GRADED LIE ALGEBRAS AND INTERSECTION COHOMOLOGY

G. Lusztig

To Toshiaki Shoji on the occasion of his 60th birthday

Introduction

Let (G, ι) be a pair consisting of a reductive connected algebraic group G over $\mathbf{C}$ and a homomorphism of algebraic groups $\iota : \mathbf{C}^* \to G$. The centralizer G^ι of $\iota(\mathbf{C}^*)$ in G acts naturally (with finitely many orbits) on the n-eigenspace L_nG of $\mathrm{Ad}(\iota(\mathbf{C}^*))$ on the Lie algebra of G. (Here $n \in \mathbf{Z} - \{0\}$.) If $\mathcal{O}$ is a G^ι-orbit on L_nG and $\mathcal{L}$ is a G^ι-equivariant irreducible local system on $\mathcal{O}$ then the intersection cohomology complex $K = IC(\bar{\mathcal{O}}, \mathcal{L})$ is defined and we are interested in the problem of computing, for any G^ι-orbit $\mathcal{O}'$ contained in $\bar{\mathcal{O}}$ and any G^ι-equivariant irreducible local system $\mathcal{L}'$ on $\mathcal{O}'$, the multiplicity $m_{i;\mathcal{L},\mathcal{L}'}$ of $\mathcal{L}'$ in the local system obtained by restricting to $\mathcal{O}'$ the i-th cohomology sheaf of K.

The main purpose of this paper is to give an algorithm to produce combinatorially a square matrix whose entries are polynomials with coefficients given by the multiplicities $m_{i;\mathcal{L},\mathcal{L}'}$. Note that we do not have a purely combinatorial proof of the fact that the algorithm does not break down. We can only prove that by using geometry. But this will not prevent a computer from carrying out the algorithm.

The method of this paper relies heavily on [L5] where many of the needed geometric results are proved. Note that in [L5] another purely algebraic description of the multiplicities above was obtained, which however did not provide an algorithm for computing them.

While the existence of the algorithm above has an intrinsic interest, it also implies (by results in [CG], [L2, 10.7]) a solution of a problem in representation theory, namely that of computing the multiplicities with which simple modules of an affine Hecke algebra appear in a composition series of certain "standard modules".

At the same time, as a biproduct of the algorithm we find a way to compute the dimensions of weight spaces in certain standard modules over an affine Hecke algebra. (See 4.6.)

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In Section 1 we describe the algorithm. In Section 3 we show (based on the geometric preliminaries in Section 2) that the algorithm in Section 1 is correct and it indeed leads to the desired matrix of multiplicities. In Section 4 we define among other things a partial order on the set of isomorphism classes of irreducible G^ι-equivariant local systems on the various orbits in L_nG. In 4.7 we give a formulation of our results in terms of a canonical basis and two PBW-bases which generalizes the theory of canonical bases [L4] in the plus part of a quantized enveloping algebra of type A_n.

Notation. The cardinal of a finite set S is denoted by $|S|$.

Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate.

Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \geq 0$. All algebraic varieties are assumed to be over $\mathbf{k}$.

We fix a prime number l invertible in $\mathbf{k}$. Let $\bar{\mathbf{Q}}_l$ be an algebraic closure of the field of l-adic numbers. We will say "local system" instead of "$\bar{\mathbf{Q}}_l$-local system". If $\mathcal{F}$ is an irreducible local system (or its isomorphism class) over a subvariety Y of an algebraic variety X we set $S_{\mathcal{F}} = Y$.

For a connected affine algebraic group H let U_H be the unipotent radical of H, $\underline{H} = H/U_H$, LH the Lie algebra of H, Z_H the centre of H.

