REMARKS ON SPRINGER’S REPRESENTATIONS

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INTRODUCTION

0.1. Let \( k \) be an algebraically closed field of characteristic exponent \( p \geq 1 \). Let \( G \) be a connected reductive algebraic group over \( k \) and let \( g \) be the Lie algebra of \( G \). Let \( \mathcal{U}_G \) be the variety of unipotent elements of \( G \) and let \( \mathcal{N}_g \) be the variety of nilpotent elements of \( g \) (we say that \( x \in g \) is nilpotent if for some/any closed imbedding \( G \subset GL(k^n) \), the image of \( x \) under the induced map of Lie algebras \( g \rightarrow \text{End}(k^n) \) is nilpotent as an endomorphism). Note that \( G \) acts on \( \mathcal{U}_G \) and \( g \) by the adjoint action. Let \( X_{\mathcal{U}_G} \) (resp. \( X_g \)) be the set of \( G \)-orbits on \( \mathcal{U}_G \) (resp. on \( \mathcal{N}_g \)).

We fix a prime number \( l \), \( l \neq p \). Let \( \hat{X}_{\mathcal{U}_G} \) (resp. \( \hat{X}_g \)) be the set of pairs \((\mathcal{O}, \mathcal{L})\) where \( \mathcal{O} \in X_{\mathcal{U}_G} \) (resp. \( \mathcal{O} \in X_g \)) and \( \mathcal{L} \) is an irreducible \( G \)-equivariant \( \mathcal{Q}_l \)-local system on \( \mathcal{O} \) up to isomorphism. Let \( W \) be the Weyl group of \( G \). For any Weyl group \( W \) let \( \text{Irr}(W) \) be the set of isomorphism classes of irreducible representations of \( W \) over \( \mathcal{Q}_l \).

In [Sp], Springer defined (assuming that \( p = 1 \) or \( p \gg 0 \)) natural injective maps \( S_G : \text{Irr}(W) \rightarrow \hat{X}_{\mathcal{U}_G} \), \( S_g : \text{Irr}(W) \rightarrow \hat{X}_g \) (each of these two maps determines the other since in this case we have canonically \( \hat{X}_{\mathcal{U}_G} = \hat{X}_g \)). In [L2] a new definition of the map \( S_G \) (based on intersection homology) was given which applies without restriction on \( p \). A similar method can be used to define \( S_g \) without restriction on \( p \) (see [X] and 2.2 below); note that in general \( \hat{X}_{\mathcal{U}_G}, \hat{X}_g \) cannot be identified. Now for any \( \mathcal{O} \in X_{\mathcal{U}_G} \) (resp. \( \mathcal{O} \in X_g \)), \((\mathcal{O}, \mathcal{Q}_l)\) is in the image of \( S_G \) (resp. \( S_g \)) hence there is a well defined injective map \( S'_G : X_{\mathcal{U}_G} \rightarrow \text{Irr}(W) \) (resp. \( S'_g : X_g \rightarrow \text{Irr}(W) \)) such that for any \( \mathcal{O} \in X_{\mathcal{U}_G} \) (resp. \( \mathcal{O} \in X_g \)) we have \( S'_G(\mathcal{O}) = E \) (resp. \( S'_g(\mathcal{O}) = E \)) where \( E \in \text{Irr}(W) \) is given by \( S_G(E) = (\mathcal{O}, \mathcal{Q}_l) \) (resp. \( S_g(E) = (\mathcal{O}, \mathcal{Q}_l) \)). Let \( \mathcal{G}_G \) be the image of \( S'_G : X_{\mathcal{U}_G} \rightarrow \text{Irr}(W) \). Let \( \mathcal{G}_g \) be the image of \( S'_g : X_g \rightarrow \text{Irr}(W) \).

In [L5], we gave an apriori definition (in the framework of Weyl groups) of the subset \( \mathcal{G}_G \) of \( \text{Irr}(W) \) which parametrizes the unipotent \( G \)-orbits in \( G \). In this paper we give an apriori definition (in a similar spirit) of the subset \( \mathcal{G}_g \) of \( \text{Irr}(W) \) which parametrizes the nilpotent \( G \)-orbits in \( g \). (See Proposition 3.2.) This relies heavily on work of Spaltenstein [S2],[S3] and on [HS]. As an application we define a natural injective map from the set of unipotent \( G \)-orbits in \( G \) to the set of nilpotent \( G \)-orbits in \( g \) (see 3.3); this maps preserves the dimension of an orbit.
In [Se], Serre asked whether a power \( u^n \) (where \( n \) is an integer not divisible by \( p \), \( p \geq 2 \)) of a unipotent element \( u \in G \) is conjugate to \( u \) under \( G \). This is well known to be true when \( p > 0 \). In \( \S 2 \) we answer positively this question in general using the theory of Springer’s representations; we also discuss an analogous property of nilpotent elements.

I wish to thank J.-P. Serre for his interesting questions and comments.

