have analogs for $X = G/K$.

Inversion formula [19]. For $f \in \mathcal{D}(X)$ we have

$$f(x) = \frac{1}{w} \int_{\mathfrak{a}^*} \int_B \tilde{f}(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} \left|c(\lambda)\right|^{-2} d\lambda\, db$$

where $c(\lambda)$ is Harish-Chandra's c-function.

Plancherel formula [20]. The map $f \to \tilde{f}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times B)$:

$$\int_X \left|f(x)\right|^2 dx = \int_{\mathfrak{a}_+^* \times B} \left|\tilde{f}(\lambda, b)\right|^2 \left|c(\lambda)\right|^{-2} d\lambda\, db. \tag{8.2}$$

Paley–Wiener theorem [21]. The map $f \to \tilde{f}$ maps the space $\mathcal{D}(X)$ onto the space of smooth $\varphi(\lambda, b)$ on $\mathfrak{a}_c^* \times B$ which are holomorphic on $\mathfrak{a}_c^*$ of exponential type (uniformly in B) satisfying the invariance condition

$$\int_B \varphi(\lambda, b) e^{(i\lambda+\rho)(A(x,b))}\, db \text{ is } W\text{-invariant in } \lambda. \tag{8.3}$$

For the next result we refer to Eguchi's paper for full explanations of notation.

The Schwartz theorem [8]. Let $0 < p \leqslant 2$ and $\mathcal{S}^p(X) \subset L^p(X)$ the corresponding Schwartz space. Let $\varepsilon = 2/p - 1$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)$ the space of functions which are holomorphic in the "tube" $\mathfrak{a}_\varepsilon^* \times B$, are rapidly decreasing and satisfy (8.3). Then $f \to \tilde{f}$ is a bijection of $\mathcal{S}^p(X)$ onto $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)$.

These results leave out the space $L^1(X)$ and one should think that a self-respecting Fourier transform should be defined here.

We shall now show (modifying a bit the proof of [31]) that this can be done and that a strong analog of the classical Riemann–Lebesgue lemma holds for $\tilde{f}$ in (8.1).

Let $C(\rho)$ denote the convex hull in $\mathfrak{a}^*$ of the set $\{s\rho\colon\ s \in W\}$ of Weyl group transforms of ρ.

Theorem 8.1. *Let $f \in L^1(B)$. Then there exists a subset $B' \subset B$ with $B - B'$ of measure 0 such that for each $b \in B'$*

(i) *$\tilde{f}(\lambda, b)$ is defined for λ in the tube $\mathfrak{a}^* + iC(\rho)$ and holomorphic in its interior.*
(ii) *$\lim_{\xi \to \infty} \tilde{f}(\xi + i\eta, b) = 0$ uniformly for $\eta \in C(\rho)$.*

Proof. Let $\lambda = \xi + i\eta$ where $\xi \in \mathfrak{a}^*$, $\eta \in C(\rho)$. Then

$$(8.4) \qquad \int_B \left|\tilde{f}(\lambda, b)\right| db \leqslant \int_X \left|f(x)\right| \int_B e^{(\eta+\rho)(A(x,b))}\, db\, dx.$$

The integral over B is the spherical function $\varphi_{-i\eta}$ which is bounded by 1 [30]. Thus

$$\left\|\tilde{f}(\lambda, \cdot)\right\|_1 \leqslant \|f\|_1$$

and for each $\lambda \in \mathfrak{a}^* + iC(\rho)$, $\tilde{f}(\lambda, b)$ exists for all b in a subset $B_\lambda \subset B$ of full invariant measure. Let

$$B' = B'(f) = \bigcap_{s \in W} B_{is\rho}.$$

For the statements (i) and (ii) we may assume $f \geqslant 0$ in (8.1). Since $b \in B_{is\rho}$ for each $s \in W$ we have

$$(8.5) \qquad \int_X f(x) e^{(s\rho+\rho)(A(x,b))}\, dx < \infty.$$

Fix $b \in B'$, $\eta \in C(\rho)$. Then

$$(8.6) \qquad \int_X f(x) e^{(\rho+\eta)(A(x,b))}\, dx = \sum_{\sigma \in W} \int_{X_\sigma} f(x) e^{(\rho+\eta)(A(x,b))}\, dx$$

where

$$X_\sigma = \left\{x \in X \colon \sigma\big(A(x,b)\big) \in \overline{\mathfrak{a}^+}\right\}.$$