1. An algorithm

1.1. Throughout this paper we assume that we are given a connected reductive algebraic group G and a homomorphism of algebraic groups $\iota : \mathbf{k}^* \to G$. We assume that either $p = 0$ or p is sufficiently large (as in the last paragraph of [L5, 2.1(a)]). We set $G^\iota = \{g \in G; g\iota(t) = \iota(t)g \quad \forall t \in \mathbf{k}^*\}$, a connected reductive subgroup of G. We have $LG = \oplus_{n\in\mathbf{Z}} L_nG$ where

$$L_nG = \{x \in LG; \mathrm{Ad}(\iota(t))x = t^n x \quad \forall t \in \mathbf{k}^*\}.$$

More generally, for any closed connected subgroup H of G that is normalized by $\iota(\mathbf{k}^*)$ we set $H^\iota = H \cap G^\iota$; we have $LH = \oplus_{n\in\mathbf{Z}} L_nH$ where $L_nH = LH \cap L_nG$. For $n \in \mathbf{Z}$, the adjoint action of G on LG restricts to an action of G^ι on L_nG.

1.2. In the remainder of this paper we fix a subset Δ of $\mathbf{Z}$ consisting of two non-zero integers whose sum is 0. We assume that either $p = 0$ or $\Delta \subset (-p, p)$.

We say that (G, ι) is *rigid* if for some/any $n \in \Delta$ there exists a homomorphism of algebraic groups $\gamma : SL_2(\mathbf{k}) \to G$ such that $\gamma\begin{pmatrix} t^n & 0 \\ 0 & t^{-n}\end{pmatrix} = \iota(t^2) \mod Z_G$ for any $t \in \mathbf{k}^*$. In this case, let C_G^ι be the nilpotent G-orbit in LG such that the corresponding unipotent class in G contains $\gamma(u)$ for any unipotent element $u \in SL_2(\mathbf{k}) - \{1\}$.

Let $\mathcal{P}$ be the variety of parabolic subgroups of G. Let $\mathcal{P}^\iota = \{P \in \mathcal{P}; \iota(\mathbf{k}^*) \subset P\}$. If $P \in \mathcal{P}^\iota$ then ι gives rise to a homomorphism

$\mathbf{k}^* \to \underline{P}$, $t \mapsto$ (image of $\iota(t)$ under $P \to \underline{P}$),

denoted again by ι. Hence $L_n\underline{P}$, $\underline{P}^\iota$ are well defined in terms of this ι; we have $L_n\underline{P} = L_nP/L_nU_P$.

1.3. Let $\mathcal{T}_G^{cu}$ (resp. $\mathcal{T}_G^{pr}$) be the set of isomorphism classes of G-equivariant irreducible local systems on some nilpotent orbit in LG which are *cuspidal* (resp. *primitive*) in the sense of [L1, 2.2] (resp. [L5, 2.7]).

We have $\mathcal{T}_G^{cu} \subset \mathcal{T}_G^{pr}$. The classification of local systems in $\mathcal{T}_G^{pr}$ can be deduced from the known classification of local systems in $\mathcal{T}_G^{cu}$. For example, if G is simple of type E_8 then $\mathcal{T}_G^{pr}$ consists of two objects: one is in $\mathcal{T}_G^{cu}$ and one is $\bar{\mathbf{Q}}_l$ over the G-orbit $\{0\}$. Let

$$\mathcal{J}_G = \{(P, \mathcal{E}); P \in \mathcal{P}^\iota, \mathcal{E} \in \mathcal{T}_{\underline{P}}^{cu}, (\underline{P}, \iota) \text{ is rigid }, C_{\underline{P}}^\iota = S_{\mathcal{E}}\}.$$

Now G^ι acts on $\mathcal{J}_G$ by $g : (P, \mathcal{E}) \mapsto (gPg^{-1}, \mathrm{Ad}(g)_!\mathcal{E})$. Let $\underline{\mathcal{J}}_G$ be the set of orbits of this action. Let K_G be the $\mathbf{Q}(v)$-vector space with basis $(\mathbf{I}_{\mathcal{S}})_{\mathcal{S} \in \underline{\mathcal{J}}_G}$.

