TIGHT MONOMIALS IN QUANTIZED ENVELOPING ALGEBRAS

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1.We consider a finite graph with set of vertices $I \neq \emptyset$ and set of edges Ω. We assume given an orientation for our graph, i.e two maps $h \mapsto h'$ and $h \mapsto h''$ from Ω to I such that the ends of h are h', h''. (We allow the possibility that Ω contains loops, that is, elements h such that $h' = h''$.) We will show that by a slight extension of the method of [L2] one can associate naturally to this graph an algebra U^- over $\mathbf{Z}[v, v^{-1}]$ (v is an indeterminate) provided with a canonical basis B. In the case where there are no loops this reduces to the construction of [L2] of an integral form of the Drinfeld-Jimbo quantized enveloping algebra corresponding to our graph; in the case where the graph consists of one vertex and one loop, our construction gives the classical Hall algebra (see [M]).

The fact that the Hall algebra arises in this way is not surprising: after all, Ringel's definition [R] of algebras starting from quivers of type A, D, E was an extension of the definition of Hall algebras. What is perhaps surprising is that the theory of canonical bases seems to make sense in a context more general than that of [L2]. For instance it is not clear how to define these canonical bases in the framework of Kashiwara's approach [K].

The simplest elements in the algebra we construct are certain elements $F_i^{(a)}$ for various $i \in I$ and $a \in \mathbf{N}$. These are in fact elements of B.

(If there are no loops at i, the elements $F_i^{(a)}, F_i^{(a')}$ are related in a simple way: a multiple of the a'-th power of the first one is equal to a multiple of the a-th power of the second one; this is no longer true if there are loops at i: our elements commute but are unrelated if there is exactly one loop and may not commute if there are at least two loops.)

By definition, a *monomial* in our algebra is an element of the form

(a) $$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \dots F_{i_m}^{(a_m)}$$

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where $i_1,\dots,i_m$ are in I and $a_1,\dots,a_m$ are in $\mathbf{N}$.

We say that a monomial is *tight* if it belongs to B; we say that a monomial is *semi-tight* if it is a linear combination of elements in B with constant coefficients (necessarily in $\mathbf{N}$).

We shall give a criterion (involving the positivity of a certain quadratic form) for a monomial to be tight or semi-tight (see no.6). As an application, we show that in the case where the graph consists of one vertex and at least two loops, all monomials are tight; hence in this case, all elements of B are monomials. (In the case where the graph consists of one vertex and one loop, all monomials are semi-tight, a known result.) We shall also show that for types A_2, A_3, A_4 there is an abundance of tight monomials (see no. 13,14,15); it would be interesting to know to what extent is this a general phenomenon (see no.16).

2. Before defining the algebra U^-, we shall define its basis B. The definition follows closely that in [L2]. B is a disjoint union of finite sets B_ν where ν runs over the set of functions $I\to\mathbf{N}$.

We fix for each i a complex vector space $\mathbf{V}_i$ of dimension $\nu(i)$. Let $\mathbf{V}=\oplus_i\mathbf{V}_i$. We choose a sequence $\mathbf{i}=(i_1,\dots,i_m)$ in I and a sequence $\mathbf{a}=(a_1,\dots a_m)$ in $\mathbf{N}$ such that $\sum_{p:i_p=i}a_p=\nu(i)$ for all i.

We will define (see no.3) a proper morphism $\pi:\tilde F\to E$ between two algebraic varieties, with $\tilde F$ depending on $\mathbf{i},\mathbf{a}$ and

$$E=\oplus_{h\in\Omega}\ \mathrm{Hom}\ (\mathbf{V}_{h'},\mathbf{V}_{h''})$$

depending only of ν; the variety $\tilde F$ will be smooth hence the direct image of the constant sheaf $\mathbf{C}$ under π will be (by the decomposition theorem [BBD]) isomorphic to a direct sum of simple perverse sheaves (with shifts) on E; for any simple perverse sheaf b on E and any integer d, let $M(b,\mathbf{i},\mathbf{a},d)$ be the number of times that the shift $b[d]$ appears as a summand in that direct image.

Let B_ν be the set of isomorphism classes of simple perverse sheaves b on E such that $M(b,\mathbf{i},\mathbf{a},d)\ne 0$ for some d and some $\mathbf{i},\mathbf{a}$ as above. Then B_ν is a finite set.

3. Let F be the variety consisting of all flags

$$\phi=(\mathbf{V}=\mathbf{V}^0\supset\mathbf{V}^1\supset\dots\supset\mathbf{V}^m)$$

where $\mathbf{V}^p$ are I-graded subspaces of $\mathbf{V}$ such that for any $p\in[1,m]$, the I-graded vector space $\mathbf{V}^{p-1}/\mathbf{V}^p$ is zero in degrees $i\ne i_p$ and has dimension a_p in degree $i=i_p$.

By definition, $\tilde F$ is the variety of all pairs (x,ϕ) where $x=(x_h)\in E$ and $\phi=(\mathbf{V}=\mathbf{V}^0\supset\mathbf{V}^1\supset\dots\supset\mathbf{V}^m=0)\in F$ is such that $x_h(\mathbf{V}^p_{h'})\subset\mathbf{V}^p_{h''}$ for all $p\in[0,m]$ and all $h\in\Omega$. Let $\pi:\tilde F\to E$ be the first projection. This is a proper map since F is complete; $\tilde F$ is smooth, since it is a vector bundle over the smooth variety F via the second projection.

This completes the definition of B_ν hence that of B.

If we apply the previous definitions with $\mathbf{i} = (i), \mathbf{a} = (a)$ (sequences with one term) and ν defined by $\nu(i) = a$ and $\nu(j) = 0$ for $j \neq i$, the morphism π is an isomorphism hence the constant sheaf $\mathbf{C}$ on E is an element of B_ν; we denote it $F_i^{(a)}$. (If, in addition, $a = 0$, we have that $F_i^{(a)}$ is independent of i; we denote it 1.)

4. Let U^- be the free $\mathbf{Z}[v, v^{-1}]$-module with basis B. We may define a structure of associative algebra with 1 on this module by repeating word by word the procedure in [L2, 3.1, 10.8]. From the definitions it follows that the product of two elements of B is a linear combination of elements of B with coefficients in $\mathbf{N}[v, v^{-1}]$.

