Network Design and Game Theory
Spring 2008
Mid-Term Exam

Instructor: Mohammad T. Hajiaghayi

March 31, 2008

1. Show that the *max-cut* problem is a special case of the unique coverage problem. (Remember, in the max-cut problem for a given graph with nonnegative edge-weights, we want to partition its vertices into two sets so as to maximize the total weight of edges between them.) **(20 points)**

2. In this problem, we consider the *node-weighted Steiner tree* problem. In this problem we are given a graph G with nonnegative weights assigned to its nodes. Let A be a subset of the nodes of G, called *terminals*, whose weights are zero . The goal in node-weighted Steiner tree is to find a connected subgraph (sub-tree) of G containing all the nodes of A so as to minimize the sum of the weights of nodes in this subgraph.

   - Show that the (edge-weighted) Steiner tree problem considered in the class is a special case of the node-weighted Steiner tree problem **(10 points)**.
   - Show that the node-weighted Steiner tree is $\Omega(\log n)$-hard[1], unless $P = NP$, by a reduction from set cover. **(20 points)**

[1] Indeed there is a sophisticated algorithm for this problem with an $O(\log n)$ approximation factor. Thus the hardness result that you show in this part is indeed tight.

3. In this problem, we design algorithms for the *prize-collecting Steiner forest* problem in several steps.

   - First show that the weighted vertex cover problem, in which given a graph G with nonnegative weights assigned to its nodes, one asks for a set of nodes with minimum sum of weights such that every edge of G has at least one endpoint in C, has a 2-approximation algorithm. **(10 points)**
   - In the prize-collecting Steiner forest problem, given a graph $G = (V, E)$, a set of pairs $\mathcal{P} = \{(s1, t1), (s2, t2), \ldots, (s_\ell, t_\ell)\}$, a non-negative cost function $c : E \to Q_+$, and finally a non-negative penalty function $\pi : P \to Q_+$, our goal is a minimum-cost way of buying a set of edges and paying the penalty for those pairs which are not connected via bought edges. Show that this problem on trees has a 2-approximation algorithm (Hint: you may use the previous part of this problem). **(20 points)**
   - Show that prize-collecting Steiner forest in general graphs has an $O(\log n)$ approximation algorithm. **(20 points)**
   - Show that prize-collecting Steiner forest in trees indeed has a polynomial-time algorithm. **(20 extra points that you may ignore it)**