# The Satisfiability Threshold of Random 3-SAT Is at Least 3.52

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**Abstract**

We prove that a random 3-SAT instance with clause-to-variable density less than 3.52 is satisfiable with high probability. The proof comes through an algorithm which selects (and sets) a variable depending on its degree and that of its complement.

## 1 Introduction

There is much interest in understanding "phase transitions" in mathematics, computer science, and mathematical physics, and in particular the $k$-SAT phase transition. In the standard model for random $k$-SAT, a random $k$-CNF formula $F(n, cn)$ with $n$ variables and density $c$ has $m = cn$ random clauses independently selected uniformly at random, with replacement, from among the $2^k \binom{n}{k}$ proper clauses of length $k$. The Satisfiability Threshold Conjecture asserts that for each $k \geqslant 2$, there exists a constant $c_k$ such that for all constants $c < c_k$, $F(n, cn)$ is a.a.s. (asymptotically almost surely) satisfiable, while for $c > c_k$ it is a.a.s. unsatisfiable.

The case of 2-SAT is well understood, with Chvátal and Reed [CR92], Geordt [Goe96], and Fernandez de la Vega [FdlV92] independently proving that $c_2 = 1$, and Bollobás, Borgs, Chayes, Kim, and Wilson [BBC$^+$] determined the "scaling window" to be $1 + \Theta(n^{-1/3})$. For $k > 2$, the conjecture remains open. Friedgut proved that for any $k$ and $n$ there is sharp threshold

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$c_k(n)$, leaving open whether $c_k(n)$ has a limit $c_k$. With the threshold behavior not understood, considerable attention has been devoted to proving density bounds below which a formula is a.a.s. satisfiable ("lower bounds" on the putative threshold) and bounds above which it is a.a.s. unsatisfiable ("upper bounds"). For $k = 3$, it is conjectured that $c_3 \approx 4.2$, and the best upper bound is 4.596 [JSV00].[1]

Existing lower bounds for 3-SAT are all algorithmically based. (By contrast, new lower bounds for $k$-SAT are based on the second-moment method [AM02, AP03].) The earliest such bound of 1.63, due to Broder, Frieze and Upfal [BFU93] was based on the "pure-literal rule": successively setting to True literals whose complement does not appear, "reducing" the formula, and repeating. The next, 3.003, due to Frieze and Suen [FS96], used the "shortest-clause rule", setting True a random literal from a random shortest clause. Skipping over a bound of 3.145 by Achlioptas [Ach00], a bound of 3.14 due to Achlioptas and Sorkin [AS00] again selects a literal from a shortest clause, but when the literal is from a 2-clause (the case of interest) it sets the literal either True or False depending (optimally) on the number of other occurrences of the literal and its negation in 2- and 3-clauses; [AS00] extends this to a version which optimally chooses to set one or two literals at a time, and sets them optimally, for a bound of 3.26. [AS00] suggests that better bounds may require looking at literal-degree information, in some way harking back to [BFU93]. This approach was taken up by Kaporis, Kirousis, and Lalas [KKL02], whose algorithm sets a variable of largest degree (a "1-parameter heuristic") to give a bound of 3.42. It was clear that the same approach could be exploited further.

## 2 Result, significance, and open problems

In this paper, we choose a variable according to its degree and that of its complement (a 2-parameter heuristic), to get a bound of 3.52. Kaporis and Lalas [KL], using a similar but not identical heuristic, independently obtained the same bound at around the same time. The purpose of this short abstract is twofold.

First, since the bounds from all heuristics of this sort rely on numerical calculations (notably, solutions to differential equations), it is important to put on record that our calculations and those of [KL] independently justify a value of 3.52, and that we reproduce the 3.42 bound of [KKL02].

[1] A bound of 4.506 due to Dubois, Boufkhad, and Mandler [DBM00] has not appeared in journal-refereed form.