For example,

$$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \dots F_{i_m}^{(a_m)} = \sum_{b \in B; d \in \mathbf{Z}} M(b, \mathbf{i}, \mathbf{a}, d) v^{d+d_0} b$$

(see no.2) where d_0 is an integer independent of b, d such that the right hand side of the last equality is stable under $v^p \mapsto v^{-p}$.

Remarks. (a) We can form a semidirect product $U^{\le 0}$ of U^- with the group algebra of the commutative group $\mathbf{Z}^I$; this is then a Hopf algebra, just as in [L2], in which both the multiplication and comultiplication satisfy very strong positivity properties with respect to B.

(b) When our graph has exactly one vertex and one loop, the algebra U^- may be identified with the Hall algebra with its canonical basis. This follows easily from the results in [L1] where the link between the Hall algebra and perverse sheaves was established.

5. For $i \in I$, let $Z(i)$ be the set of all $p \in [1, m]$ such that $i_p = i$.

Let $\mathbf{i} = (i_1, \dots, i_m)$ be a sequence in I and let $\mathbf{a} = (a_1, \dots a_m)$ be a sequence in $\mathbf{N}$.

We will associate to $\mathbf{i}$ a real vector space P with a quadratic form.

Let P' be the real vector space with coordinate functions z_i^{pq} indexed by triples (i, p, q) such that $i \in I$ and $p, q \in Z(i)$. We shall define z_i^{pq} to be the 0 linear function whenever $i \in I$ and $p, q \in [1, m]$ are not both contained in $Z(i)$.

Given $s \in [1, m]$, let $i \in I$ be defined by $s \in Z(i)$; we consider the linear forms $\xi_s = \sum_r z_i^{rs}$ and $\xi'_s = \sum_r z_i^{sr}$ on P'.

Let P be the subspace of P' defined by the equations $\xi_s = \xi'_s$ for all $s \in [1, m]$.

Let $T_{\mathbf{a}}$ be the finite subset of P' consisting of all points $z = (z_i^{pq})$ with coordinates in $\mathbf{N}$ such that $\xi_s(z) = \xi'_s(z) = a_s$ for all $s \in [1, m]$. (We have $T_{\mathbf{a}} \subset P$.) There is a distinguished vector $z_{\mathbf{a}} \in T_{\mathbf{a}}$; it has coordinates $z_i^{pq} = 0$ if $p \neq q$ and $z_i^{pp} = a_p$ if $p \in Z(i)$.

We consider the quadratic form

$$Q = \sum_h \sum_{\substack{r \le p \\ q < s}} z_{h'}^{pq} z_{h''}^{rs} - \sum_i \sum_{\substack{r \le p \\ q < s}} z_i^{pq} z_i^{rs}$$

on P'.

Clearly, $Q(z_{\mathbf{a}}) = 0$.

Note that if $z = (z_i^{pq}) \in P'$ satisfies $\xi'_s(z) = a_s$ for all $s \in [1, m]$, then we have

$$\sum_i \sum_{\substack{r \le p \\ q < s}} z_i^{pq} z_i^{rs} = -\sum_i \sum_{\substack{r \le p \\ s \le q}} z_i^{pq} z_i^{rs} + \sum_i \sum_{\substack{r,p \in Z(i) \\ r \le p}} a_r a_p$$

and similarly

$$\sum_h \sum_{\substack{r \le p \\ q < s}} z_{h'}^{pq} z_{h''}^{rs} = -\sum_h \sum_{\substack{r \le p \\ s \le q}} z_{h'}^{pq} z_{h''}^{rs} + \sum_h \sum_{\substack{r \in Z(h'') \\ p \in Z(h') \\ r \le p}} a_r a_p;$$

hence we have

$$\begin{aligned} Q(z) = &-\sum_h \sum_{\substack{r \le p \\ s \le q}} z_{h'}^{pq} z_{h''}^{rs} + \sum_i \sum_{\substack{r \le p \\ s \le q}} z_i^{pq} z_i^{rs} \\ &+\sum_h \sum_{\substack{r \in Z(h'') \\ p \in Z(h') \\ r \le p}} a_r a_p - \sum_i \sum_{\substack{r,p \in Z(i) \\ r \le p}} a_r a_p. \end{aligned} \tag{a}$$

6. We can now state the following result.

Theorem. *(a) If the quadratic form Q takes only values > 0 on $T_{\mathbf{a}} - \{z_{\mathbf{a}}\}$, then the monomial 1(a) is tight.*

(b) If the quadratic form Q takes only values ≥ 0 on $T_{\mathbf{a}}$, then the monomial 1(a) is semi-tight.

The proof will be given in no.17.

Corollary. *Assume that our graph consists of one vertex i and of $l \geq 1$ loops.*

(a) If $l = 1$, then all monomials in U^- are semi-tight.

(b) If $l \geq 2$, then all monomials in U^- are tight.

Let $\mathbf{i}, \mathbf{a}$ be as in no.5; we have $i_p = i$ for all p. Let $z = (z_i^{pq})$ be a point of $T_{\mathbf{a}}$. The value of the quadratic form Q at z is $(l-1)\sum_{\substack{r \le p \\ q < s}} z_i^{pq} z_i^{rs}$. This is certainly ≥ 0. Assume that this value is 0 and that $l \geq 2$. Then

$$z_i^{pq} z_i^{rs} = 0 \tag{c}$$

for all indices $r \leq p; q < s$ in $[1, m]$. We will show that

$$z_i^{kl} = 0 \text{ for all } k \neq l. \tag{d}$$

Assume that this is already known whenever $\min(k,l) < n$ for a certain integer $n \in [1, m-1]$. If $z_i^{nl} \neq 0$ and $l > n$, then $z_i^{pn} z_i^{nl} = 0$ for all $p \geq n$ (by (c)) so that $z_i^{pn} = 0$ for all $p \geq n$. By our assumption we have also $z_i^{pn} = 0$ for all $p < n$. Thus $z_i^{pn} = 0$ for all p. But then $0 = \sum_p z_i^{pn} = \sum_p z_i^{np}$ (by the definition of $T_{\mathbf{a}}$)). Since all terms of the last sum are ≥ 0, it follows that they are all zero, contradicting $z_i^{nl} \neq 0$. Thus, we have $z_i^{nl} = 0$ for all $l > n$.