Replace $\eta(A(x,b))$ by $(\sigma\eta)\,(\sigma(A(x,b)))$ and let $(\sigma\eta)^+$ be the element in $\overline{\mathfrak{a}^*_+}$, which is W-conjugate to $\sigma\eta$. Then since $(\sigma\eta)^+ - \sigma\eta \geqslant 0$ on $\mathfrak{a}^+$ we have

$$\int_{X_\sigma} f(x) e^{(\rho+\eta)(A(x,b))}\, dx \leqslant \int_{X_\sigma} f(x) e^{(\rho+(\sigma\eta)^+)(\sigma(A(x,b)))}\, dx.$$

Now by Lemma 8.3, Ch. IV in [24]

$$\overline{\mathfrak{a}^*_+} \cap C(\rho) = \overline{\mathfrak{a}^*_+} \cap (\rho + {}^-\mathfrak{a}^*),$$

where

$$^{-}\mathfrak{a}^* = \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \mu \rangle \leqslant 0 \text{ for } \mu \in \mathfrak{a}^*_+\}.$$

Thus

$$(\sigma\eta)^+ \in \overline{\mathfrak{a}^*_+} \cap (\rho + {}^{-}\mathfrak{a}^*),$$

whence

$$(8.7) \qquad (\sigma\eta)^+ - \rho \leqslant 0 \quad \text{on } \mathfrak{a}^+.$$

Thus the last integral is bounded by

$$\int_{X_\sigma} f(x) e^{(\rho + \sigma^{-1}\rho)(A(x,b))}\, dx < \infty$$

by (8.5). This shows by (8.6) that if $b \in B'$ and $\lambda \in \mathfrak{a}^* + iC(\rho)$ the integral (8.1) is absolutely convergent. The holomorphy statement follows by Morera's theorem. This proves (i).

For part (ii) we use the Radon transform (6.1). Since $f \in L^1(X)$, $\hat{f}(\xi)$ exist for almost all $\xi \in \Xi$ (see [20, II, §1]). Since $(kM, a) \to ka \cdot \xi_0$ is a diffeomorphism of K/M onto Ξ we write $\hat{f}(kM, a)$ for $\hat{f}(ka \cdot \xi_0)$. Enlarging B' to another subset of B of full invariant measure we may assume $\hat{f}(b, a)$ exists for $b \in B'$ and almost all a. Now we have

$$(8.8) \qquad \int_X f(x)\, dx = \int_{AN} f(anK)\, da\, dn$$

for suitable Haar measures on A and N. Applying this to the function $x \to f(k \cdot x)$ with $kM = b \in B'$ we get

$$(8.9) \qquad \int_X f(x)\, dx = \int_A \hat{f}(kM, a)\, da$$

so since $A(an \cdot o) = \log a$,

$$(8.10) \qquad \tilde{f}(\lambda, kM) = \int_A \hat{f}(kM, a) e^{(\rho+\eta)(\log a)} e^{-i\xi(\log a)}\, da$$
$$= \sum_{s \in W} \int_{s^{-1}A^+} \hat{f}(kM, a) e^{(\rho+\eta)(\log a)} e^{-i\xi(\log a)}\, da.$$

Now $a \in s^{-1}A^+$ implies $sa \in A^+$ and

$$\eta(\log a) = (s\eta)(s \log a) \leqslant (s\eta)^+(s \log a) \leqslant \rho(s \log a)$$

by (8.7). Thus on $s^{-1}A^+$,

$$\hat{f}(kM,a)e^{(\rho+\eta)(\log a)} \leqslant \hat{f}(kM,a)e^{(\rho+s^{-1}\rho)(\log a)}. \tag{8.11}$$