Second, our heuristic (and that of [KL]) efficiently solves denser random instances than any other theoretically justified algorithms; since solving 3-SAT instances is of practical importance, our algorithm may be of practical utility. In that regard, a few remarks. Our heuristic succeeds only with a probability that is asymptotically bounded away from 0, but exploiting a standard one-step backtracking trick brings the asymptotic probability to 1. Also, the algorithms commonly used in practice are Davis-Putnam-type backtracking procedures, quite different from the "greedy" approaches taken in all the works described above. However, it is easy to imagine using the present heuristic as a selection rule for a Davis-Putnam algorithm, preserving the heuristic's theoretically justified behavior on random instances, while gaining the Davis-Putnam algorithm's guarantee of a correct answer on arbitrary instances.

Not present in this short abstract is a rigorous justification of our proof methodology (fairly easy and familiar), nor of the numerical calculations, which to be done rigorously would require theoretically derived Lipschitz bounds on a derivative, and interval-arithmetic calculations employing those bounds, along with a few other technicalities.

Future work could include consideration of a variable's number of appearances and that of its complement separately in 2-clauses and 3-clauses (a 4-parameter heuristic), which is analyzable in the same framework. In at least the 2- and 4-parameter versions of literal-degree heuristics (as opposed to the 1-parameter version), it is not clear how best to select a next literal: does a $(2,3)$ literal (2 positive appearances, 3 negative) trump a $(4,5)$, or vice-versa? In the 4-parameter version, it is also not clear how best to set a chosen literal; this was the question answered in [AS00] for the non-degree-spectrum case. An optimal solution to these questions would be a most interesting theoretical contribution, and could also give significant improvements in the bounds.

## 3 Algorithm

We call a variable with $i$ positive and $j$ negative appearances an $(i,j)$-variable. Our algorithm is defined as follows.

**Algorithm A**
*Input:* A 3-CNF formula.
**begin**

**while** there exists an unset variable
    **choose** an $(i,j)$-degree variable using a selection rule
    **set** $v$ True if $i < j$ and False otherwise

**while** there exists a unit clause
**set** a literal of an arbitrary unit clause True.
**if** an empty clause is generated report *failure*; otherwise report *success*
**end**

The best selection rule we found was this. If there is a "pure" variable (one with $i = 0$ or $j = 0$), select it. Otherwise, choose a variable with maximum discrepancy $|i - j|$, breaking ties in favor of maximal $i + j$. (This identifies a unique unordered pair $\{i, j\}$, and all variables with those degrees are indistinguishable to the algorithm.) This selection rule satisfies formulas up to density 3.52. Other selection rules we tried were less good. Working as above but breaking ties in favor of *minimal* $i + j$ only worked up to density 3.50. Selecting by maximum $i/j$ instead of $i - j$ only worked up to 3.44. Selecting by maximum $\max\{i, j\}$ is equivalent to the approach of [KKL02], and we reproduce their 3.42.

# 4 Analysis

In truth, the "natural" algorithm above is not the one analyzed. Rather than making $\Theta(n)$ iterations, the analyzed algorithm makes a constant number of iterations, in each of which it sets $\Theta(n)$ variables with common degree $\{i, j\}$.

It is easily verified that during the algorithm, the formula remains uniformly random conditioned on its degree sequence. To make the calculations finite, we truncate the degree sequence at some value $h$ ($h = 31$ in our calculations). Then with $n$ the original number of variables, for $i, j < h$ we let $n_{i,j}$ be $1/n$th the current number of variables of degree $(i, j)$; $n_{h,j}$ (and $n_{i,h}$) the similar value for variables with $\geqslant h$ positive (negative) appearances; and $n_{h,h}$ that for variables with $\geqslant h$ positive and negative appearances. Setting a single $(i, j)$-variable produces straightforwardly computable expected changes $\Delta$ (detailed in Appendix A) to the $h^2$-dimensional vector $S$ of values $n_{i,j}$, and each element of $\Delta$ has order only $O(1/n)$, so the differential equation method (see for example [Wor95]) can be used to prove that, as we set $\Theta(n)$ variables with common degree $(i, j)$, the vector $n_{i,j}$ almost surely almost exactly follows a trajectory described by the solution of a differential equation corresponding to $\Delta$.