1. Combinatorics

1.1. For \( k \in \mathbb{N} \) let \( \mathcal{E}_k = \{a_* = (a_0, a_1, \ldots, a_k) \in \mathbb{N}^{k+1}; a_0 \leq a_1 \leq \cdots \leq a_k \} \). For \( a_* \in \mathcal{E}_k \) let \( |a_*| = \sum_i a_i \). For \( a'_* \in \mathcal{E}_k \) we set \( a_* + a'_* = (a_0 + a'_0, a_1 + a'_1, \ldots, a_k + a'_k) \). For any \( n \in \mathbb{N} \) let \( \mathcal{E}_k^n = \{a_* \in \mathcal{E}_k; |a_*| = n\} \). We have an imbedding \( \mathcal{E}_k^n \rightarrow \mathcal{E}_{k+1}^n \). (\( a_0, a_1, \ldots, a_k \)) \( \mapsto (0, a_0, a_1, \ldots, a_k) \). This is a bijection if \( k \) is sufficiently large with respect to \( n \). For \( n \in \mathbb{N} \) let
\[
\mathcal{C}_k^n = \{(a_*, a'_*) \in \mathcal{E}_k \times \mathcal{E}_k; |a_*| + |a'_*| = n\},
\]
\[
\mathcal{D}_k^n = \{(a_*, a'_*) \in \mathcal{C}_k^n; \text{ either } |a_*| > |a'_*| \text{ or } a_* = a'_*\}.
\]
Here \( k \) is large (relative to \( n \)), fixed. Let
\[
b_1\mathcal{C}_k^n = \{ (a_*, a'_*) \in \mathcal{C}_k^n; a'_i \leq a_i + 2 \ \forall i \in [0, k] \},
b_2\mathcal{C}_k^n = \{ (a_*, a'_*) \in \mathcal{C}_k^n; a'_i \leq a_i + 2 \ \forall i \in [0, k], a_i \leq a'_{i+1} \ \forall i \in [0, k-1] \},
c_1\mathcal{C}_k^n = \{ (a_*, a'_*) \in \mathcal{C}_k^n; a'_i \leq a'_{i+1} + 1 \ \forall i \in [0, k-1], a'_i \leq a_i + 1 \ \forall i \in [0, k] \},
d_1\mathcal{C}_k^n = \{ (a_*, a'_*) \in \mathcal{C}_k^n; a'_i \leq a_i \ \forall i \in [0, k] \},
d_1\mathcal{D}_k^n = \{ (a_*, a'_*) \in \mathcal{D}_k^n; a'_i \leq a_i \ \forall i \in [0, k] \},
d_2\mathcal{D}_k^n = \{ (a_*, a'_*) \in \mathcal{D}_k^n; a'_i \leq a_i + 1 \ \forall i \in [0, k-1] \},
\]
\[
b_2\mathcal{C}_k^n \subset b_1\mathcal{C}_k^n \subset b_2\mathcal{C}_k^n,
c_1\mathcal{C}_k^n \subset b_2\mathcal{C}_k^n \subset c_1\mathcal{C}_k^n,
d_1\mathcal{C}_k^n \subset d_2\mathcal{C}_k^n \subset d_1\mathcal{C}_k^n.
\]
The following statements are obvious. If \( (a_*, a'_*) \in \mathcal{C}_k^m \), \( (b_*, b'_*) \in \mathcal{C}_k^{m'} \) then \( (a_* + b_*, a'_* + b'_*) \in \mathcal{C}_k^{m+m'} \). If \( (a_*, a'_*) \in b_1\mathcal{C}_k^m \), \( (b_*, b'_*) \in d_1\mathcal{D}_k^m \), then \( (a_* + b_*, a'_* + b'_*) \in b_2\mathcal{C}_k^{m+m'} \). If \( (a_*, a'_*) \in d_2\mathcal{D}_k^m \), \( (b_*, b'_*) \in d_2\mathcal{D}_k^{m'} \) then \( (a_* + b_*, a'_* + b'_*) \in d_1\mathcal{C}_k^{m+m'} \).

In the following result we assume that \( k \) is large relative to \( n \).

**Proposition 1.2.** (a) Let \( (c_*, c'_*) \in \mathcal{C}_k^n \). Then either \( (c_*, c'_*) \in c_1\mathcal{C}_k^n \) or there exist \( m \geq 1, m' \geq 1 \) such that \( m + m' = n \) and \( (a_*, a'_*) \in \mathcal{C}_k^m \), \( (b_*, b'_*) \in \mathcal{C}_k^{m'} \) such that \( (c_*, c'_*) = (a_* + b_*, a'_* + b'_*) \).

(b) Let \( (c_*, c'_*) \in b_1\mathcal{C}_k^n \). Then either \( (c_*, c'_*) \in b_1\mathcal{C}_k^n \) or there exist \( m \geq 0, m' \geq 2 \) such that \( m + m' = n \) and \( (a_*, a'_*) \in b_1\mathcal{C}_k^m \), \( (b_*, b'_*) \in d_1\mathcal{D}_k^{m'} \), such that \( (c_*, c'_*) = (a_* + b_*, a'_* + b'_*) \).

(c) Let \( (c_*, c'_*) \in d_1\mathcal{C}_k^n \). Then either \( (c_*, c'_*) \in d_1\mathcal{C}_k^n \) or there exist \( m \geq 2, m' \geq 2 \) such that \( m + m' = n \) and \( (a_*, a'_*) \in d_1\mathcal{D}_k^m \), \( (b_*, b'_*) \in d_1\mathcal{D}_k^{m'} \) such that \( (c_*, c'_*) = (a_* + b_*, a'_* + b'_*) \).
We prove (a). Assume first that $c_s < c_{s+1}$ for some $s \in [0, k-1]$. Define $(b_s, b'_s) \in C^k_r$, $r = k-s > 0$, by $b_i = 1$ for $i \in [s+1, k]$, $b_i = 0$ for $i \in [0, s]$, $b'_i = 0$ for $i \in [0, k]$. Define $(a_s, a'_s) \in C^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [s+1, k]$, $a_i = c_i$ in $[0, s]$, $a'_i = c'_i$. We have $a_s + b_s = c_s$, $a'_s + b'_s = c'_s$. If $r < n$ we see that (a) holds. If $r = n$ then $(c_s, c'_s) = (b_s, b'_s) \in C^1 C^n_k$ and (a) holds again.

Next we assume that $c'_s < c'_{s+1}$ for some $s \in [0, k-1]$. Define $(b_s, b'_s) \in C^k_r$, $r = k-s > 0$, by $b_i = 0$ for $i \in [s+1, k]$, $b'_i = 1$ for $i \in [s+1, k]$, $b'_i = 0$ for $i \in [0, s]$. Define $(a_s, a'_s) \in C^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [s+1, k]$, $a'_i = c'_i$ for $i \in [0, s]$. We have $a_s + b_s = c_s$, $a'_s + b'_s = c'_s$. If $r < n$ we see that (a) holds. If $r = n$ then $(c_s, c'_s) = (b_s, b'_s) \in C^1 C^n_k$ and (a) holds again.

Finally we assume that $c_0 = c_1 = \cdots = c_r$; $c'_0 = c'_1 = \cdots = c'_r$. Since $k$ is large we can assume that $c_0 = 0$, $c'_0 = 0$. Then $n = 0$ and $(c_s, c'_s) \in C^1 C^n_k$.

We prove (b). If $n = 0$ we have clearly $(c_s, c'_s) \in b^1 C^k_k$. Hence we can assume that $n > 0$ and that the result is true when $n$ is replaced by $n' \in (0, n-1]$.