For $b = kM \in B'$ the integral in (8.1) is absolutely convergent so by (8.9) the function

$$a \to \hat{f}(kM,a)e^{(\rho+\eta)(\log a)} \tag{8.12}$$

belongs to $L^1(A)$. The first part of (8.10) combined with the Riemann–Lebesgue lemma for the Fourier transform on A shows that for each $\eta \in C(\rho)$

$$\lim_{\xi\to\infty} \tilde{f}(\lambda, b) = 0.$$

For the uniform convergence in (ii) we use the second part of (8.10). Let f_n be positive in $\mathcal{D}(X)$ such that $f_n \to f$ a.e. and $f_n(x) \leqslant f(x)$. In (8.10) and (8.11) we replace f by the function $g_n = f - f_n$. Then

$$\begin{aligned}\left|\tilde{g}_n(\lambda, kM)\right| &\leqslant \sum_{s\in W} \int_{s^{-1}A^+} \hat{g}_n(kM,a)e^{(\rho+\eta)(\log a)}\,da \\ &\leqslant \sum_{s\in W} \int_{s^{-1}A^+} \hat{g}_n(kM,a)e^{(\rho+s^{-1}\rho)(\log a)}\,da,\end{aligned}$$

which tends to 0 as $n \to \infty$ by (8.9), (8.12) and the dominated convergence theorem. Thus given $\varepsilon > 0$ we can fix N such that $|\tilde{g}_N(\lambda, kM)| < \varepsilon$ for all $\lambda \in \mathfrak{a}^* + iC_\rho$. By the Paley–Wiener theorem for $\mathcal{D}(X)$ there is an L such that $|\tilde{f}_N(\xi + i\eta, kM)| \leqslant \varepsilon$ for $|\xi| > L$ and $\eta \in C(\rho)$. Since $\tilde{g}_N = \tilde{f} - \tilde{f}_N$ this proves (ii). □

Remark. Another version of (ii) involving the L^1 norm over B is given in [45].

9. SPECTRAL ANALYSIS ON X

A theorem of Schwartz [48] states that if f is a function in $\mathcal{E}(\mathbf{R})$ $(f \not\equiv 0)$ the closed subspace of $\mathcal{E}(X)$ (in its usual Fréchet space topology) generated by all the translates of f contains an exponential $e^{\mu x}$ for some $\mu \in \mathbf{C}$.

We shall now give the proof from [32] of the following analog of Schwartz's theorem.

Theorem 9.1. *Let $X = G/K$ have rank one and $f \neq 0$ a function in $\mathcal{E}(X)$. Then the closed subspace V_f of $\mathcal{E}(X)$ generated by the G-translates of f contains a function*

$$x \to e_{\mu,b}(x) = e^{\mu(A(x,b))}$$

for some $\mu \in \mathfrak{a}_c^*$.

For this we consider for $\lambda \in \mathfrak{a}_c^*$ the *Poisson transform*

$$\mathcal{P}_\lambda : F(b) \to f(x), \quad F \in L^1(B),$$

where

$$f(x) = \int_B e^{(i\lambda+\rho)(A(x,b))} F(b)\, db. \tag{9.1}$$

The element λ is said to be *simple* if $\mathcal{P}_\lambda$ is injective. The simplicity criterion for λ (see [22]) implies that for each $\lambda \in \mathfrak{a}_c^*$ one of the transforms $s\lambda$ $(s \in W)$ is simple. Consider now the spherical function φ_λ (7.3) which can also be written

$$\varphi_\lambda(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}\, db.$$

We know from [25, III, Lemma 2.3] that if $-\lambda$ is simple then the closed space $\mathcal{E}_{(\lambda)}(X) \subset \mathcal{E}(X)$ generated by the G-translates of φ_λ contains the space $\mathcal{P}_\lambda(L^2(B))$.

