Fast approximation schemes for $K_{3,3}$-minor-free or K_5-minor-free graphs ⋆

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Abstract

We prove that any graph excluding one of K_5 or $K_{3,3}$ as a minor has local treewidth bounded by $3k + 4$. As a result, we can design *practical* polynomial-time approximation schemes for both minimization and maximization problems on these classes of non-planar graphs.

1 Introduction

The genesis of our result lies in a marvellous result by Baker [Bak94] for designing polynomial-time approximation schemes for problems defined on planar graphs. The technique decomposes a planar embedding by successively deleting outer faces; vertices are given a level number corresponding to the iteration in which they are deleted. Removing only those vertices with level number congruent to i mod k results in a k-outerplanar graph; there are k choices of i, and every vertex is in exactly $k-1$ of the resulting k-outerplanar

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graphs. Many NP-complete problems (such as maximum independent set, minimum dominating set, and minimum vertex cover) can be solved exactly on k-outerplanar graphs by dynamic programming, and it can be shown that some choice of i results in a solution for the original graph that is within a factor of $(1 + 1/k)$ of the optimal.

A planar graph is a graph excluding both $K_{3,3}$ and K_5 as minors. Our results extend Baker's by permitting only one of these to be excluded as a minor from the class of input graphs, while retaining the practical nature of Baker's work (no huge hidden constants). In order to explain our methods, it is necessary to survey previous work making use of [DST96,Epp00] or varying [BP92,Che98] Baker's idea.

Baker's technique can be extended by replacing bounded outerplanarity with bounded local treewidth. A k-tree is either a $(k + 1)$-clique or contains a vertex v whose neighbors induce a clique in G of size at most k and whose removal results in a k-tree. The *treewidth* of a graph G, $\mathrm{tw}(G)$ is the minimum k such that G is the subgraph of a k-tree. As with k-outerplanar graphs, a wide range of NP-complete problems can be solved in linear time for graphs of bounded treewidth (see e.g. [Arn85,Bod96,Bod97]). The decomposition by deleting every kth face is replaced by deleting every kth level of a breadth-first tree of G, provided that the treewidth of the resulting graphs is a function of k.

To provide this guarantee, we define the *local treewidth* of a graph G by using the function $\mathrm{ltw}^G : \mathbb{N} \to \mathbb{N}$ that associates with every $r \in \mathbb{N}$ the maximal treewidth of an r-neighborhood in G. We set $\mathrm{ltw}^G(k) = \max_{v \in V(G)}\{\mathrm{tw}(G[N_G^k(v)])\}$, and we say that a graph class $\mathcal{C}$ has *bounded local treewidth* when there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ and $k \in \mathbb{N}$, $\mathrm{ltw}^G(k) \leq f(k)$. (We adopt the notation $G[S]$ for the subgraph of G induced by the vertices in set S, and $N_G^k(S)$ for the vertices of distance k from some vertex of $S \subseteq V(G)$; if $S = \{v\}$ we simply use the notation $N_G^k(v)$.)

A graph is called an *apex graph* if deleting one vertex produces a planar graph. A graph G is a *minor* of a graph H if H can be obtained from a subgraph of G by contracting edges. A graph class $\mathcal{C}$ is a *minor-closed* class if the minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is *H-minor free* if $H \notin \mathcal{C}$. Eppstein [Epp00] showed that a minor-closed graph class $\mathcal{C}$ has bounded local treewidth if and only if $\mathcal{C}$ is H-minor free for some apex graph H.

As a consequence, a wide number of graph theoretic (minimization or maximization) problems such as maximum independent set, maximum triangle matching, minimum dominating set, minimum edge dominating set and minimum vertex cover have polynomial-time approximation schemes when the

input is a graph excluding some apex graph as a minor. These schemes can be constructed simply by replacing the k-outerplanar graphs in Baker's work with the bounded treewidth graphs defined by deleting levels of the breadth-first search tree, and by replacing the dynamic programming on k-outerplanar graphs with dynamic programming on the tree decompositions. However, the design of such schemes provides practical approximation algorithms only for those graph classes where the local treewidth is a simple function (preferably linear) of r.