Assume first that we can find $0 < t \leq s \leq k$ such that $c'_j = c_j + 2$ for $j \in [s+1, k]$, $c'_j < c_j + 2$ for $j \in [t, s]$, $c_{t-1} < c_t$. Note that if $s < k$ then $c'_s < c'_{s+1}$; indeed, $c'_s = c_s - 2 \leq c_{s+1} - 2 = c'_{s+1}$. Define $(b_s, b'_s) \in D^k_r$, $r = 2k - t - s + 1 > 0$ by $b_i = 1$ for $i \in [t, k]$, $b_i = 0$ for $i \in [0, t-1]$, $b'_i = 1$ for $i \in [s+1, k]$, $b'_i = 0$ for $i \in [0, s]$. Define $(a_s, a'_s) \in b^1 C^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [t, k]$, $a_i = c_i$ for $i \in [0, t-1]$, $a'_i = c'_i$ for $i \in [s+1, k]$, $a'_i = c'_i$ for $i \in [0, s]$. We have $a_s + b_s = c_s$, $a'_s + b'_s = c'_s$. If $r \geq 2$ we see that (b) holds. If $r = 1$ then $t = s = k$ and $a_k = c_k - 1$, $a_i = c_i$ for $i \in [0, k-1]$, $a'_i = c'_i$ for $i \in [0, k]$. The induction hypothesis is applicable to $(a_s, a'_s) \in b^1 C^k_{n-1}$. If $(a_s, a'_s) \in b^1 C^k_{n-1}$ then clearly $(c_s, c'_s) \in b^1 C^k_{n-1}$ and (b) holds. If $(a_s, a'_s) \notin b^1 C^k_{n-1}$ then we can find $m, m' \geq 2$ such that $m + m' = n - 1$ and $(a_s, a'_s) \in b^1 C^k_{n-1}$, $(b_s, b'_s) \in D^k_{r'}$ such that $(a_s, a'_s) = (a_s + b_s, a'_s + b'_s)$. Then $(c_s, c'_s) = (a_s + b_s, a'_s + b'_s + b'_s)$ where $(a_s, a'_s) \in b^1 C^m_m$, $(b_s + b_s, b'_s + b'_s) \in D^{m+1}_{r'}$ so that (b) holds.

Next we assume that $c_i > 0$ for some $i$. Then we have $0 = c_0 = c_1 = \cdots = c_{l-1} = c_l$ for some $l \in [0, k]$. If $c'_s < c_s + 2$ for some $s \in [l, k]$ then we can assume that $s$ is maximum possible with this property and there are two possibilities. Either $c'_i < c_i + 2$ for all $i \in [l, s]$ and then by the previous paragraph (with $t = l$) we see that (b) holds; or $c'_i = c_i + 2$ for some $i \in [l, s]$ and letting $t = 1$ be the largest such $i$ we have $0 < t \leq s$, $c'_j < c_j + 2$ for $j \in [l, s]$, $c'_j = c_j + 2$ for $j \in [s+1, k]$ and $c_{t-1} = c'_t - 2 \leq c'_t - c_t$. Using again the previous paragraph we see that (b) holds. Thus we may assume that $c'_i = c_i + 2$ for all $i \in [l, k]$. Assume in addition that $c'_s < c'_{s+1}$ for some $s \in [l, k-1]$. We can assume that $s$ is maximum possible so that $c'_s < c'_{s+1} = \cdots = c'_k$. We have $c_{s+1} = c_{s+1} - 2 > c'_s - 2 = c_s$ hence $c_s < c_{s+1}$. Define $(b_s, b'_s) \in D^k_r$, $r = 2k - 2s \geq 2$, by $b_i = 1$ for $i \in [s+1, k]$, $b_i = 0$ for $i \in [0, s]$, $b'_i = 1$ for $i \in [s+1, k]$, $b'_i = 0$ for $i \in [0, s]$. Define $(a_s, a'_s) \in b^1 C^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [s+1, k]$, $a_i = c_i$ for $i \in [0, s]$, $a'_i = c'_i - 1$ for $i \in [s+1, k]$, $a'_i = c'_i$ for $i \in [0, s]$. We have $a_s + b_s = c_s$, $a'_s + b'_s = c'_s$. We see that (b) holds. Thus we can assume that $c'_i = c'_{i+1} = \cdots = c'_k = N + 2$ so that $c_l = c_{l+1} = \cdots = c_k = N$. 

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Note that $c_i' \leq 2$ for $i \in [0, l - 1]$. We have $(c_*, c_*)' = b_1C_k^n$ so that (b) holds.

Finally we assume that $c_0 = c_1 = \cdots = c_k = 0$. Then $c_i' \leq 2$ for $i \in [0, k]$ and $(c_*, c_*)' \in b_1C_k^n$ so that (b) holds. This completes the proof of (b).

We prove (c). If $n = 0$ we have clearly $(c_*, c_*)' \in d_1D_k^n$. Hence we can assume that $n > 0$ and that the result is true when $n$ is replaced by $n' \in [0, n - 1]$.

Assume first that we can find $0 \leq t \leq s \leq k$ such that $c_j' = c_j$ for $j \in [s + 1, k]$, $c_j' < c_j$ for $j \in [t, s]$, $c_{l-1} < c_l$. Note that if $s < k$ then $c_s' < c_0'$; indeed, $c_s' < c_s \leq c_{s+1} = c_{s+1}'$. Define $(b_*, b_*') \in d_1D_r^n$, $r = 2k - t - s + 1 > 0$ by $b_i = 1$ for $i \in [t, k]$, $b_i = 0$ for $i \in [0, t - 1]$, $b_i' = 1$ for $i \in [s + 1, k]$, $b_i' = 0$ for $i \in [0, s]$. Define $(a_*, a_*') \in d_1D_{n-r}$ by $a_i = c_i - 1$ for $i \in [t, k]$, $a_i = c_i$ for $i \in [0, t - 1]$, $a_i' = c_i' - 1$ for $i \in [s + 1, k]$, $a_i' = c_i$ for $i \in [0, s]$. We have $a_* + b_* = c_*$, $a_*' + b_*' = c_*'$. If $n - 2 \geq r \geq 2$ we see that (c) holds. If $r = 1$ then $t = s = k$ and $c_k = c_k - 1$, $a_i = c_i$ for $i \in [0, k - 1]$, $a_i' = c_i'$ for $i \in [0, k]$. The induction hypothesis is applicable to $(a_*, a_*') \in d_1D_{n-r}$. If $(a_*, a_*') \in d_1D_{n-1}$ then clearly $(c_*, c_*)' = d_1D_{n-1}$ and (c) holds.