Similarly, if $z_i^{ln} \neq 0$ and $l > n$, then $z_i^{ln} z_i^{np} = 0$ for all $p > n$ (by (c)) so that $z_i^{np} = 0$ for all $p > n$. By our assumption we have also $z_i^{np} = 0$ for all $p < n$. Thus $z_i^{np} = 0$ for all $p \neq n$. But then $0 = -z_i^{nn} + \sum_p z_i^{np} = -z_i^{nn} + \sum_p z_i^{pn}$ (by the definition of $T_{\mathbf{a}}$)). Hence $\sum_{p:p\neq n} z_i^{pn} = 0$. Since all terms of the last sum are ≥ 0, it follows that they are all zero, contradicting $z_i^{ln} \neq 0$. Thus, we have $z_i^{ln} = 0$ for all $l > n$. We have shown that our assumption implies $z_i^{kl} = 0$ for any $k \neq l$ such that $\min(k,l) = n$. This proves (d) by induction. Thus we have $z = z_{\mathbf{a}}$. The corollary is proved.

7. Is it true that the elements $F_i^{(a)}$ $(i \in I, a \in \mathbf{N})$ generate the algebra U^-? By the previous corollary, this is true when $|I| = 1$ and there are at least two loops; it is also true when $|I| = 1$ and there is exactly one loop (as one easily proves). In the case when I is arbitrary and there are no loops, this is again true, by the results of [L2].

In nos. 8-11 we will set up a machinery which will be used in nos. 12-16 to calculate tight and semitight monomials in various examples.

8. Let $(\mathbf{i}, \mathbf{a})$ be as in no.5.

For any two indices $s < s'$ in $[1, m]$ such that $i_s = i_{s'} = i$ and such that $p \notin Z(i)$ whenever $s < p < s'$, we set

$$N(s,s') = (l_i - 1)(a_s + a_{s'}) + \sum_{\substack{i' \in I \\ i' \neq i}} \sum_{\substack{p \in Z(i') \\ s<p<s'}} e_{ii'} a_p$$

where l_i is the number of edges joining i with i (loops) and $e_{ii'}$ is the number of (unoriented) edges joining i with i'.

More generally, given $s < s'$ in $[1, m]$ such that $i_s = i_{s'} = i$, we set

$$N(s,s') = N(s_0, s_1) + N(s_1, s_2) + \cdots + N(s_{k-1}, s_k)$$

where $\{s = s_0 < s_1 < \cdots < s_k = s'\}$ are the elements of $Z(i) \cap [s, s']$ in increasing order. This is compatible with the previous definition.

9. For some computations it is convenient to rewrite the quadratic form Q (on P) by eliminating the variables z_i^{pp}.

Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a sequence in I. We will associate to $\mathbf{i}$ a real vector space $\bar{P}$ with a quadratic form. Let $\bar{P}'$ be the real vector vector space with coordinate functions z_i^{pq} indexed by triples (i, p, q) such that $i \in I$ and $p, q \in Z(i)$ satisfy $p \neq q$. We shall define z_i^{pq} to be the 0 linear function whenever $i \in I$ and $p \neq q \in [1, m]$ are not both contained in $Z(i)$.

Let $\bar P$ be the subspace of $\bar P'$ defined by the equations

$$\sum_{r:r\ne s} z_i^{rs}=\sum_{r:r\ne s} z_i^{sr} \quad (s\in[1,m]).$$

Let $\bar P^+$ be the set of all $z=(z_i^{pq})\in\bar P$ such that $z_i^{pq}\in\mathbf N$ for all (i,p,q), $\quad p\ne q$.
We consider the quadratic form

$$\begin{aligned}\bar Q=&\sum_i\sum_{r\le p<s\le q} z_i^{pq}z_i^{rs}+\sum_i\sum_{q<s\le p<r} z_i^{pq}z_i^{rs}\\ &-\sum_h\sum_{r\le p<s\le q} z_{h'}^{pq}z_{h''}^{rs}-\sum_h\sum_{q<s\le p<r} z_{h'}^{pq}z_{h''}^{rs}\end{aligned}\tag{b}$$

on $\bar P'$. By restriction we get a quadratic form (denoted again $\bar Q$) on $\bar P$.
Given a sequence $\mathbf a=(a_1,\dots,a_m)$ in $\mathbf N$, we consider the linear form

$$\sum_i\sum_{\substack{r,s\\ s<r}} N(s,r)z_i^{rs}$$

on $\bar P'$. (See no.8.) By restriction we get a linear form $L_{\mathbf a}$ on $\bar P$.
Consider the affine linear map $\mu:\bar P'\to P'$ such that given by $z_i^{pq}\circ\mu=toz_i^{pq}$ if $p\ne q$ and $z_i^{pp}\circ\mu=a_p-\sum_{q:q\ne p} z_i^{pq}$. This map is clearly injective. It carries $\bar P$ into P.

Lemma. *(a) For any $p\in[1,m]$ the equation*

$$\sum_{\substack{r,s\\ r\le p<s}} z_i^{rs}=\sum_{\substack{r,s\\ s\le p<r}} z_i^{rs}$$

holds on $\bar P$.
(b) Any $z\in T_{\mathbf a}-\{z_{\mathbf a}\}$, is the image of a (unique) point in $\bar P^+-\{0\}$. Moreover, $\mu(0)=z_{\mathbf a}$.
(c) For any $z\in\bar P$ we have $Q(\mu(z))=\bar Q(z)+L_{\mathbf a}(z)$.