For a connected affine algebraic group $\mathcal{G}$ let $X_{\mathcal{G}}$ be the variety of Borel subgroups of $\mathcal{G}$ and let $\mathrm{rk}(\mathcal{G})$ be the dimension of a maximal torus of $\mathcal{G}$. We set

$$e_{\mathcal{G}} = \sum_j \dim H^{2j}(X_{\mathcal{G}}, \bar{\mathbf{Q}}_l) v^{2j}, \quad \vartheta_{\mathcal{G}} = (1 - v^2)^{\mathrm{rk}(\mathcal{G})} e_{\mathcal{G}} \in \mathbf{Z}[v^2].$$

If $\mathcal{F} \in \mathcal{T}_G^{pr}$ and $\mathcal{G}$ is the connected centralizer in G of some element in $S_{\mathcal{F}}$, we set $r_{\mathcal{F}} = v^{-\dim X_{\mathcal{G}}} e_{\mathcal{G}} \in \mathcal{A}$.

1.4. Let $Q \in \mathcal{P}^\iota$. Associating to the $\underline{Q}^\iota$-orbit of $(P', \mathcal{E}) \in \mathcal{J}_{\underline{Q}}$ the G^ι-orbit of $(P, \mathcal{E}) \in \mathcal{J}_G$ (where P is the inverse image of P' under $Q \to \underline{Q}$ and $\underline{P}', \underline{P}$ are identified in the obvious way) defines a map $a_Q^G : \underline{\mathcal{J}}_{\underline{Q}} \to \underline{\mathcal{J}}_G$. We define a $\mathbf{Q}(v)$-linear map $f_Q^G : K_{\underline{Q}} \to K_G$ by $\mathbf{I}_{\mathcal{S}'} \mapsto \mathbf{I}_{a_Q^G(\mathcal{S}')}$ for any $\mathcal{S}' \in \underline{\mathcal{J}}_{\underline{Q}}$.

1.5. We define a map $\mu : \mathcal{J}_G \to \mathcal{T}_G^{pr}$ by $(P, \mathcal{E}) \mapsto \mathcal{F}$ where $\mathcal{F}$ is as follows. We choose a Levi M of P and we identify M with $\underline{P}$ in the obvious way. Then $\mathcal{E}$ becomes a local system $\mathcal{E}_M$ on a nilpotent M-orbit D in LM. Let C be the nilpotent G-orbit in LG that contains D and let $\mathcal{F}$ be the unique G-equivariant local system on C such that $\mathcal{F}|_D = \mathcal{E}_M$. (See [L5, 2.7].) Then $\mathcal{F} \in \mathcal{T}_G^{pr}$ is clearly independent of the choice of M.

For $\mathcal{F} \in \mathcal{T}_G^{pr}$ let $\mathcal{J}_G^{\mathcal{F}} = \mu^{-1}(\mathcal{F})$. Then $\mathcal{J}_G^{\mathcal{F}}$ is G^ι-stable; let $\underline{\mathcal{J}}_G^{\mathcal{F}}$ be the set of G^ι-orbits on $\mathcal{J}_G^{\mathcal{F}}$. We have a partition $\underline{\mathcal{J}}_G = \sqcup_{\mathcal{F}} \underline{\mathcal{J}}_G^{\mathcal{F}}$ and a direct sum decomposition

$$K_G = \oplus_{\mathcal{F}} K_G^{\mathcal{F}}$$

where $\mathcal{F}$ runs over $\mathcal{T}_G^{pr}$ and $K_G^{\mathcal{F}}$ is the subspace of K_G spanned by $\{\mathbf{I}_{\mathcal{S}}; \mathcal{S} \in \underline{\mathcal{J}}_G^{\mathcal{F}}\}$.