Next we assume that $c_i > 0$ for some $i$. Then we have $0 = c_0 = c_1 = \cdots = c_{l-1} < c_l$ for some $l \in [0, k]$. If $c_s' < c_s$ for some $s \in [l, k]$ then we can assume that $s$ is maximum possible with this property and there are two possibilities. Either $c_i' < c_i$ for all $i \in [l, s]$ and then by the previous paragraph (with $t = 1$) we see that (c) holds; or $c_i' = c_i$ for some $i \in [l, s]$ and letting $t - 1$ be the largest such $i$ we have $0 < t \leq s$, $c_j' < c_j$ for $j \in [t, s]$, $c_j' = c_j$ for $j = t - 1$ and $c_{t-1} = c_{t-1}' \leq c_i' < c_i$; using again the previous paragraph we see that (c) holds. Thus we may assume that $c_i' = c_i$ for all $i \in [l, k]$. Assume in addition that $c_s' < c_{s+1}'$ for some $s \in [l, k - 1]$. We can assume that $s$ is maximum possible so that $c_s' < c_{s+1}' = \cdots = c_k'$. We have $c_{s+1}' = c_{s+1}' > c_s' = c_s$ hence $c_s < c_{s+1}$. Define $(b_*, b_*') \in d_1D_{n-r}$, $r = 2k - 2s \geq 2$, by $b_i = 1$ for $i \in [s + 1, k]$, $b_i = 0$ for $i \in [0, s]$, $b_i' = 1$ for $i \in [s + 1, k]$, $b_i' = 0$ for $i \in [0, s]$. Define $(a_*, a_*') \in d_1D_{n-r}$ by $a_i = c_i - 1$ for $i \in [s + 1, k]$, $a_i = c_i$ for $i \in [0, s]$, $a_i' = c_i' - 1$ for $i \in [s + 1, k]$, $a_i' = c_i'$ for $i \in [0, s]$. We have $a_* + b_* = c_*$, $a_*' + b_*' = c_*'$. If $r \leq n - 2$ we see that (c) holds. If $r = n - 1$ then $a_0 = 0$ for $i \in [0, k - 1]$, $a_0 = 0$, $a_i' = 0$ for $i \in [0, k]$; hence $c_i = 1$ for $i \in [s + 1, k - 1]$, $c_k = 2$, $c_i = 0$ for $i \in [0, s]$, $c_i' = 1$ for $i \in [s + 1, k]$, $c_i' = 0$ for $i \in [0, s]$. Hence $(c_*, c_*)' \in d_1D_k^n$. Note that $c_i' = 0$ for $i \in [0, l - 1]$. We have $(c_*, c_*)' \in d_1D_k^n$ so that (c) holds. Finally we assume that $c_0 = c_1 = \cdots = c_k = 0$. Then $c_i' = 0$ for $i \in [0, k]$. In this case we have $n = 0$ and $(c_*, c_*)' \in d_1D_k^n$ so that (c) holds. This completes the
proof of (c).

2. On Serre’s questions

2.1. For any affine algebraic group $H$ over $k$ we denote by Lie $H$ the Lie algebra of $H$. For any $O \in \mathcal{X}_G$ (or $O \in \mathcal{X}_g$) we set $d_O = 2 \dim B - \dim O$.

2.2. We recall the definition of Springer’s representations following [L2]. Let $B$ be the variety of Borel subgroups of $G$. Let $\tilde{B} = \{(g, B) \in G \times B; g \in B\}$ and let $f : \tilde{B} \to G$ be the first projection. Let $K = f!\mathcal{Q}_t$. In [L2] it was observed that $K$ is an intersection cohomology complex on $G$ coming from a local system on the open dense subset of $G$ consisting on regular semisimple elements. Moreover $W$ acts naturally on this local system and hence, by ”analytic continuation”, on $K$. In particular, if $O \in \mathcal{X}_G$ and $i \in \mathbb{Z}$ then $W$ acts naturally on the $i$-th cohomology sheaf $\mathcal{H}^i K|_O$ of $K|_O$, an irreducible $G$-equivariant local system on $O$; hence if $L$ is an irreducible $G$-equivariant local system on $O$ then $W$ acts naturally on the $\mathcal{Q}_t$-vector space $\text{Hom}(L, \mathcal{H}^i K|_O)$. We denote this $W$-module (with $i = d_O$) by $V_{O,L}$. As shown in [L4], $V_{O,L}$ is either 0 or of the form $\mathcal{Q}_l \otimes E$ where $E \in \text{Irr}(W)$; moreover any $E \in \text{Irr}(W)$ arises in this way from a unique $(O, L)$ and $E \mapsto (O, L)$ is an injective map

$$S_G : \text{Irr}(W) \to \mathcal{X}_g.$$

We would like to define a similar map from $\text{Irr}(W)$ to $\mathcal{X}_g$. Let $\tilde{B}' = \{(x, B) \in g \times B; x \in \text{Lie } B\}$ and let $f' : \tilde{B}' \to g$ be the first projection. Let $K' = f'!\mathcal{Q}_t$.

Now if $p$ is small the set of regular semisimple elements in $g$ may be empty (this is the case for example if $G = SL_2(k)$, $p = 2$) so the method of [L4] cannot be used directly. However, T.Xue [X] has observed that the method of [L4], [L2] can be applied if $G$ is a classical group of adjoint type and $p = 2$ (in that case the set of regular semisimple elements in $g$ is open dense in $g$). More generally for any $G$ which is adjoint, the set of regular semisimple elements in $g$ is open dense in $g$. (Here is a proof. We must only check that if $T$ is a maximal torus of $G$ and $t = \text{Lie } T$ then the set $t_{\text{reg}}$ of regular semisimple elements in $t$ is open dense in $t$. Let $Y = \text{Hom}(k^*, T)$. We have $t = k \otimes Y$. Now $t_{\text{reg}}$ is the set of all $x \in t$ such that for any root $\alpha : t \to k$ we have $\alpha(x) \neq 0$. It is enough to show that any root $\alpha : t \to k$ is $\neq 0$. We have $\alpha = 1 \otimes \alpha_0$ where $\alpha_0 : Y \to \mathbb{Z}$ is a well defined homomorphism. It is enough to show that $\alpha_0$ is surjective. This follows from the adjointness of $G$.) As in the group case it now follows that $K'$ is an intersection cohomology complex on $g$ coming from a local system on $g_{\text{reg}}$. Moreover $W$ acts naturally on this local system and hence, by ”analytic continuation”, on $K'$. In particular, if $O \in \mathcal{X}_g$ and $i \in \mathbb{Z}$ then $W$ acts naturally on the $i$-th cohomology sheaf $\mathcal{H}^i K'|_O$ of $K'|_O$, an irreducible $G$-equivariant local system on $O$; hence if $L$ is an irreducible $G$-equivariant local system on $O$ then $W$ acts naturally on the $\mathcal{Q}_t$-vector space $\text{Hom}(L, \mathcal{H}^i K'|_O)$. We denote this $W$-module (with $i = d_O$) by $V_{O,L}$. As in [L4], [X], $V_{O,L}$ is either 0 or of the form $\mathcal{Q}_l \otimes E$ where $E \in \text{Irr}(W)$; moreover any $E \in \text{Irr}(W)$ arises in this way from a unique $(O, L)$ and $E \mapsto (O, L)$
is an injective map