The (easy) verification of (a),(b) is left to the reader. We verify (c). Let $z=(z_i^{pq}|p\ne q)\in\bar P$.
Using 5(a), we have

$$
\begin{aligned}
Q(\mu(z)) = \sum_i \Bigg(& \sum_{\substack{r\le p\\ s\le q\\ p\ne q\\ r\ne s}} z_i^{pq} z_i^{rs} + \sum_{\substack{r\le p\\ s\le p\\ r\ne s\\ p\in Z(i)}} (a_p - \sum_{\substack{q\\ q\ne p}} z_i^{pq}) z_i^{rs} + \sum_{\substack{r\le p\\ r\le q\\ p\ne q\\ r\in Z(i)}} z_i^{pq}(a_r - \sum_{\substack{s\\ s\ne r}} z_i^{rs}) \\
& + \sum_{\substack{r\le p\\ r\in Z(i)\\ p\in Z(i)}} (a_p - \sum_{\substack{q\\ q\ne p}} z_i^{pq})(a_r - \sum_{\substack{s\\ s\ne r}} z_i^{rs}) - \sum_{\substack{r,p\in Z(i)\\ r\le p}} a_r a_p \Bigg) \\
- \sum_h \Bigg(& \sum_{\substack{r\le p\\ s\le q\\ p\ne q\\ r\ne s}} z_{h'}^{pq} z_{h''}^{rs} - \sum_{\substack{r\le p\\ s\le p\\ r\ne s\\ p\in Z(h')}} (a_p - \sum_{\substack{q\\ q\ne p}} z_{h'}^{pq}) z_{h''}^{rs} - \sum_{\substack{r\le p\\ r\le q\\ p\ne q\\ r\in Z(h'')}} z_{h'}^{pq}(a_r - \sum_{\substack{s\\ s\ne r}} z_{h''}^{rs}) \\
& - \sum_{\substack{r\le p\\ r\in Z(h'')\\ p\in Z(h')}} (a_p - \sum_{\substack{q\\ q\ne p}} z_{h'}^{pq})(a_r - \sum_{\substack{s\\ s\ne r}} z_{h''}^{rs}) + \sum_{\substack{r\in Z(h'')\\ p\in Z(h')\\ r\le p}} a_r a_p \Bigg).
\end{aligned}
$$

The last equality can be expressed as

$$Q(\mu(z)) = A + B' + B''$$

where A is linear in the $z_i^{pq} \quad (p \neq q)$:

$$A = \sum_{\substack{i\\ r\ne s}} \Bigg(- \sum_{\substack{p\in Z(i)\\ r\le p<s}} a_p - \sum_{\substack{p\in Z(i)\\ s<p\le r}} a_p + \sum_{\substack{h,p\in Z(h'')\\ h'=i\\ s<p\le r}} a_p + \sum_{\substack{h,p\in Z(h')\\ h''=i\\ r\le p<s}} a_p \Bigg) z_i^{rs},$$

and B', B'' are quadratic in the $z_i^{pq} \quad (p \neq q)$:

$$B' = \sum_i \Bigg(\sum_{\substack{r\le p\\ s\le q\\ p\ne q\\ r\ne s}} z_i^{pq} z_i^{rs} - \sum_{\substack{r\le p\\ s\le p\\ p\ne q\\ r\ne s}} z_i^{pq} z_i^{rs} - \sum_{\substack{r\le p\\ r\le q\\ p\ne q\\ r\ne s}} z_i^{pq} z_i^{rs} + \sum_{\substack{r\le p\\ p\ne q\\ r\ne s}} z_i^{pq} z_i^{rs} \Bigg),$$

$$B'' = \sum_h \Bigg(- \sum_{\substack{r\le p\\ s\le q\\ p\ne q\\ r\ne s}} z_{h'}^{pq} z_{h''}^{rs} + \sum_{\substack{r\le p\\ s\le p\\ p\ne q\\ r\ne s}} z_{h'}^{pq} z_{h''}^{rs} + \sum_{\substack{r\le p\\ r\le q\\ p\ne q\\ r\ne s}} z_{h'}^{pq} z_{h''}^{rs} - \sum_{\substack{r\le p\\ p\ne q\\ r\ne s}} z_{h'}^{pq} z_{h''}^{rs} \Bigg).$$

Using (a), we can write

$$\begin{aligned}A &= -\sum_{\substack{i\\ p\in Z(i)}} a_p\Big(\sum_{r\le p<s} z_i^{rs}+\sum_{s<p\le r} z_i^{rs}\Big)+\sum_{\substack{h\\ p\in Z(h')}} a_p\sum_{r\le p<s} z_{h''}^{rs}+\sum_{\substack{h\\ p\in Z(h'')}} a_p\sum_{s<p\le r} z_{h'}^{rs}\\
&= -\sum_{\substack{i\\ p\in Z(i)}} a_p\Big(\sum_{s\le p<r} z_i^{rs}+\sum_{s<p\le r} z_i^{rs}\Big)+\sum_{\substack{h\\ p\in Z(h')}} a_p\sum_{s\le p<r} z_{h''}^{rs}+\sum_{\substack{h\\ p\in Z(h'')}} a_p\sum_{s<p\le r} z_{h'}^{rs}\\
&= \sum_i\sum_{\substack{r,s\in Z(i)\\ s<r}} c_{i,r,s} z_i^{rs}\end{aligned}$$

where

$$c_{i,r,s}=-\sum_{\substack{p\in Z(i)\\ s\le p<r}} a_p-\sum_{\substack{p\in Z(i)\\ s<p\le r}} a_p+\sum_{\substack{h;p\in Z(h')\\ s\le p<r\\ h''=i}} a_p+\sum_{\substack{h;p\in Z(h'')\\ s<p\le r\\ h'=i}} a_p.$$

Note that $c_{i,r,s}=N(s,r)$ (see no.8) hence

$$A=L_{\mathbf{a}}(z).$$

We now compute the quadratic term B''. We split each sum in B'' into two sums: one defined by the condition $p<q$ and one defined by the condition $p>q$. In the following calculation we omit writing summation over h, the condition $r\ne s$ as well as $z_{h'}^{pq}z_{h''}^{rs}$ after each summation symbol:

$$\begin{aligned}B'' &= -\sum_{\substack{r\le p<q\\ s\le q}}+\sum_{\substack{r\le p<q\\ s\le p}}+\sum_{r\le p<q;s}-\sum_{r\le p<q;s}-\sum_{\substack{s\le q<p\\ r\le p}}+\sum_{\substack{q<p\\ r\le p\\ s\le p}}+\sum_{r\le q<p;s}-\sum_{\substack{q<p\\ r\le p;s}}\\
&= -\sum_{r\le p<s\le q}+\sum_{\substack{q<s\le p\\ r\le p}}-\sum_{q<r\le p;s}=-\sum_{r\le p<s\le q}+\sum_{q<s<r\le p}+\sum_{q<r<s\le p}+\sum_{r\le q<s\le p}\\
&-\sum_{q<r\le p<s}-\sum_{q<r<s\le p}-\sum_{q<s<r\le p}-\sum_{s\le q<r\le p}\\
&= -\sum_{r\le p<s\le q}+\sum_{r\le q<s\le p}-\sum_{q<r\le p<s}-\sum_{s\le q<r\le p}.\end{aligned}$$