Eppstein was able to prove that for any surface S of (orientable or non-orientable) genus equal to γ, there exists a constant c such that for all the graphs G embeddable in S and for all $k \geq 0$, $\mathrm{ltw}^G(k) \leq c \cdot \gamma \cdot k$. Unfortunately, the constants involved in Eppstein's work are immense, resulting in polynomial-time approximation schemes that are unusable for any practical purpose. So far, the only graph class with local treewidth small enough to give practical PTAS is the class of planar graphs; Bodlaender [Bod98] proved that for planar G, $\mathrm{ltw}^G(k) \leq 3k - 1$.

In this paper we generalize Bodlaender's result to H-minor free graphs when H is either K_5 or $K_{3,3}$. Chen [Che98], by a different method, decomposes such graphs into covers of bounded treewidth; our results improve his, and we will discuss them in more detail at the end. We will prove first that if an apex graph H is a single-crossing graph (can be drawn on the plane with at most one crossing) then the local treewidth of graphs excluding H as a minor is bounded by $3k + c_H$ where c_H is a constant depending only on H.

Our results are based on a characterization of our graph classes in terms of a graph summation operation. Given graphs G_1 and G_2 with disjoint vertex sets and an integer $m \geq 0$, we let X_i $(i = 1, 2)$ be a set of m pairwise adjacent vertices in G_i, and let G_i' $(i = 1, 2)$ be obtained from G_i by deleting some (possibly none) edges with both endpoints in X_i. For a bijection $g : X_1 \to X_2$, the *m-sum* G of G_1 and G_2 (denoted $G = G_1 \oplus_m G_2$ or simply $G = G_1 \oplus G_2$) is the graph obtained from the union of G_1' and G_2' by identifying x with $g(x)$ for all $x \in X_1$. We will call the images of the vertices of X_1 in $G_1 \oplus_m G_2$ the *join set*.

Our main result will make use of the following three easy lemmas that we present without proof in this extended abstract.

Lemma 1 *For any graph G and subgraph G' of G, $\mathrm{ltw}^{G'}(k) \leq \mathrm{ltw}^G(k)$, for any $k \geq 0$.*

Lemma 2 *For any two graphs G and H, $\mathrm{tw}(G \oplus H) \leq \max\{\mathrm{tw}(G), \mathrm{tw}(H)\}$.*

Lemma 3 *For any graph G, any set R of pairwise adjacent vertices of $V(G)$, any $v \in R$, and any $k \geq 0$, $\mathrm{tw}(G_1[N^k_{G_1}(R)]) \leq \mathrm{tw}(G_1[N^{k+1}_{G_1}(v)])$.*

The next lemma shows how local treewidth bounds are preserved by m-sums.

Lemma 4 *If G_1 and G_2 are graphs where $\mathrm{ltw}^{G_i}(k) \leq f(k)$, and $H = G_1 \oplus_m G_2$, then $\mathrm{ltw}^H(k) \leq f(k)$.*

Proof Let v be any vertex in $V(H)$ and X the join set of $G_1 \oplus_m G_2$. Without loss of generality, we assume that $v \in G_1$. We will prove that for any $k \geq 0$ the following property $\mathcal{P}_k$ holds: $\mathrm{tw}(N_H^r(v)) \leq f(k)$.

As $\mathcal{P}_k$ is obvious when $m = 0$, we assume that $m > 0$. We will first verify $\mathcal{P}_k$ for the values of k where $N_H^k(v)$ does not contain vertices from G_2. Indeed, in this case $H[N_H^k(v)]$ is a subgraph of G_1 and $\mathcal{P}_k$ holds as $\mathrm{tw}(N_H^k(v)) = \mathrm{tw}(N_{G_i}^k(v)) \leq f(k)$.