For $\mathcal{F} \in \mathcal{T}_G^{pr}$ let $Y^{\mathcal{F}}$ be the set of all $((P, \mathcal{E}), (P', \mathcal{E}')) \in \mathcal{J}_G^{\mathcal{F}} \times \mathcal{J}_G^{\mathcal{F}^*}$ such that P, P' have a common Levi. Now G^ι acts diagonally on $Y^{\mathcal{F}}$; let $\underline{Y}^{\mathcal{F}}$ be the set of orbits. Define $\tau : \underline{Y}^{\mathcal{F}} \to \mathbf{Z}$ by

$$\Omega \mapsto \tau(\Omega) = \dim \frac{L_n U_{P'} + L_n U_P}{L_n U_{P'} \cap L_n U_P} - \dim \frac{L_0 U_{P'} + L_0 U_P}{L_0 U_{P'} \cap L_0 U_P}$$

where $((P, \mathcal{E}), (P', \mathcal{E}'))$ is any element of the G^ι-orbit Ω and $n \in \Delta$. (The fact that this definition is independent of the choice of n in Δ is seen as in [L5, 16.3].)

1.6. For $(P,\mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$ we choose a Levi M of P that contains $\iota(\mathbf{k}^*)$; let $\tilde{P}$ be the unique parabolic subgroup with Levi M such that $P \cap \tilde{P} = M$. We have $(\tilde{P},\tilde{\mathcal{E}}) \in \mathcal{J}_G^{\mathcal{F}}$ for a unique $\tilde{\mathcal{E}}$. Although $\tilde{P}$ is not uniquely defined by P, its G^ι-orbit is uniquely defined by the G^ι-orbit of $(P,\mathcal{E})$ (since M is uniquely defined by P up to the conjugation action of U_P^ι). Thus $(P,\mathcal{E}) \mapsto (\tilde{P},\tilde{\mathcal{E}})$ induces a well defined involution $\mathcal{S} \mapsto \tilde{\mathcal{S}}$ of $\underline{\mathcal{J}}_G^{\mathcal{F}}$.

Similarly, for $((P,\mathcal{E}),(P',\mathcal{E}')) \in Y^{\mathcal{F}}$ we choose a common Levi M of P and P' that contains $\iota(\mathbf{k}^*)$ and let $\tilde{P}$ be the unique parabolic subgroup with Levi M such that $P \cap \tilde{P} = M$. We have $((\tilde{P},\tilde{\mathcal{E}}),(P',\mathcal{E}')) \in Y^{\mathcal{F}}$ for a unique $\tilde{\mathcal{E}}$. Again the G^ι-orbit of $((\tilde{P},\tilde{\mathcal{E}}),(P',\mathcal{E}'))$ is uniquely defined by the G^ι-orbit of $((P,\mathcal{E}),(P',\mathcal{E}'))$ (since M is uniquely defined by P,P' up to the conjugation action of $U_{P\cap P'}^\iota$). Thus $((P,\mathcal{E}),(P',\mathcal{E}')) \mapsto ((\tilde{P},\tilde{\mathcal{E}}),(P',\mathcal{E}'))$ induces a well defined involution $\Omega \mapsto \tilde{\Omega}$ of $\underline{Y}^{\mathcal{F}}$. As in the proof of [L5, 16.4(c)] we have

(a) $$\tau(\Omega) + \tau(\tilde{\Omega}) = c_{\mathcal{F}}$$

where
$$c_{\mathcal{F}} = \dim L_n G - \dim L_0 G - \dim L_n \underline{P} + \dim L_0 \underline{P}$$
for any $(P,\mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$. (If $(P,\mathcal{E}')$ is another pair in $\mathcal{J}_G^{\mathcal{F}}$ then there exists an isomorphism $\underline{P} \xrightarrow{\sim} \underline{P}'$ which is compatible with ι so that $c_{\mathcal{F}}$ depends only on $\mathcal{F}$).