Coming to the proof of the theorem we conclude from the Bagchi–Sitaram result (7.4) that the space V_f^K of K-invariants in V_f contains a spherical function φ_λ. By the simplicity result quoted, either λ or $-\lambda$ is simple so we can take $-\lambda$ simple. Thus by the conclusion above, V_f contains the space $\mathcal{P}_\lambda(L^2(B))$. Now by [25, III, Exercise B1, pp. 371 and 570],

$$e^{(i\lambda+\rho)A(x,eM)} = \sum_{\delta \in \hat{K}_M} d(\delta)\varphi_{\lambda,\delta}(x), \tag{9.2}$$

with δ and $\hat{K}_M$ as in (6.5) and

$$\varphi_{\lambda,\delta}(x) = \int_K e^{(i\lambda+\rho)(A(x,k))} \langle \delta(k)v, v\rangle\, dk.$$

Thus $\varphi_{\lambda,\delta} \in V_f$ so since (9.2) converges in the topology of $\mathcal{E}(X)$ the theorem follows.

Remark. Since Schwartz's theorem fails for $\mathbf{R}^n$ $(n > 1)$ the proof above via the Bagchi–Sitaram theorem is limited to the case of rank $X = 1$. However, this does not rule out the possibility that Theorem 9.1 might remain valid for X of higher rank.

10. FURTHER RESULTS ON THE FOURIER TRANSFORM

A result of Hardy's [16] shows limitations on how fast a function on $\mathbf{R}^n$ and its Fourier transform can decay at ∞. Precisely, if

$$\left|f(x)\right| \leqslant Ae^{-\alpha|x|^2}, \quad \left|\tilde{f}(u)\right| \leqslant Be^{-\beta|u|^2}, \quad \alpha, \beta > 0,$$

and if $\alpha\beta > 1/4$ then $f = 0$. Sitaram and Sundari [53] proved an analog for a class of spaces X and Sengupta [49] extended this to all X. Many other variations of the result have been proved by Ray and Sarkar, Cowling, Sitaram and Sundari, Narayanan and Ray, Shimeno, Thangavelu. (See References.)

The following classical result is closely related to Wiener's Tauberian theorem. *Let $f \in L^1(\mathbf{R}^n)$ such that $\tilde{f}(u) \neq 0$ for all $u \in \mathbf{R}^n$. Then the translates of f span a dense subspace of* $\mathbf{R}^n$. Many papers deal with analogies of this result for semisimple Lie groups and symmetric spaces. See [10,46,51,52,45,36] for a sample.

The polar coordinate representation $(kM, a) \to kaK$ of X identifies X with $K/M \times A^+$ up to a null set. Thus one might interpret the Plancherel formula (8.2) as identifying X with its "dual". But in contrast to $\mathbf{R}^n$ where the Fourier transform is essentially equal to its inverse, the Fourier transform

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda+\rho)(A(x,b))}\, dx, \tag{10.1}$$

and the inverse

$$(\mathcal{F}^{-1}\varphi)(x) = \int_{\mathfrak{a}^* \times B} \varphi(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} \left|c(\lambda)\right|^2 d\lambda\, db \tag{10.2}$$

are quite different. Hence it is a natural problem to prove the analog of the Paley–Wiener theorem for $\mathcal{F}^{-1}$.

This was done by A. Pasquale [38] for the spherical transform for X of rank one or the case of G complex, and by N. Andersen [1] in general. Let L denote the Laplacian on X.

Theorem 10.1. *The image of $\mathcal{F}^{-1}(\mathcal{D}(\mathfrak{a}^* \times B))$ consists of the functions f on X satisfying*

$$\left(1 + d(o, x)\right)^m L^n f \in L^2(X) \quad \textit{for all } m, n \in \mathbf{Z}^+$$

and

$$\lim_{n\to\infty} \left\| \left(L + \langle \rho, \rho \rangle\right)^n \right\|_2^{1/2n} < \infty.$$

Another characterization was given by Pesenson [39], namely

$$\|L^\sigma f\|_2 \leqslant \left(\omega^2 + |\rho|^2\right)^\sigma \|f\|_2 \quad \text{for all } \sigma > 0.$$

REFERENCES

[1] Andersen N.B. – Real Paley–Wiener theorems for the inverse Fourier transform on a Riemannian symmetric space, Pacific J. Math. **213** (2004) 1–13.

[2] Anker J. – The spherical Fourier transform of rapidly decreasing functions—a simple proof of a characterization due to Harish-Chandra, Helgason, Trombi and Varadarajan, J. Funct. Anal. **96** (1991) 331–349.