So instead of selecting an $(i, j)$-variable as in Algorithm A, we use the same selection rule to select a pair $(i, j)$, $i, j \leqslant h$, and we set $n \cdot \min\{\delta, n_{i,j}\}$ $(i, j)$-variables at once. Here $\delta$ is a value of our choosing, which could vary from round to round, but which we fixed at $10^{-6}$. Each such round (including the unit-clause steps it implies) can be described by the differential equation method, and our analysis simply consists of simulating the differential equations for a constant number of rounds. It is clear that after some number of rounds, all the values in $S$ can be made arbitrarily small, and at that point we apply the main theorem of Cooper, Frieze, and Sorkin [CFS02] to show that the remaining formula is satisfiable a.a.s. (The positive side of their result has a natural algorithmic interpretation, so our procedure remains algorithmic to the end.)

# 5 Differential Equations

In this section, we describe the differential equations for the case in which we have a 2-dimensional table for keeping the expected number of variables with $k < h$ positive appearances and $l < h$ negative appearances. Since $dt$ (the "time parameter" described before) is very small, w.l.o.g. we can assume all values of $S$ remain fixed during a round. Using these values, we obtain the new value of $S$ after a round. Suppose we set a variable from cell $n_{k,l}$ True (in other words, set a $(k,l)$-degree variable True). Then writing $\delta$ to denote the *expected* increase to a parameter, for such a "free move" we have:

$$\begin{aligned}
\delta m_3 &= -k\frac{3 \cdot m_3}{\Sigma_m} - l\frac{3 \cdot m_3}{\Sigma_m}, \\
\delta m_2 &= -k\frac{2 \cdot m_2}{\Sigma_m} - l\frac{2 \cdot m_2}{\Sigma_m} + l\frac{3 \cdot m_3}{\Sigma_m}, \\
\delta m_1 &= l\frac{2 \cdot m_2}{\Sigma_m}, \text{ and} \\
\delta n_{ij} &= -\psi(i,k)\psi(j,l) - k \cdot \frac{2 \cdot 1 \cdot m_2 + 3 \cdot 2 \cdot m_3}{\Sigma_m} \cdot \\
&\quad \cdot \frac{(i+j) \cdot n_{ij} - (i+1) \cdot n_{i+1,j} - (j+1) \cdot n_{i,j+1}}{\Sigma_n},
\end{aligned}$$

where $\Sigma_m = \Sigma_n = 2 \cdot m_2 + 3 \cdot m_3$ is the total density of appearances of all variables and $m_1$ is the expected number of unit clauses generated by this free move. Here $\psi(x,y) = 1$ if $x = y$ and zero otherwise. Note also that since by definition $m_1 = 0$ at the start of a round, at the end, $m_1 = \delta m_1$ as given above.

After a free move, we have a number of "forced moves" in which the literals in all $m_1$ unit clauses must be set True to satisfy our formula. A literal in a unit clause is a variable from cell $(k+1,l)$ with probability $\frac{(k+1)n_{k+1,l}}{\Sigma_n}$, or the negation of a variable from cell $(k,l+1)$ with probability