1.7. Let $\mathcal{F} \in \mathcal{T}_G^{pr}$. In (a), (b) below we give a "combinatorial" interpretation of the sets $\underline{\mathcal{J}}_G^{\mathcal{F}}$ and $\underline{Y}^{\mathcal{F}}$. We may assume that $S_{\mathcal{F}} \cap L_n G \neq \emptyset$; otherwise both our sets are empty.

Let M be the centralizer in G of some maximal torus of the connected centralizer in G^ι of some element in $S_{\mathcal{F}} \cap L_n G$. Then $\iota(\mathbf{k}^*) \subset M$ and M is independent of the choices (up to G^ι-conjugacy) since, by [L5, 14.5], $S_{\mathcal{F}} \cap L_n G$ is a single G^ι-orbit. Let $\underline{X} = \{P \in \mathcal{P}; M \text{ is a Levi of } P\}$. If $P \in \underline{X}$ then $(P,\mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$ for a unique $\mathcal{E}$, see [L5, 11.6(c)]. We have an imbedding $\underline{X} \to \mathcal{J}_G^{\mathcal{F}}$, $P \mapsto (P,\mathcal{E})$ and an imbedding $\underline{X} \to \mathcal{J}_G^{\mathcal{F}^*}$, $P \mapsto (P,\mathcal{E}^*)$. Let $N_G M$ be the normalizer of M in G. It is known that the conjugation action of $N_G M/M$ on $\underline{X}$ is simply transitive. Note that $(N_G M)^\iota/M^\iota$ is naturally a subgroup of $N_G M/M$. We show:

(a) *the map j_1 : (set of $(N_G M)^\iota/M^\iota$-orbits on $\underline{X}$) $\to \underline{\mathcal{J}}_G^{\mathcal{F}}$ induced by the imbedding $\underline{X} \to \mathcal{J}_G^{\mathcal{F}}$ is bijective;*

(b) *the map j_2 : (set of $(N_G M)^\iota/M^\iota$-orbits on $\underline{X} \times \underline{X}$) $\to \underline{Y}^{\mathcal{F}}$ (orbits for diagonal action) induced by the imbedding $\underline{X} \times \underline{X} \to Y^{\mathcal{F}}$ is bijective.*

Let $(P,\mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$. We can find a Levi M' of P that contains $\iota(\mathbf{k}^*)$. Using [L5, 11.4] we see that there exists $g \in G^\iota$ such that $gM'g^{-1} = M$. Then $gPg^{-1} \in \underline{X}$. We see that j_1 is surjective.

Now let $P,P' \in \underline{X}$ be such that $P' = gPg^{-1}$ for some $g \in G^\iota$. Then M and $M' = g^{-1}Mg$ are Levi subgroups of P that contain $\iota(\mathbf{k}^*)$. There is a unique $u \in U_P$ such that $uMu^{-1} = M'$. For any $t \in \mathbf{k}^*$ we set $u' = \iota(t)u\iota(t)^{-1} \in U_P$; we have

$M'=\iota(t)M'\iota(t)^{-1}=u'(\iota(t)M\iota(t)^{-1})u'^{-1}=u'Mu'^{-1}$.
By the uniqueness of u we have $u'=u$. Thus $u\in U_P^\iota$. Let $g'=gu$. Then $g'\in G^\iota$, $P'=g'Pg'^{-1}$, $M=g'^{-1}Mg'$. Thus, $g'\in(N_GM)^\iota$. We see that j_1 is injective. This proves (a).

Let $((P,\mathcal{E}),(P',\mathcal{E}'))\in Y^{\mathcal{F}}$. We can find a common Levi M' of P,P' that contains $\iota(\mathbf{k}^*)$. As in the proof of (a) we can find $g\in G^\iota$ such that $gM'g^{-1}=M$. Then $(gPg^{-1},gP'g^{-1})\in\underline{X}\times\underline{X}$. We see that j_2 is surjective.