[3] Bagchi S.C., Sitaram A. – Spherical mean-periodic functions on semisimple Lie groups, Pacific J. Math. **84** (1979) 241–250.
[4] van den Ban E.P., Schlichtkrull H. – Asymptotic expansions and boundary values of eigenfunctions on a Riemannian symmetric space, J. Reine Angew. Math. **380** (1987) 108–165.
[5] Beerends R.J. – The Fourier Transform of Harish-Chandra's c-function and inversion of the Abel transform, Math. Ann. **277** (1987) 1–23.
[6] Berenstein C.A., Casadio Tarabusi A.E. – Inversion formulas for k-dimensional Radon transform in real hyperbolic spaces, Duke Math. J. **62** (1991) 613–631.
[7] Cowling M., Sitaram A., Sundari M. – Hardy's uncertainty principle on semisimple Lie groups, Pacific J. Math. **192** (2000) 293–296.
[8] Eguchi M. – Asymptotic Expansions of Eisenstein Integrals and Fourier transform on symmetric spaces, J. Funct. Anal. **34** (1979) 167–216.
[9] Eguchi M., Hashizume M., Okamoto K. – The Paley–Wiener theorem for distributions on symmetric spaces, Hiroshima Math. J. **3** (1973) 109–120.
[10] Ehrenpreis L., Mautner F. – Some properties of the Fourier transform on semisimple Lie groups I, Ann. of Math. **61** (1955) 406–439.
[11] Funk P. – Über eine geometrische Anvendung der Abelschen Integralgleichung, Math. Ann. **77** (1916) 129–135.
[12] Gelfand I.M., Graev M.I. – The geometry of homogeneous spaces, group representations in homogeneous spaces and questions in integral geometry related to them, Amer. Math. Soc. Transl. **37** (1964).
[13] Gindikin S.G., Karpelevic F.I. – On a problem in integral geometry, Chebotarev Mem. Vol. Kazan Univ. 1964; Selecta Math. Sovietica **1** (1981) 169–184.
[14] Gonzalez F., Quinto E.T. – Support theorems for Radon transforms on higher rank symmetric spaces, Proc. Amer. Math. Soc. **122** (1994) 1045–1052.
[15] Harish-Chandra – Spherical functions on a semisimple Lie group II, Amer. J. Math. **80** (1958) 553–613.
[16] Hardy G.H. – A theorem concerning Fourier transforms, J. London Math. Soc. **8** (1933) 227–231.
[17] Helgason S. – Duality and Radon transforms for symmetric spaces, Amer. J. Math. **85** (1963) 667–692.
[18] Helgason S. – A duality in integral geometry. Some generalizations of the Radon transform, Bull. Amer. Math. Soc. **70** (1964) 435–446.
[19] Helgason S. – Radon–Fourier transforms on symmetric spaces and related group representations, Bull. Amer. Math. Soc. **71** (1965) 757–763.
[20] Helgason S. – A duality for symmetric spaces with applications to group representations, Adv. Math. **5** (1970) 1–154.
[21] Helgason S. – The surjectivity of invariant differential operators on symmetric spaces, Ann. of Math. **98** (1973) 451–480.
[22] Helgason S. – A duality for symmetric spaces with applications to group representations II. Differential equations and eigenspace representations, Adv. Math. **22** (1976) 187–219.
[23] Helgason S. – Ranges of Radon transforms, in: Proc. Symp. Appl. Math., vol. 27, Amer. Math. Soc., Providence, RI, 1982, pp. 63–70.
[24] Helgason S. – Groups and Geometric Analysis, Academic Press, San Diego, CA, 1984, Amer. Math. Soc., Providence, RI, 2000.
[25] Helgason S. – Geometric Analysis on Symmetric Spaces, Math. Surveys and Monographs, vol. 39, Amer. Math. Soc., Providence, RI, 1994.
[26] Helgason S. – The Radon Transform, Birkhäuser, Boston, 1999.
[27] Helgason S., Non-Euclidean analysis, in: Proc. Janos Bolyai Memorial Conference, Budapest, July 2002.
[28] Helgason S. – Differential operators on homogeneous spaces, Acta Math. **102** (1959) 239–299.
[29] Helgason S. – Fundamental solutions of invariant differential operators on symmetric spaces, Amer. J. Math. **86** (1964) 565–601.
[30] Helgason S., Johnson K. – The bounded spherical functions on symmetric spaces, Adv. Math. **3** (1969) 586–593.