It remains to verify $\mathcal{P}_k$ for the values of k where $N_H^k(v)$ contains vertices from both G_1 and G_2. We now define k' as the smallest $j \geq 0$ such that $R = N_{G_1}^j(v) \cap X \neq \emptyset$, and we observe that $H[N_H^k(v)]$ is the $|X_i|$-sum of $G_1[N_{G_1}^k(v)]$ and $G_2[N_{G_2}^{k-k'}(R)]$ with the vertices in X as join set. By Lemma 2, $\mathrm{tw}(H[N_H^k(v)]) \leq \max\{\mathrm{tw}(G_1[N_{G_1}^k(v)]), \mathrm{tw}(G_2[N_{G_2}^{k-k'}(R)])\} \leq \max\{f(k), \mathrm{tw}(G_2[N_{G_2}^{k-k'}(R)])\}$. In the special case where $k' = 0$, $R = \{v\}$ and $\mathcal{P}_k$ holds, as $\mathrm{tw}(G_2[N_{G_2}^{k-k'}(v)]) \leq f(k)$. If $k' > 0$ then $\mathrm{tw}(G_2[N_{G_2}^{k-k'}(R)]) \leq \mathrm{tw}(G_2[N_{G_2}^{k-1}(R)]) \leq \mathrm{tw}(G_2[N_{G_2}^{k-1}(R)]) \leq f(k)$ by Lemma 3. □

We are now ready to prove our main result on the bounded local treewidth of single-crossing graphs.

Theorem 5 *For any single-crossing graph H there is an integer $c_H \geq 0$ such that for any graph excluding H as a minor and for all $r \geq 0$, $\mathrm{ltw}^G(r) \leq 3r + c_H$.*

Proof Robertson and Seymour [RS93] proved that for any single-crossing graph H there is an integer $c_H \geq 0$ such that every graph excluding H as a minor may obtained by 0-,1-,2-, and 3-sums, starting from planar graphs and graphs of treewidth at most c_H. Let G be a graph excluding H. By Robertson and Seymour's result, there is a sequence of graphs $L = (G_0, \ldots, G_t)$ consisting of either planar graphs or graphs of treewidth at most c_H and a sequence of indices $(m_1, \ldots, m_t), \forall_{i=1,\ldots,t} k_j \leq 3$ such that $G = G_0 \oplus_{m_1} G_1 \oplus_{m_2} \cdots \oplus_{m_{t-1}} G_{t-1} \oplus_{m_t} G_t$. We will use induction on t. If $t = 0$, and G_0 is planar, the result is a consequence of Bodlaender's result bounding the local treewidth of planar graphs. If $t = 0$ and G_0 has treewidth at most c_H then the induction base follows from the easy fact that $\forall_{k \geq 0}\ \mathrm{lwt}^G(k) \leq \mathrm{tw}(G)$. For the inductive step, suppose that $H = G_0 \oplus_{m_1} G_1 \oplus_{m_2} \cdots \oplus_{m_{t-1}} G_{t-1}$ has local treewidth at most $3r + c_H$. By Lemma 4, we conclude that $\mathrm{ltw}^G(r) = \mathrm{ltw}^{H \oplus_{m_t} G_t}(r) \leq 3r + c_H$. □

Wagner [Wag37] showed that a graph has no minor isomorphic to $K_{3,3}$ (K_5) if and only if it can be obtained by 0-,1-, and 2-sums (0-,1-,2-, and 3-sums), starting from planar graphs and K_5 (V_8). By V_8 we denote the graph obtained from a cycle of length 8 by joining each pair of diagonally opposite vertices by an edge. As both K_5 and V_8 have treewidth four, the value of the constant c_H in the proof of the previous theorem is four, and we have the following result.

Theorem 6 *If G is a K_5-minor free or $K_{3,3}$-minor free graph then* $\mathrm{ltw}^G(k) \leq 3k+4$. $\square$

Chen [Che98] gives a different and more complicated construction of k different subgraphs of the input graph, each of bounded treewidth, such that every vertex of the input graph is in exactly $k-1$ of them. Our construction improves his in three respects. First, for K_5-minor free graphs, his construction yields treewidth at most $6k-7$ and ours yields treewidth at most $3k+4$. This results in immediate improvement in the running time of polynomial-time approximation schemes; as an example, to approximate maximum independent set within a factor of $1-1/k$, Chen's construction yields an algorithm with time $O(k2^{24k-28}n)$ and ours gives time $O(k2^{12k+16}n)$ (both methods require $O(n^2)$ time to construct the k subgraphs). Second, for a technical reason, Chen's construction will not work for minimization problems (as discussed in the last section of his paper) but ours will. Finally, our approach offers a blueprint for generalization to other excluded minors through structure theorems such as Wagner's.

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