\[ S_\mathfrak{g} : \text{Irr}(W) \to \hat{X}_\mathfrak{g}. \]

If \( G \) is not assumed to be adjoint, let \( G_{\text{ad}} \) be the adjoint group of \( G \) and let \( \mathfrak{g}_{\text{ad}} = \text{Lie } G_{\text{ad}}. \) The obvious map \( \pi : \mathfrak{g} \to \mathfrak{g}_{\text{ad}} \) induces a bijective morphism \( \mathcal{N}_\mathfrak{g} \to \mathcal{N}_{\mathfrak{g}_{\text{ad}}} \) and a bijection \( \mathcal{X}_\mathfrak{g} \to \mathcal{X}_{\mathfrak{g}_{\text{ad}}}. \) Now any \( G_{\text{ad}} \)-equivariant irreducible \( \mathbb{Q}_l \)-local system on a \( G_{\text{ad}} \)-orbit in \( \mathcal{N}_{\mathfrak{g}_{\text{ad}}} \) can be viewed as an irreducible \( G \)-equivariant \( \mathbb{Q}_l \)-local system on the corresponding \( G \)-orbit in \( \mathcal{N}_\mathfrak{g}. \) This yields an injective map \( \mathcal{X}_{\mathfrak{g}_{\text{ad}}} \to \hat{X}_\mathfrak{g}. \) We define an injective map \( S_\mathfrak{g} : \text{Irr}(W) \to \hat{X}_\mathfrak{g} \) as the composition of the last map with \( S_{\mathfrak{g}_{\text{ad}}}. \)

2.3. For any \( u \in U_G \), let \( \mathcal{B}_u = \{ B \in \mathcal{B} ; u \in B \} \) and let \( \mathcal{O} \) be the \( G \)-orbit of \( u \) in \( U_G \). Note that \( \mathcal{B}_u \) is a non-empty subvariety of \( \mathcal{B} \) of dimension \( d_{\mathcal{O}}/2 \), see [S1]. Using this and the definition of \( S_G \) we see that \( (\mathcal{O}, \mathbb{Q}_l) \) is in the image of \( S_G \). Hence there is a well defined injective map \( S'_G : \mathcal{X}_G \to \text{Irr}(W) \) such that for any \( \mathcal{O} \in \mathcal{X}_G \) we have \( S'_G(\mathcal{O}) = E \) where \( E \in \text{Irr}(W) \) is given by \( S_G(E) = (\mathcal{O}, \mathbb{Q}_l) \).

Similarly, for any \( x \in \mathcal{N}_\mathfrak{g} \) let \( \mathcal{B}_x = \{ B \in \mathcal{B} ; x \in \text{Lie } B \} \) and let \( \mathcal{O} \) be the \( G \)-orbit of \( x \) in \( \mathcal{N}_\mathfrak{g} \). Note that \( \mathcal{B}_x \) is a non-empty subvariety of \( \mathcal{B} \) of dimension \( d_{\mathcal{O}}/2 \), see [HS]. Using this and the definition of \( S_\mathfrak{g} \) we see that \( (\mathcal{O}, \mathbb{Q}_l) \) is in the image of \( S_\mathfrak{g} \). Hence there is a well defined injective map \( S'_\mathfrak{g} : \mathcal{X}_\mathfrak{g} \to \text{Irr}(W) \) such that for any \( \mathcal{O} \in \mathcal{X}_\mathfrak{g} \) we have \( S'_\mathfrak{g}(\mathcal{O}) = E \) where \( E \in \text{Irr}(W) \) is given by \( S_\mathfrak{g}(E) = (\mathcal{O}, \mathbb{Q}_l) \).

The maps \( S'_G, S'_\mathfrak{g} \) can be described directly as follows. For \( i \in \mathbb{Z} \), we may identify \( H^i(\mathcal{B}) \) (\( l \)-adic cohomology) with the stalk of \( H^iK \) at \( 1 \in G \) hence the \( W \)-action on \( K \) induces a \( W \)-action on the vector space \( H^i(\mathcal{B}) \). If \( \mathcal{O} \in \mathcal{X}_G \) and \( u \in \mathcal{O} \) then the inclusion \( \mathcal{B}_u \to \mathcal{B} \) induces a linear map \( f_u : H^{d_{\mathcal{O}}}(\mathcal{B}) \to H^{d_{\mathcal{O}}}(\mathcal{B}_u) \) whose kernel is \( W \)-stable; hence there is an induced action of \( W \) on the image \( I_u \) of \( f_u \). The \( W \)-module \( I_u \) is of the form \( \mathbb{Q}_l \otimes E \) for a well defined \( E \in \text{Irr}(W) \). We have \( S'_G(\mathcal{O}) = E \). Similarly, if \( \mathcal{O} \in \mathcal{X}_\mathfrak{g} \) and \( x \in \mathcal{O} \) then the inclusion \( \mathcal{B}_x \to \mathcal{B} \) induces a linear map \( \phi_x : H^{d_{\mathcal{O}}}(\mathcal{B}) \to H^{d_{\mathcal{O}}}(\mathcal{B}_x) \) whose kernel is \( W \)-stable; hence there is an induced action of \( W \) on the image \( I_x \) of \( \phi_x \). The \( W \)-module \( I_x \) is of the form \( \mathbb{Q}_l \otimes E \) for a well defined \( E \in \text{Irr}(W) \). We have \( S'_\mathfrak{g}(\mathcal{O}) = E \).