Using the definition of $\bar P$, we see that

(d) $$\sum_{r\le q<s\le p}+\sum_{r\le q<p<s}=\sum_{r\le q<s;q<p}=\sum_{s\le q<r;q<p}=\sum_{s\le q<r\le p}+\sum_{s\le q<p<r}$$

and

(e)
$$\sum_{q<r\le p<s} + \sum_{r\le q<p<s} = \sum_{\substack{r\le p<s\\ q<p}} = \sum_{\substack{s\le p<r\\ q<p}} = \sum_{q<s\le p<r} + \sum_{s\le q<p<r}.$$

Substracting (e) from (d) we obtain

$$\sum_{r\le q<s\le p} - \sum_{q<r\le p<s} = \sum_{s\le q<r\le p} - \sum_{q<s\le p<r}.$$

Introducing this in the last expression for B'', we obtain

$$B'' = -\sum_{r\le p<s\le q} + \sum_{s\le q<r\le p} - \sum_{q<s\le p<r} - \sum_{s\le q<r\le p}.$$

Performing cancellations we obtain

$$B'' = -\sum_{h}\sum_{r\le p<s\le q} z_{h'}^{pq} z_{h''}^{rs} - \sum_{h}\sum_{q<s\le p<r} z_{h'}^{pq} z_{h''}^{rs}.$$

An identical computation gives

$$B' = \sum_{i}\sum_{r\le p<s\le q} z_{i}^{pq} z_{i}^{rs} + \sum_{i}\sum_{q<s\le p<r} z_{i}^{pq} z_{i}^{rs}.$$

Thus we have
$$A + B' + B'' = \bar{Q}(z) + L_{\mathbf{a}}(z).$$

The Lemma is proved.

10. Using the previous Lemma we see that the Theorem in no.6 has the following

Corollary. *Let* $\mathbf{i}, \mathbf{a}$ *be as in no.5.*

(a) If the non-homogeneous quadratic form $\bar{Q} + L_{\mathbf{a}}$ *takes only values* > 0 *on* $\bar{P}^+ - \{0\}$*, then the monomial 1(a) is tight.*

(b) If the non-homogeneous quadratic form $\bar{Q} + L_{\mathbf{a}}$ *takes only values* ≥ 0 *on* $\bar{P}^+$*, then the monomial 1(a) is semi-tight.*

11. Assume that i, j are distinct elements of I, that there are no loops at i and j and that there are exactly $e \ge 1$ edges joining i and j. Let $\mathbf{i} = (i_1, i_2, i_3) = (i, j, i)$; let $\mathbf{a} = (a_1, a_2, a_3)$. Now $\bar{P}$ is one dimensional with the coordinate function $z = z_i^{13} = z_i^{31}$. The (non-homogeneous) quadratic form on $\bar{P}$ is

$$z^2 + (ea_2 - a_1 - a_3)z.$$

If $ea_2 \ge a_1 + a_3$, then this quadratic form takes a value > 0 at $z > 0$; hence the monomial $F_i^{(a_1)} F_j^{(a_2)} F_i^{(a_3)}$ is tight. Similarly we see that, if $ea_2 = a_1 + a_3 - 1$, then the monomial $F_i^{(a_1)} F_j^{(a_2)} F_i^{(a_3)}$ is semi-tight. (The monomials which enter in the Serre relation are of this type).

12. *Affine A_1.* Assume that $I = \{i, j\}$ and that $\Omega = \{h_1, h_2\}$ with h_1, h_2 joining i with j. Let $\mathbf{i} = (i_1, i_2, i_3, i_4) = (i, j, i, j)$. Let $\mathbf{a} = (a_1, a_2, a_3, a_4)$. Now $\bar{P}$ is two dimensional with the coordinate functions $z_i = z_i^{13} = z_i^{31}$ and $z_j = z_j^{24} = z_j^{42}$. The (non-homogeneous) quadratic form on $\bar{P}$ is

$$(z_i - z_j)^2 + (2a_2 - a_1 - a_3)z_i + (2a_3 - a_2 - a_4)z_j.$$

If

(a) $$2a_2 \geq a_1 + a_3 \text{ and } 2a_3 \geq a_2 + a_4,$$

then this quadratic form takes a value ≥ 0 at (z_i, z_j) with $z_i \geq 0, z_j \geq 0$; hence the monomial $F_i^{(a_1)} F_j^{(a_2)} F_i^{(a_3)} F_j^{(a_4)}$ is semi-tight.

If the inequalities (a) are satisfied and at least one of them is strict, then the quadratic form takes a value > 0 at (z_i, z_j) with $z_i \geq 0, z_j \geq 0$, $(z_i, z_j) \neq (0, 0)$; hence the monomial $F_i^{(a_1)} F_j^{(a_2)} F_i^{(a_3)} F_j^{(a_4)}$ is tight.