$\frac{(l+1)n_{k,l+1}}{\Sigma_n}$. In either case, in the rest of the formula (excepting the unit clauses) that variable has degree $(k, l)$. Thus the expected number $\rho$ of new unit clauses produced by one such forced move (the Malthus parameter in our Galton-Watson process) is

$$\sum_{0 \leqslant k', l' < h} \frac{(k'+1)n_{k'+1,l'}}{\Sigma_n} l' \frac{2 \cdot m_2}{\Sigma_m} + \frac{(l'+1)n_{k',l'+1}}{\Sigma_n} k' \frac{2 \cdot m_2}{\Sigma_m},$$

where $l' \frac{2 \cdot m_2}{\Sigma_m}$ (see parameter $m_1$ defined above) is the density of new unit clauses after setting a $(k', l')$-degree variable True (which happens with probability $\frac{(k'+1)n_{k'+1,l'}}{\Sigma_n}$) and $k' \frac{2 \cdot m_2}{\Sigma_m}$ is the density of new unit clauses after setting a $(k', l')$-degree variable False (which happens with probability $\frac{(l'+1)n_{k',l'+1}}{\Sigma_n}$). For such a forced move, the expected parameter changes are:

$$\delta' m_3 = \sum_{0 \leqslant k', l' < h} \frac{(k'+1)n_{k'+1,l'}}{\Sigma_n} \delta M_3(k', l', \mathrm{T}) + \frac{(l'+1)n_{k',l'+1}}{\Sigma_n} \delta M_3(k', l', \mathrm{F}),$$

$$\delta' m_2 = \sum_{0 \leqslant k', l' < h} \frac{(k'+1)n_{k'+1,l'}}{\Sigma_n} \delta M_2(k', l', \mathrm{T}) + \frac{(l'+1)n_{k',l'+1}}{\Sigma_n} \delta M_2(k', l', \mathrm{F}),$$

where $\delta M_3(k', l', \mathrm{T})$ has exactly the same formula as $\delta m_3$ defined above, likewise for $\delta M_2(k', l', \mathrm{T})$ and $\delta m_2$, and where by symmetry $\delta M_3(k', l', \mathrm{F}) = \delta M_3(l', k', \mathrm{T})$ and $\delta M_2(k', l', \mathrm{F}) = \delta M_2(l', k', \mathrm{T})$.

Finally, for each $i$ and $j$,

$$\begin{aligned} \delta' n_{i,j} = \sum_{0 \leqslant k', l' < h} & \frac{(k'+1)n_{k'+1,l'}}{\Sigma_n} (\delta N_{i,j}(k', l', \mathrm{T}) \\ & - \psi(i, k'+1) \cdot \psi(j, l')) \\ & + \frac{(l'+1)n_{k',l'+1}}{\Sigma_n} (\delta N_{i,j}(k', l', \mathrm{F}) - \psi(i, k') \cdot \psi(j, l'+1)), \end{aligned}$$

where

$$\begin{aligned} \delta N_{i,j}(k, l, \mathrm{T}) = -k \cdot & \frac{2 \cdot 1 \cdot m_2 + 3 \cdot 2 \cdot m_3}{\Sigma_m} \cdot \\ & \cdot \left( \frac{(i+j) \cdot n_{i,j}}{\Sigma_n} - \frac{(i+1) \cdot n_{i+1,j}}{\Sigma_n} - \frac{(j+1) \cdot n_{i,j+1}}{\Sigma_n} \right) \text{ and} \\ \delta N_{i,j}(k, l, \mathrm{F}) = \delta N_{i,j} & (l, k, \mathrm{T}). \end{aligned}$$

We note that the formula for $\delta' n_{i,j}$ can be obtained by considering the flow which goes in or out for cell $n_{i,j}$.

Reasoning via the Galton-Watson process, we know that the expected number of forced moves is $\frac{m_1}{1-\rho}$. Thus the new expected value of $S$ after setting a small fraction $dt$ of variables from cell $n_{k,l}$ True is: $S + dt(\delta m_2, \delta m_3, \delta n) + dt\frac{m_1}{1-\rho}(\delta' m_2, \delta' m_3, \delta' n)$. If we set a variable from cell $n_{k,l}$ False, the expected changes can be obtained by just swapping the role of $k$ and $l$ in the above description.

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