Let \( \mathfrak{S}_G \) be the image of \( S'_G : \mathcal{X}_G \to \text{Irr}(W) \). Let \( \mathfrak{S}_\mathfrak{g} \) be the image of \( S'_\mathfrak{g} : \mathcal{X}_\mathfrak{g} \to \text{Irr}(W) \).

2.4. Any automorphism \( a : G \to G \) induces a Lie algebra automorphism \( a' : \mathfrak{g} \to \mathfrak{g} \) and an automorphism \( \mathfrak{a}_G \) of \( W \) as a Coxeter group. Now \( a \) (resp. \( a' \)) induces a permutation \( \mathcal{O} \to a(\mathcal{O}) \) (resp. \( \mathcal{O} \to a'(\mathcal{O}) \)) of \( \mathcal{X}_G \) (resp. \( \mathcal{X}_\mathfrak{g} \)) denoted again by \( a \) (resp. \( a' \)). Also \( \mathfrak{a}_G \) induces in an obvious way a permutation of \( \text{Irr}(W) \) denoted again by \( a \). From the definitions we see that

\[
\mathfrak{a}_G S'_G = S'_G a, \quad a S'_{\mathfrak{g}} = S'_G a'.
\]

Let \( x \mapsto x^p \) be the \( p \)-th power map \( \mathfrak{g} \to \mathfrak{g} \) (if \( p > 1 \)) and the 0 map \( \mathfrak{g} \to \mathfrak{g} \) (if \( p = 1 \)). The \( r \)-th iteration of this map is denoted by \( x \mapsto x^{p^r} \); this restricts to a map \( \mathcal{N}_\mathfrak{g} \to \mathcal{N}_\mathfrak{g} \) which is 0 for \( r \gg 0 \). The following result answers questions of Serre [Se].
Proposition 2.5. (a) Let $u \in U_G$ and let $n \in \mathbb{Z}$ be such that $nn' = 1$ in $k$ for some $n' \in \mathbb{Z}$. Then $u^n$ and $u$ are $G$-conjugate.

(b) Let $x \in N_\mathfrak{g}$ and let $x' = a_0x + a_1x^p + a_2x^{p^2} + \ldots$ where $a_0, a_1, a_2, \ldots \in k$, $a_0 \neq 0$ (so that $x' \in N_\mathfrak{g}$). Then $x'$, $x$ are $G$-conjugate.

We prove (a). Let $O$ be the $G$-orbit of $u$ and let $O'$ be the $G$-orbit of $u' := u^n$. Clearly, $B_u \subset B_{u'}$. Since $u'$ is a power of $u$ we have also $B_{u'} \subset U$ hence $B_u = B_{u'}$. From $\dim B_u = \dim B_{u'}$ we see that $d_O = d_{O'}$. The map $f_u : H^{d_O}(B) \to H^{d_{O'}}(B_u)$ in 2.3 remains the same if $u$ is replaced by $u'$. From the description of $S'_G$ given in 2.3 we deduce that $S'_G(O) = S'_G(O')$. Since $S'_G$ is injective we deduce that $O = O'$. This proves (a).

We prove (b). Let $O$ be the $G$-orbit of $x$ and let $O'$ be the $G$-orbit of $x'$. Clearly, $B_x \subset B_{x'}$. Since $x = a_0x' + a_1x^{p} + a_2x^{p^2} + \ldots$ with $a_0, a_1, a_2, \ldots \in k$, $a_0 = a_0^{-1}$, we have $B_{x'} \subset B_x$ hence $B_{x'} = B_x$. From $\dim B_x = \dim B_{x'}$ we see that $d_O = d_{O'}$. The map $\phi_x : H^{d_O}(B) \to H^{d_{O'}}(B_{x'})$ in 2.3 remains the same if $x$ is replaced by $x'$. From the description of $S'_G$ given in 2.3 we deduce that $S'_G(O) = S'_G(O')$. Since $S'_G$ is injective we deduce that $O = O'$. This proves (b).

Parts (a),(b) of the following result answer questions of Serre [Se]; the proof of (b) below (assuming (a)) is due to Serre [Se].

Proposition 2.6. Let $c : G \to G$ be an automorphism such that for some maximal torus $T$ of $G$ we have $c(t) = t^{-1}$ for all $t \in T$. Let $\bar{c} : \mathfrak{g} \to \mathfrak{g}$ be the automorphism of $\mathfrak{g}$ induced by $c$.

(a) For any $u \in U_G$, $c(u), u$ are $G$-conjugate.

(b) For any $g \in G$, $c(g), g^{-1}$ are $G$-conjugate.

(c) For any $x \in N_\mathfrak{g}$, $\bar{c}(x), x$ are $G$-conjugate.

(d) For any $x \in \mathfrak{g}$, $\bar{c}(x), -x$ are $G$-conjugate.

We prove (a). Let $\zeta : W \to W$ be the automorphism induced by $c$. If $B \in B$ contains $T$ then $T \subset c(B)$ and $B, c(B)$ are in relative position $w_0$, the longest element of $W$. Hence if $B, B'$ in $B$ contain $T$ and are in relative position $w \in W$ then $c(B), c(B')$ contain $T$ and are in relative position $w_0 w w_0^{-1}$. They are also in relative position $\zeta(w)$. It follows that $\zeta(w) = w_0 w w_0^{-1}$ for all $w \in W$. Hence the induced permutation $\zeta : \text{Irr}(W) \to \text{Irr}(W)$ is the identity map. Let $O$ be the $G$-orbit of $u \in U_G$. Then $c(O)$ is the $G$-orbit of $c(u)$. By 2.4 we have $S'_G(c(O)) = \zeta(S'_G(O)) = S'_G(O)$. Since $S'_G$ is injective it follows that $O = c(O)$. This proves (a).