13. Assume now that $I = \{i, j, k\}$ and that $\Omega = \{h_1 : i \to j; h_2 : k \to j\}$. (Thus, we are in type A_3.)

Proposition. *The following monomials in U^- are tight:*

(a) $$F_i^{(a)} F_k^{(b)} F_j^{(c)} F_i^{(d)} F_k^{(e)} F_j^{(f)}$$

for $c \geq a + d, \quad c \geq b + e, \quad d + e \geq c + f;$

(b) $$F_j^{(f)} F_k^{(e)} F_i^{(d)} F_j^{(c)} F_k^{(b)} F_i^{(a)}$$

for $c \geq a + d, \quad c \geq b + e, \quad d + e \geq c + f;$

(c) $$F_i^{(a)} F_j^{(b)} F_i^{(c)} F_k^{(d)} F_j^{(e)} F_i^{(f)}$$

for $b \geq a + c, \quad e \geq c + f, \quad c + d \geq b + e;$

(d) $$F_k^{(a)} F_j^{(b)} F_k^{(c)} F_i^{(d)} F_j^{(e)} F_k^{(f)}$$

for $b \geq a + c, \quad e \geq c + f, \quad c + d \geq b + e;$

(e) $$F_j^{(a)} F_i^{(b)} F_j^{(c)} F_k^{(d)} F_j^{(e)} F_i^{(f)}$$

for $c+e \geq b+f, \quad b \geq a+c, \quad d \geq c+e$;

(f) $$F_j^{(a)} F_k^{(b)} F_j^{(c)} F_i^{(d)} F_j^{(e)} F_k^{(f)}$$

for $c+e \geq b+f, \quad b \geq a+c, \quad d \geq c+e$;

(g) $$F_i^{(f)} F_j^{(e)} F_k^{(d)} F_j^{(c)} F_i^{(b)} F_j^{(a)}$$

for $c+e \geq b+f, \quad b \geq a+c, \quad d \geq c+e$;

(h) $$F_k^{(f)} F_j^{(e)} F_i^{(d)} F_j^{(c)} F_k^{(b)} F_j^{(a)}$$

for $c+e \geq b+f, \quad b \geq a+c, \quad d \geq c+e$.

The proof is an application of Corollary 10. We first consider case (a).

Let $\mathbf{i} = (i_1, i_2, i_3, i_4, i_5, i_6) = (i, k, j, i, k, j)$. Let $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$.

Now $\bar{P}$ is three dimensional with the coordinate functions $z_i = z_i^{14} = z_i^{41}$, $z_k = z_k^{25} = z_k^{52}$ and $z_j = z_j^{36} = z_j^{63}$.

The non-homogeneous quadratic form on $\bar{P}$ is

$$z_i^2 + z_j^2 + z_k^2 - z_i z_j - z_j z_k + (a_3 - a_1 - a_4) z_i + (a_3 - a_2 - a_5) z_k + (a_4 + a_5 - a_3 - a_6) z_j.$$

If

$$a_3 \geq a_1 + a_4, \quad a_3 \geq a_2 + a_5, \quad a_4 + a_5 \geq a_3 + a_6,$$

then the quadratic form takes a value > 0 at (z_i, z_j, z_k) with $z_i \geq 0, z_j \geq 0, z_k \geq 0$ (not all three zero); hence the monomial $F_i^{(a_1)} F_k^{(a_2)} F_j^{(a_3)} F_i^{(a_4)} F_k^{(a_5)} F_j^{(a_6)}$ is tight.

Next we consider case (c).

Let $\mathbf{i} = (i_1, i_2, i_3, i_4, i_5, i_6) = (i, j, i, k, j, i)$. Let $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$. The non-homogeneous quadratic form on $\bar{P}$ is

$$\begin{aligned}
&- z_i^{31} z_j^{52} - z_i^{36} z_j^{25} + z_i^{31} z_i^{63} + z_i^{36} z_i^{16} + z_i^{16} z_i^{13} + (z_i^{13})^2 + (z_i^{16})^2 + (z_i^{36})^2 + (z_j^{25})^2 \\
&+ (a_2 - a_1 - a_3) z_i^{31} + (a_2 + a_5 - a_1 - 2a_3 - a_6) z_i^{61} + (a_5 - a_3 - a_6) z_i^{63} \\
&+ (a_3 + a_4 - a_2 - a_5) z_j^{52}.
\end{aligned}$$

There are 5 independent variables: the 8 variables above satisfy the linear relations:

$$z_i^{13} + z_i^{63} = z_i^{31} + z_i^{36}, \quad z_i^{16} + z_i^{13} = z_i^{61} + z_i^{63}, \quad z_j^{25} = z_j^{52}.$$

If

$$a_2 \geq a_1 + a_3, \quad a_5 \geq a_3 + a_6, \quad a_3 + a_4 \geq a_2 + a_5,$$

then the quadratic form takes values >0 on $\bar P^+ - \{0\}$, hence the corresponding monomial is tight.

Next we consider case (e).

Let $\mathbf{i} = (i_1, i_2, i_3, i_4, i_5, i_6) = (j, i, j, k, j, i)$. Let $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$. The non-homogeneous quadratic form on $\bar P$ is

$$\begin{aligned}&- z_i^{26} z_j^{13} - z_i^{26} z_j^{15} + z_j^{31} z_j^{53} + z_j^{35} z_j^{15} + z_j^{15} z_j^{13} + (z_j^{13})^2 + (z_j^{15})^2 + (z_j^{35})^2 + (z_i^{26})^2 \\ &+ (a_3 + a_5 - a_2 - a_6) z_i^{62} + (a_2 - a_1 - a_3) z_j^{31} + (a_2 + a_4 - a_1 - 2a_3 - a_5) z_j^{51} \\ &+ (a_4 - a_3 - a_5) z_j^{53}.\end{aligned}$$

There are again 5 independent variables: the 8 variables above satisfy the linear relations:

$$z_j^{13} + z_j^{53} = z_j^{31} + z_j^{35}, \quad z_j^{15} + z_j^{13} = z_j^{51} + z_j^{53}, \quad z_i^{26} = z_i^{62}.$$

If

$$a_3 + a_5 \ge a_2 + a_6, \quad a_2 \ge a_1 + a_3, \quad a_4 \ge a_3 + a_5,$$

then the quadratic form takes values >0 on $\bar P^+ - \{0\}$, hence the corresponding monomial is tight.

It is easy to see that if a monomial is tight, then reading it from right to left gives another tight monomial. In this way, (b) follows from (a) and (g) follows from (e).

Finally, (d) follows from (c), (f) follows from (e) and (h) follows from (g) applying the non-trivial automorphism of our graph.

Remark. A similar proof shows that the monomial $F_i^{(a)} F_k^{(b)} F_j^{(c)} F_i^{(d)} F_k^{(e)} F_j^{(f)}$ with $c \ge a + d, \quad c \ge b + e, \quad d + e = c + f - 1$, is semi-tight.

14. The 8 types of monomials in the previous proposition correspond to the 8 possible reduced decompositions of the longest element in the Weyl group of type A_3 (we identify reduced decompositions which differ only by switching two neighbouring commuting reflections).