Now let P,P_1,P',P_1' in $\underline{X}$ be such that $P'=gPg^{-1},P_1'=gP_1g^{-1}$ for some $g\in G^\iota$. Then M and $M'=g^{-1}Mg$ are Levi subgroups of $P\cap P_1$ that contain $\iota(\mathbf{k}^*)$. There is a unique $u\in U_{P\cap P_1}$ such that $uMu^{-1}=M'$. As in the proof of (a) we see using the uniqueness of u that $u\in U_{P\cap P_1}^\iota$. Let $g'=gu$. Then $g'\in G^\iota$, $P'=g'Pg'^{-1}$, $P_1'=g'P_1g'^{-1}$, $M=g'^{-1}Mg'$. Thus, $g'\in(N_GM)^\iota$. We see that j_2 is injective. This proves (b).

1.8. Define a symmetric $\mathbf{Q}(v)$-bilinear form $(:):K_G\times K_G\longrightarrow\mathbf{Q}(v)$ by setting (for $\mathcal{S}\in\underline{\mathcal{J}}_G^{\mathcal{F}}$, $\mathcal{S}'\in\underline{\mathcal{J}}_G^{\mathcal{F}'}$):

$$(\mathbf{I}_{\mathcal{S}}:\mathbf{I}_{\mathcal{S}'})=0\text{ if }\mathcal{F}'\ne\mathcal{F}^*,$$

$$(\mathbf{I}_{\mathcal{S}}:\mathbf{I}_{\mathcal{S}'})=\frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}}\sum_{\substack{\Omega\in\underline{Y}^{\mathcal{F}}\\ \pi_1(\Omega)=\mathcal{S},\pi_2(\Omega)=\mathcal{S}'}}(-v)^{\tau(\Omega)}\text{ if }\mathcal{F}'=\mathcal{F}^*;$$

here $\pi_1:\underline{Y}^{\mathcal{F}}\longrightarrow\underline{\mathcal{J}}_G^{\mathcal{F}}$, $\pi_2:\underline{Y}^{\mathcal{F}}\longrightarrow\underline{\mathcal{J}}_G^{\mathcal{F}'}$ are the obvious projections and $P\in\mathcal{P}$ is such that $(P,\mathcal{E})\in\mathcal{S}$ for some $\mathcal{E}$.

Let $^-:\mathbf{Q}(v)\longrightarrow\mathbf{Q}(v)$ be the $\mathbf{Q}$-algebra involution such that $\overline{v^m}=v^{-m}$ for all $m\in\mathbf{Z}$. Define a $\mathbf{Q}$-linear involution $\beta:K_G\longrightarrow K_G$ by

$$\beta(\rho\mathbf{I}_{\mathcal{S}})=\overline{\rho}\mathbf{I}_{\mathcal{S}}$$

for all $\rho\in\mathbf{Q}(v),\mathcal{S}\in\underline{\mathcal{J}}_G$. Define a $\mathbf{Q}(v)$-linear involution $\sigma:K_G^{\mathcal{F}}\longrightarrow K_G^{\mathcal{F}}$ by $\sigma(\mathbf{I}_{\mathcal{S}})=\mathbf{I}_{\tilde{\mathcal{S}}}$ (see 1.6) for all $\mathcal{S}\in\underline{\mathcal{J}}_G^{\mathcal{F}}$. From 1.6(a) we see that, for $\xi\in K_G^{\mathcal{F}}$, $\xi'\in K_G^{\mathcal{F}'}$, we have

(a) $$\overline{(\beta(\xi):\beta(\xi'))}=(-v)^{c_{\mathcal{F}}}(\sigma(\xi):\xi').$$

Let

$$\mathcal{R}_G=\{\xi\in K_G;(\xi:K_G)=0\}.$$

From (a) we see that $\beta(\mathcal{R}_G)\subset\mathcal{R}_G$. Clearly, if $Q\in\mathcal{P}^\iota$ then for $\xi\in K_{\underline{Q}}$ we have

(b) $$f_Q^G(\beta(\xi))=\beta(f_Q^G(\xi)).$$

Moreover,

(c) $$f_Q^G(\mathcal{R}_{\underline{Q}})\subset\mathcal{R}_G.$$

See 3.5 for a proof.