Following [Se], we prove (b) by induction on $\dim(G)$. If $\dim G = 0$ the result is trivial. Now assume that $\dim G > 0$. Write $g = su = us$ with $s$ semisimple, $u$ unipotent. If the result holds for $g_1 \in G$ then it holds for any $G$-conjugate of $g_1$. Hence by replacing $g$ by a conjugate we can assume that $s \in T$ so that $c(s) = s^{-1}$. Let $Z(s)^0$ be the connected centralizer of $s$, a connected reductive subgroup of $G$ containing $T$. Note that $c$ restricts to an automorphism of $Z(s)^0$ of the same type as $c : G \to G$. Moreover we have $g \in Z(s)^0$. If $Z(s)^0 \neq G$ then by the induction hypothesis we see that $c(g), g^{-1}$ are conjugate under $Z(s)^0$
Proposition 3.2. We can now state the following result. When \( r \) (a subset of \( W \)) is an integer then \( c(u) \) is central in \( G \). Hence \( c(u), u^{-1} \) are conjugate in \( G \). In other words, for some \( h \in G \) we have \( c(u) = hu^{-1}h^{-1} \). Since \( s \) is central in \( G \) and \( c(s) = s^{-1} \) we have \( c(s) = hs^{-1}h^{-1} \). It follows that \( c(g) = c(s)c(u) = hs^{-1}h^{-1}hu^{-1}h = hs^{-1}u^{-1}h^{-1} = hg^{-1}h^{-1} \). This proves (b).

The proof of (c) is completely similar to that of (a); it uses \( S_L^p \) instead of \( S_G \). The proof of (d) is completely similar to that of (b); it uses (c) and 2.5(b) instead of (b) and 2.5(a).

3. A parametrization of the set of nilpotent \( G \)-orbits in \( \mathfrak{g} \)

3.1. Let \( V \) be a finite dimensional \( \mathbb{Q} \)-vector space. Let \( R \subset V^* = \text{Hom}(V, \mathbb{Q}) \) be a (reduced) root system and let \( W \subset \text{GL}(V) \) be the Weyl group of \( R \). Let \( \Pi \) be a set of simple roots for \( R \). Let \( \Theta = \{ \beta \in R; \beta - \alpha \notin R \cup \{0\} \text{ for all } \alpha \in \Pi \} \). For any integer \( r \geq 1 \) let \( \mathcal{A}_r \) be the set of all \( J \subset \Pi \cup \Theta \) such that \( J \) is linearly independent in \( V^* \) and \( \sum_{\alpha \in \Pi} Z_{\alpha}/\sum_{\beta \in J} Z_{\beta} \) is finite of order \( r \) for some \( k \in \mathbb{N} \). For \( J \in \mathcal{A}_r \) let \( W_J \) be the subgroup of \( W \) generated by the reflections with respect to the roots in \( J \). For any \( E \in \text{Irr}(W) \) let \( b_E \) be the smallest integer \( \geq 0 \) such that \( E \) appears with multiplicity \( m_E > 0 \) in the \( b_E \)-th symmetric power of \( V \) regarded as a \( W \)-module. Let \( \text{Irr}(W)^\dagger = \{ E \in \text{Irr}(W); m_E = 1 \} \). Replacing here \( (V, W) \) by \( (V, W_J) \) with \( J \in \mathcal{A}_r \), we see that \( b_E \) is defined for any \( E \in \text{Irr}(W_J) \) and that \( \text{Irr}(W_J)^\dagger \) is defined. For \( J \in \mathcal{A}_r \) and \( E \in \text{Irr}(W_J)^\dagger \) there is a unique \( \tilde{E} \in \text{Irr}(W) \) such that \( \tilde{E} \) appears with multiplicity 1 in \( \text{Ind}_{W_J}^W E \) and \( b_{\tilde{E}} = b_E \); moreover, we have \( \tilde{E} \in \text{Irr}(W)^\dagger \). We set \( \tilde{E} = j_{W_J}^WE \). Define \( S^1_W \subset \text{Irr}(W)^\dagger \) as in \([L5, 1.3]\). 

Replacing \( (V, W) \) by \( (V, W_J) \) with \( J \in \mathcal{A}_r \), we obtain a subset \( S^1_{W_J} \subset \text{Irr}(W_J)^\dagger \). For any integer \( r \geq 1 \) let \( S^1_W \) be the set of all \( E \in \text{Irr}(W) \) such that \( E = j_{W_J}^W E_1 \) for some \( J \in \mathcal{A}_r \) and some \( E_1 \in S^1(W_J) \) (see \([L5, 1.3]\)). If \( r = 1 \) this agrees with the earlier definition of \( S^1_W \) since in this case \( W_J = W \) for any \( J \in \mathcal{A}_r \). For any integer \( r \geq 1 \) we define a subset \( T^r_W \) of \( \text{Irr}(W)^\dagger \) by induction on \( |W| \) as follows. If \( W = \{1\} \) we set \( T^r_W = \text{Irr}(W) \). If \( W \neq \{1\} \) then \( T^r_W \) is the set of all \( E \in \text{Irr}(W) \) such that either \( E \in S^1_W \) or \( E = j_{W_J}^W E_1 \) for some \( J \in \mathcal{A}_r \) with \( W_J \neq W \) and some \( E_1 \in T^r(W_J) \). From the definition it is clear that \( S^1_W \subset S_W \subset T^r_W \).

When \( r = 1 \) we have \( S_W = T^1_W \).

We apply these definitions in the case where \( r = p, V = \mathbb{Q} \otimes Y_G \) (with \( T \) being "the maximal torus" of \( G \) and \( Y_G = \text{Hom}(k^*, T) \)), \( R \) is "the root system" of \( G \) (a subset of \( V^* \)) with its canonical set of simple roots and \( W = W \) viewed as a subgroup of \( GL(V) \). Then the subsets \( S^p_W \subset S^p_W \subset T^p_W \) of \( \text{Irr}(W) \) are defined. We can now state the following result.

Proposition 3.2. \((a) \) We have \( S_G = S^p_W \).

\((b) \) We have \( S_W = T^p_W \).

For (a) see \([L5, 1.4]\). The proof of (b) is given in 3.5.
Corollary 3.3. There is a unique (injective) map \( \tau : X_G \to X_{g} \) such that \( S'_G(\xi) = S'_{g}(\tau(\xi)) \) for all \( \xi \in X_G \).

The existence and uniqueness of \( \tau \) follows from \( \mathcal{S}_G \subset \mathcal{S}_g \) which in turn follows from 3.2 and the inclusion \( S^p_W \subset T^p_W \).

It is known that when \( p \neq 2 \) we have \( \text{card} \mathcal{S}_G = \text{card} \mathcal{S}_g \); hence in this case \( \tau \) is a bijection.