The elements of the canonical basis thus obtained are essentially distinct (except in degenerate cases). By writing them in terms of a Poincaré-Birkhoff-Witt basis, we see that their leading terms have exponents filling 8 of the 10 regions in $\mathbf{N}^6$ described in the [L3] (these 10 regions are the maximal regions where a certain piecewise-linear isomorphism is linear); each of the 8 types of monomials fills completely one of the regions. In this sense we see that 8/10 of the canonical basis consists of monomials; it is known that some elements of the canonical basis are not monomials.

15. In this section we attempt to do in the case of type A_4 what we have done for type A_3 in no.13.

Assume that $I = \{i, j, k, l\}$ and that $\Omega = \{h_1 : i \to j; h_2 : k \to j; h_3 : k \to l\}$.

Let

$$\mathbf{i} = (i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}) = (i, k, j, l, i, k, j, l, i, k).$$

Let $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$.

Proposition. *The monomial*

$$F_i^{(a_1)}F_k^{(a_2)}F_j^{(a_3)}F_l^{(a_4)}F_i^{(a_5)}F_k^{(a_6)}F_j^{(a_7)}F_l^{(a_8)}F_i^{(a_9)}F_k^{(a_{10})}$$

is semi-tight provided that

$$\begin{aligned}&a_3 \geq a_1 + a_5, \quad a_7 \geq a_5 + a_9, \quad a_3 + a_4 \geq a_2 + a_6,\\ \text{(a)} \qquad &a_7 + a_8 \geq a_6 + a_{10}, \quad a_5 + a_6 \geq a_3 + a_7, \quad a_6 \geq a_4 + a_8.\end{aligned}$$

If, in addition, at least one of the inequalities $a_3 \geq a_1 + a_5$, $a_7 \geq a_5 + a_9$ is strict and at least one of the inequalities $a_3 + a_4 \geq a_2 + a_6$, $a_7 + a_8 \geq a_6 + a_{10}$ is strict, then the monomial above is tight.

To prove the first statement, it is enough, by Corollary 10, to show that the expression

$$\begin{aligned}q =& - z_i^{51} z_j^{73} - z_i^{59} z_j^{37} - z_k^{62} z_j^{73} - z_k^{6,10} z_j^{37}\\ &- z_k^{62} z_l^{84} - z_k^{6,10} z_l^{48} + z_i^{31} z_i^{95} + z_k^{62} z_k^{10,6}\\ &+ (z_i^{15})^2 + z_i^{19} z_i^{15} + z_i^{19} z_i^{59} + (z_i^{59})^2\\ &+ (z_k^{26})^2 + z_k^{2,10} z_k^{26} + z_k^{2,10} z_k^{6,10} + (z_k^{6,10})^2 + (z_j^{37})^2 + (z_l^{48})^2\\ &+ (a_3 - a_1 - a_5) z_i^{51} + (a_3 + a_7 - a_1 - 2a_5 - a_9) z_i^{91}\\ &+ (a_7 - a_5 - a_9) z_i^{95} + (a_3 + a_4 - a_2 - a_6) z_k^{62}\\ &+ (a_3 + a_4 + a_7 + a_8 - a_2 - 2a_6 - a_{10}) z_k^{10,2} + (a_7 + a_8 - a_6 - a_{10}) z_k^{10,6}\\ &+ (a_5 + a_6 - a_3 - a_7) z_j^{73} + (a_6 - a_4 - a_8) z_l^{84}\end{aligned}$$

is ≥ 0 if

$$\begin{aligned}&z_i^{15} + z_i^{95} = z_i^{51} + z_i^{59}, \quad z_i^{51} + z_i^{91} = z_i^{15} + z_i^{19},\\ &z_k^{26} + z_k^{10,6} = z_k^{62} + z_k^{6,10}, \quad z_k^{62} + z_k^{10,2} = z_k^{26} + z_k^{2,10},\\ \text{(b)} \qquad &z_j^{37} = z_j^{73}, \quad z_l^{48} = z_l^{84},\end{aligned}$$

and the z_u^{pq} are ≥ 0.

We have $q = q_2 + q_1$ where q_1 (resp. q_2) is homogeneous of degree 1 (resp. 2) in the z_u^{pq}.

By our assumption (a),we have $q_1 \geq 0$. We must show that $q_2 \geq 0$.

Let t be the unique solution of the equation $1/t = 1 + 1/(1 - t)$ in the interval $(0, 1)$.

We have

$$\begin{aligned}
q_2 &= (z_l^{48} - (z_k^{6,10} + z_k^{62})/2)^2 + t(z_j^{37} - (z_i^{51} + z_i^{59})/(2t))^2 \\
&+ (1-t)(z_j^{37} - (z_k^{6,10} + z_k^{62})/(2-2t))^2 \\
&+ (z_i^{95} - z_i^{51})^2/4 + (3z_i^{95} - z_i^{51} - 4z_i^{59})^2/12 + t^2(z_i^{51} + z_i^{59})^2/12 \\
&+ (z_k^{10,6} - z_k^{62})^2/4 + (3z_k^{10,6} - z_k^{62} - 4z_k^{6,10})^2/12 + t^2(z_k^{62} + z_k^{6,10})^2/12 \geq 0;
\end{aligned}$$

and the first assertion of the proposition follows.

We now prove the second assertion of the proposition. If $q = 0$, then by the previous argument we have $q_1 = 0$ and $q_2 = 0$; moreover, each of the squares in q_2 must be zero. It follows that

(c) $$z_i^{15} = z_i^{51} = z_i^{59} = z_i^{95} = z_k^{26} = z_k^{62} = z_k^{6,10} = z_k^{10,6} = z_j^{37} = z_l^{48} = 0$$

Introducing this in the definition of q_1 we obtain

$$(a_3 + a_7 - a_1 - 2a_5 - a_9)z_i^{91} + (a_3 + a_4 + a_7 + a_8 - a_2 - 2a_6 - a_{10})z_k^{10,2} = 0.$$

By our assumption, the coefficients of $z_i^{91}, z_k^{10,2}$ are > 0; since $z_i^{91}, z_k^{10,2}$ are ≥ 0, it follows that $z_i^{91} = z_k^{10,2} = 0$. These equalities, together with (b) and (c), show that all $z_u^{pq} = 0$. This implies the second assertion of the proposition, by Corollary 10.