1.9. Let $n \in \Delta$. Let $P \in \mathcal{P}^\iota$ be such that $(\underline{P}, \iota)$ is rigid. We can find a Levi subgroup M of P such that $\iota(\mathbf{k}^*) \subset M$. Let s be the unique element in $[LM, LM]$ such that $[s, x] = mx$ for any $m \in \mathbf{Z}, x \in L_m M$. We have $LG = \oplus_{r \in (n/2)\mathbf{Z}} L^r G$ where $L^r G = \{x \in LG; [s, x] = rx\}$ and $LG = \oplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}} L^r_t G$ where $L^r_t G = L^r G \cap L_t G$. We say that P (as above) is n-good if

$LU_P = \oplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}; 2t/n < 2r/n} L^r_t G.$

(This implies that

$LM = \oplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}; 2t/n = 2r/n} L^r_t G$, $LP = \oplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}; 2t/n \le 2r/n} L^r_t G$.)

Note that the condition that P is n-good is independent of the choice of M.

Let $\mathfrak{P}_n$ be the set of all $P \in \mathcal{P}^\iota$ such that $(\underline{P}, \iota)$ is rigid and P is n-good. Let $\mathfrak{P}'_n = \{P \in \mathfrak{P}_n; P \ne G\}$. Now G^ι acts on $\mathfrak{P}_n, \mathfrak{P}'_n$ by conjugation. Let $\underline{\mathfrak{P}}_n, \underline{\mathfrak{P}}'_n$ be the sets of orbits of these actions. These are finite sets since G^ι acts with finitely many orbits on $\mathcal{P}^\iota$. We have $G \in \mathfrak{P}_n$ if and only if (G, ι) is rigid. Hence $\underline{\mathfrak{P}}'_n = \underline{\mathfrak{P}}_n$ if (G, ι) is not rigid, $\underline{\mathfrak{P}}_n = \underline{\mathfrak{P}}'_n \sqcup \{G\}$ if (G, ι) is rigid.

1.10. Let $n \in \Delta$. For $\eta \in \underline{\mathfrak{P}}_n$ we set

$$d_\eta = \dim L_0 G - \dim L_0 P + \dim L_n P$$

where $P \in \eta$. For $\eta, \eta' \in \underline{\mathfrak{P}}'_n$ we say that $\eta' \prec \eta$ if $d_{\eta'} < d_\eta$. We say that $\eta' \preceq \eta$ if either $\eta = \eta'$ or $\eta' \prec \eta$. Now $\preceq$ is a partial order on $\underline{\mathfrak{P}}'_n$.

1.11. Our goal is to define subsets $\mathcal{Z}^\eta_n$ of K_G (for $n \in \Delta$ and $\eta \in \underline{\mathfrak{P}}_n$)). The definition of these subsets is inductive and is based on a number of lemmas which will be verified in Section 3 (where we assume, as we may, that $\mathbf{k}$ is an algebraic closure of a finite field). If G is a torus then η must be $\{G\}$ and $\mathcal{Z}^\eta_n$ consists of the unique basis element of K_G. We now assume that G is not a torus and that the subsets $\mathcal{Z}^\eta_n$ are already defined when G is replaced by any $\underline{P}$ with $P \in \mathfrak{P}'_n$.

Our definition is based on the following scheme.

(i) We first define $\mathcal{Z}^\eta_n$ in the case where $\eta \in \underline{\mathfrak{P}}'_n$ by

$$\mathcal{Z}^\eta_n = f^G_P(\mathcal{Z}^{\{\underline{P}\}}_n)$$

where $P \in \eta$ and f^G_P is as in 1.4. (Note that $\mathcal{Z}^{\{\underline{P}\}}_n \subset K_{\underline{P}}$ is defined by the inductive assumption.) We set

$$\mathcal{Z}'_n = \cup_{\eta \in \underline{\mathfrak{P}}'_n} \mathcal{Z}^\eta_n.$$

(ii) Using (i) we define elements W^ξ_n for any $\xi \in \mathcal{Z}'_n$ by a procedure similar to the definition of the "new" basis of a Hecke algebra. (See 1.13.)