[31] Helgason S., Rawat R., Sengupta J., Sitaram A. – Some remarks on the Fourier transform on a symmetric space, Tech. Report, Ind. Stat. Inst. Bangalore, 1998.
[32] Helgason S., Sengupta J. – Preprint, 1997.
[33] John F. – Plane Waves and Spherical Means, Wiley, New York, 1955.
[34] Kostant B. – On the existence and irreducibility of certain series of representations, in: Gelfand I.M. (Ed.), Lie Groups and Their Representations, Halsted, New York, 1975, p. 231, 329.
[35] Lax P.D., Phillips R.S. – Translation representation for the solution of the non-Euclidean wave equation, Comm. Pure Appl. Math. **32** (1979) 617–667.
[36] Mohanty P., Ray S.K., Sarkar R.P., Sitaram A. – Helgason Fourier transform for symmetric spaces II, J. Lie Theory **14** (2004) 227–242.
[37] Narayanan E.K., Ray S.K. – L^p version of Hardy's theorem on semisimple Lie groups, Proc. Amer. Math. Soc. **130** (2002) 1859–1866.
[38] Pasquale A. – A Paley–Wiener theorem for the inverse spherical transform, Pacific J. Math. **193** (2000) 143–176.
[39] Pesenson I., Deconvolution of band limited functions on non-compact symmetric spaces, Preprint.
[40] Radon J. – Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, Ber. Verth. Sächs. Akad. Wiss. Leipzig. Math. Nat. kl. **69** (1917) 262–277.
[41] Ray S.K., Sarkar R.P. – Cowling–Price theorem and characterization of heat kernel of symmetric spaces, Preprint.
[42] Rouvière F. – Inverting Radon transforms; the group-theoretic approach, Enseign. Math. **47** (2001) 205–252.
[43] Rouvière F. – Geodesic Radon transforms on symmetric spaces, Preprint, 2004.
[44] Rouvière F. – Sur la transformation d'Abel de groupes de Lie semisimple de rang un, Ann. Scuola. Norm. Sup. Pisa **10** (1983) 263–290.
[45] Sarkar R., Sitaram A. – The Helgason Fourier Transform for Symmetric Spaces. C.S. Seshadri Festschrift Volume, Birkhäuser, Boston, 2003.
[46] Sarkar R.P. – Wiener Tauberian theorem for rank one symmetric spaces, Pacific J. Math. **186** (1998) 349–358.
[47] Sawyer P. – The Abel transform on symmetric spaces of noncompact type, Amer. Math. Soc. Transl. (2) **210** (2003) 331–355.
[48] Schwartz L. – Théorie generale des fonctions moyenne-periodiques, Ann. of Math. **48** (1947) 857–929.
[49] Sengupta J. – The Uncertainty Principle on Riemannian symmetric spaces of the noncompact type, Proc. Amer. Math. Soc. **128** (2000) 2493–2499.
[50] Shimeno N. – An analog of Hardy's theorem for the Harish-Chandra transform, Hiroshima Math. J. **31** (2001) 383–390.
[51] Sitaram A. – An analog of the Wiener Tauberian theorem for spherical transforms on semisimple Lie groups, Pacific J. Math. **89** (1980) 439–445.
[52] Sitaram A. – On an analog of the Wiener Tauberian theorem for symmetric spaces of the noncompact type, Pacific J. Math. **133** (1988) 197–208.
[53] Sitaram A., Sundari M. – An analog of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups, Pacific J. Math. **177** (1997) 187–200.
[54] Thangavelu S. – An Introduction to the Uncertainty Principle, Birkhäuser, Basel, 2004.
[55] Thangavelu S. – Hardy's theorem for the Helgason Fourier transform on noncompact rank one symmetric spaces, Colloq. Math. **94** (2002) 263–280.

(Received February 2005)