16. Assume now that our graph is of type A, D or E. Let $\mathbf{i} = (i_1, i_2, \dots, i_m)$ be a sequence in I which defines a reduced expression of the longest element in the corresponding Weyl group. Let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ be a sequence in $\mathbf{N}$ with the following property: for any two indices $s < s'$ in $[1, m]$ such that $i_s = i_{s'} = i$ and such that $p \notin Z(i)$ whenever $s < p < s'$, we have

(a) $$\sum_p a_p \geq a_s + a_{s'}$$

(sum over all p with $s < p < s'$ which are joined with i in the graph).

The following question is suggested by the results in no.13,15: under what circumstances is the corresponding monomial 1(a) tight or semi-tight? Note that our assumption on $\mathbf{a}$ assures the positivity of the linear part of the non-homogeneous quadratic form $\bar{Q}$. (The quadratic part is independent of $\mathbf{a}$.)

17. *Proof of the Theorem.* Let $\pi : \tilde{F} \to E$ be as in no.3. Using π, we can form the fibre product $\tilde{F} \times_E \tilde{F}$. We have a natural diagonal imbedding $\tilde{F} \subset \tilde{F} \times_E \tilde{F}$.

Let $\bar{F} = \pi(\tilde{F})$, a closed subset of E. We shall prove that under the assumption of 6(a) (resp. 6(b)), the map $\pi : \tilde{F} \to \bar{F}$ is small (resp. semi-small) in the sense of Goresky and MacPherson. The Theorem follows from this, using the definitions. It is enough to prove the statements (a),(b) below.

(a) Under the assumption of 6(a), we have $\dim(\tilde F \times_E \tilde F - \tilde F) < \dim \tilde F$.
(b) Under the assumption of 6(b), we have $\dim(\tilde F \times_E \tilde F) = \dim \tilde F$.
We shall define a partition

$$\tilde F \times_E \tilde F = \sqcup_z (\tilde F \times_E \tilde F)_z$$

into locally closed subvarieties indexed by the elements $z \in T_{\mathbf{a}}$ (see no.5).

Let $z = (z_i^{pq}) \in T_{\mathbf{a}}$. If $\phi = (\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^m = 0)$ and $\psi = (\mathbf{V} = \mathbf{W}^0 \supset \mathbf{W}^1 \supset \cdots \supset \mathbf{W}^m = 0)$ are flags in F, we say that ϕ, ψ are in relative position z if there exists a direct sum decomposition

$$\mathbf{V}_i = \oplus_{r,s \in Z(i)} U_i^{rs}$$

with $\dim U_i^{rs} = z_i^{rs}$, such that

$$\mathbf{V}^p \cap \mathbf{V}_i = \oplus_{\substack{r,s \in Z(i)\\ r \le p}} U_i^{rs}$$

for all i and $p \in Z(i)$ and

$$\mathbf{W}^p \cap \mathbf{V}_i = \oplus_{\substack{r,s \in Z(i)\\ s \le p}} U_i^{rs}$$

for all i and $p \in Z(i)$.

Any two flags in F are in a well defined relative position (a version of Bruhat decomposition for GL_n also used in [BLM,§1]). Specifying the relative position of two flags defines a partition of $F \times F$ into locally closed subvarieties $(F \times F)_z$ indexed by $z \in T_{\mathbf{a}}$ above.

Let $(\tilde F \times_E \tilde F)_z$ be the inverse image of $(F \times F)_z$ under the obvious map $\tilde F \times_E \tilde F \to F \times F$.

This defines the required partition of $\tilde F \times_E \tilde F$.

For example, the diagonal $\tilde F \subset \tilde F \times_E \tilde F$ is precisely $(\tilde F \times_E \tilde F)_{z_{\mathbf{a}}}$ (see no.5).

Now the variety $(F \times F)_z$ is a connected homogeneous space of dimension

$$\sum_i \sum_{r,p \in Z(i)} a_p a_r - \sum_i \sum_{\substack{p,q,r,s\\ r \le p\\ s \le q}} z_i^{pq} z_i^{rs},$$

(compare [BLM, 2.2].)

We fix $(\phi, \psi) \in (F \times F)_z$ and choose U_i^{rs} as above. Let Y be the variety consisting of all $x \in E$ such that $(x, \phi) \in \tilde F, (x, \psi) \in \tilde F$. The condition that $x = (x_h)$ is in Y is that

$$x_h(U_{h'}^{pq}) \subset \oplus_{\substack{r,s \in Z(h'')\\ r \le p\\ s \le q}} U_{h''}^{rs}$$

for all $h \in \Omega$ and all $p, q \in Z(h')$.

Thus, Y is a vector space of dimension

$$\dim Y = \sum_h \sum_{\substack{p,q \in Z(h') \\ r,s \in Z(h'') \\ r \le p \\ s \le q}} \dim U_{h'}^{pq} \dim U_{h''}^{rs}$$

It follows that $(\tilde{F} \times_E \tilde{F})_z$ is irreducible, of dimension

$$-\sum_i \sum_{\substack{p,q,r,s \\ r \le p \\ s \le q}} z_i^{pq} z_i^{rs} + \sum_h \sum_{\substack{p,q;r,s \\ r \le p \\ s \le q}} z_{h'}^{pq} z_{h''}^{rs} + \sum_i \sum_{r,p \in Z(i)} a_p a_r.$$

In particular, $\tilde{F} \cong (\tilde{F} \times_E \tilde{F})_{z_{\mathbf{a}}}$ is irreducible, of dimension

$$-\sum_i \sum_{\substack{p,r \in Z(i) \\ r \le p}} a_p a_r + \sum_h \sum_{\substack{p \in Z(h') \\ r \in Z(h'') \\ r \le p}} a_p a_r + \sum_i \sum_{r,p \in Z(i)} a_p a_r.$$

Substracting the previous two expressions, we obtain

$$\dim(\tilde{F}) - \dim(\tilde{F} \times_E \tilde{F})_z = Q(z).$$

(See 5(a).) This implies directly statements (a) and (b) above. The Theorem is proved.

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