3.4. For \( n \in \mathbb{N} \) let \( W_n \) be the group of all permutations of the set \( \{1, 2, \ldots, n, n', \ldots, 2', 1'\} \)
which commute with the involution \( i \mapsto i', i' \mapsto i \); let \( W'_n \) be the subgroup of \( W_n \) consisting of the even permutations. Assume that \( k \in \mathbb{N} \) is large relative to \( n \). When \( G \) is adjoint simple of type \( B_n \) or \( C_n \) \( (n \geq 2) \) we identify \( W = W_n \) in the standard way; we have a bijection \( [a_s, a'_s] \mapsto (a_s, a'_s), \text{Irr} (W) = \text{Irr}(W_n) \leftrightarrow C^n_k \) as in [L1, 2.3]; moreover, \( \text{Irr}(W) = \text{Irr}(W)^1 \), see [L1, 2.4]. When \( G \) is adjoint simple of type \( D_n \) \( (n \geq 4) \) we identify \( W = W'_n \) in the standard way; we have a surjective map \( \zeta : \text{Irr}(W)^1 \to D^n_k \) such that for any \( \rho \in \text{Irr}(W'_n) \) we have \( \zeta(\rho) = (a_s, a'_s) \) where \( (a_s, a'_s) \in D^n_k \) is such that \( \rho \) appears in the restriction of \( [a_s, a'_s] \) from \( W_n \) to \( W'_n \) (the set \( \text{Irr}(W'_n) \) is determined by [L1, 2.5]); note that \( |\zeta^{-1}(a_s, a'_s)| = 2 \) if \( a_s = a'_s \) and is 1 otherwise.

3.5. In this subsection we prove 3.2(b). We can assume that \( G \) is adjoint, simple. If \( p = 1 \) or \( p \) is a good prime for \( G \) then \( \mathcal{S}_g = \mathcal{S}_G \) hence using 3.2(a) we have \( \mathcal{S}_g = S^p_W \); in our case we have \( W_J = W \) for any \( J \in \mathcal{A}_p \) hence from the definitions we have \( S^p_W = S'_W = T^p_W \) and the result follows. In the rest of this subsection we assume that \( p \) is a bad prime for \( G \). In this case \( \mathcal{S}_g \) has been described explicitly by Spaltenstein [S2],[S3],[HS] as follows (assuming that the theory of Springer correspondence holds; this assumption can be removed in view of \([X]\) and the remarks in 2.2.)

If \( G \) is of type \( C_n, n \geq 2 \) \((p = 2)\), then we have \( \mathcal{S}_g = \text{Irr}(W) \). If \( G \) is of type \( B_n, n \geq 2 \) \((p = 2)\), then, according to \([S1]\), \( \mathcal{S}_g = \{[a_s, a'_s] \in \text{Irr}(W); (a_s, a'_s) \in bC^n_k \} \). (Here \( k \) is large and fixed.) If \( G \) is of type \( D_n, n \geq 4 \) \((p = 2)\), then \( \mathcal{S}_g = \zeta^{-1}(dD^n_k) \).
If \( G \) is of type \( G_2 \) \((p = 2 \text{ or } 3)\), of type \( F_4 \) \((p = 3)\), of type \( E_6 \) \((p = 2 \text{ or } 3)\), of type \( E_7 \) \((p = 3)\), or of type \( E_8 \) \((p = 3 \text{ or } 5)\) then \( \mathcal{S}_g = \mathcal{S}_G \). If \( G \) is of type \( E_4 \) \((p = 2)\) then \( \mathcal{S}_g = \mathcal{S}_G \cup \{13, 2_3\} \) (notation as in \([L3, 4.10]\)); note that \( b_{1_3} = 12, b_{2_3} = 4 \). If \( G \) is of type \( E_7 \) \((p = 2)\) then \( \mathcal{S}_g = \mathcal{S}_G \cup \{84'\} \) (notation as in \([L3, 4.12]\); we have \( b_{84'_{15}} = 15 \)). If \( G \) is of type \( E_8 \) \((p = 2)\) then \( \mathcal{S}_g = \mathcal{S}_G \cup \{50_x, 700_{xx}\} \) (notation as in \([L3, 4.13]\); we have \( b_{50_x} = 8, b_{700_{xx}} = 16 \)).

On the other hand, for types \( B, C, D, T^p_W \) is computed by induction using 1.2, the formulas for the maps \( J^W_W \) given in \([L6, 4.5, 5.3, 6.3]\) and the known description of \( S^p_W \); for exceptional types, \( T^p_W \) is computed by induction using the tables in \([A]\) and the known description of \( S^p_W \).

In each case, the explicitly described subset \( \mathcal{S}_g \) of \( \text{Irr}(W) \) coincides with the explicitly described subset \( T^p_W \). This completes the proof of 3.2(b).

To illustrate the inclusion \( \mathcal{S}_g \subset T^p_W \) we note that:
if $G$ is of type $E_8$ ($p = 2$) then $50, 700, 700, 50$ in $\mathcal{G}_G - \mathcal{G}_G$ are obtained by applying $j^{W} W_j$ (where $W_j$ is of type $E_7 \times A_1$) to $15'$, $84'$, $84'$, $84'$ (which belong to $T_{W_j} \mathcal{S}_{W_j} - \mathcal{S}_{W_j}^2$, $\mathcal{S}_{W_j}^2 - \mathcal{S}_{W_j}^3$, respectively);

if $G$ is of type $F_4$ ($p = 2$) then $13, 23$ in $\mathcal{G}_G - \mathcal{G}_G$ are obtained by applying $j^{W} W_j$ (where $W_j$ is of type $B_4, C_3 \times A_1$) to an object in $\mathcal{S}_{W_j}^2 - \mathcal{S}_{W_j}^3$.

3.6. If $G$ is of type $B_n$ or $C_n$, $n \geq 2$ ($p = 2$), then, according to [LS], $\mathcal{G}_G = \{[a_*, a_*'] \in \text{Irr}(W); (a_*, a_*') \in \mathbb{Z}^2 \mathcal{C}_k^n\}$. (Here $k$ is large and fixed.) If $G$ is of type $D_n$, $n \geq 4$ ($p = 2$), then according to [LS], $\mathcal{G}_G = \zeta^{-1}(d^2 D^n)$.

References