(iii) Using (i) and (ii) for n and $-n$ we define $\mathcal{Z}^\eta_n$ for $\eta \in \underline{\mathfrak{P}}_n - \underline{\mathfrak{P}}'_n$. (See 1.18.)

Lemma 1.12. *Let $n \in \Delta$.*

(a) If $\eta \in \underline{\mathfrak{P}}'_n$ and $P \in \eta$, the map $\mathcal{Z}^{\{\underline{P}\}}_n \to \mathcal{Z}^\eta_n$ given by $\xi \mapsto f^G_P(\xi)$ is bijective.

(b) The union $\cup_{\eta \in \underline{\mathfrak{P}}'_n} \mathcal{Z}^\eta_n$ is disjoint.

(c) In the setup of (a) let $\xi' \in \mathcal{Z}'_n$ (relative to $\underline{P}$ instead of G). Then $f_P^G(\xi')$ is an $\mathcal{A}$-linear combination of elements in various $\mathcal{Z}_n^{\eta'}$ (with $\eta' \in \underline{\mathfrak{P}}'_n$, $\eta' \prec \eta$) plus an element of $\mathcal{R}_G$.

(d) In the setup of (a) let $\xi_0 \in \mathcal{Z}_n^{\{\underline{P}\}}$. Then $\beta(\xi_0) - \xi_0$ is an $\mathcal{A}$-linear combination of elements in $\mathcal{Z}'_n$ (relative to $\underline{P}$) plus an element of $\mathcal{R}_{\underline{P}}$.

(e) The matrix with entries $(\xi : \xi') \in \mathbf{Q}(v)$ indexed by $\mathcal{Z}'_n \times \mathcal{Z}'_n$ is non-singular.

See 3.6, 3.7 for a proof.

1.13. Let $n \in \Delta$. We show that for any $\xi \in \mathcal{Z}'_n$ we have

(a) $$\beta(\xi) = \sum_{\xi_1 \in \mathcal{Z}'_n} a_{\xi,\xi_1} \xi_1 \mod \mathcal{R}_G$$

where $a_{\xi,\xi_1} \in \mathcal{A}$ are uniquely determined and satisfy the following conditions (where η, η_1 are given by $\xi \in \mathcal{Z}_n^{\eta}, \xi_1 \in \mathcal{Z}_n^{\eta_1}$):

$a_{\xi,\xi_1} \neq 0$ implies $\eta_1 \prec \eta$ or $\xi = \xi_1$;

$a_{\xi,\xi_1} = 1$ if $\xi = \xi_1$.

We have $\xi = f_P^G(\xi_0)$ where $\xi_0 \in \mathcal{Z}_n^{\{\underline{P}\}}$ (notation of 1.11(i)). We express $\beta(\xi_0) - \xi_0$ as in 1.12(d). Applying f_P^G and using 1.12(c), 1.8(d) and 1.8(c) we deduce that (a) holds except perhaps for the uniqueness statement. To show the uniqueness we note that the a_{ξ,ξ_1} are determined from the system of linear equations

$(\beta(\xi) : \xi_2) = \sum_{\xi_1 \in \mathcal{Z}'_n} (\xi_1 : \xi_2) a_{\xi,\xi_1}$

(with $\xi_2 \in \mathcal{Z}'_n$) whose matrix of coefficients is invertible by 1.12(e).

Using the equality $\beta^2 = 1 : K_G \longrightarrow K_G$ and the inclusion $\beta(\mathcal{R}_G) \subset \mathcal{R}_G$ (see 1.8) we see that for any ξ, ξ_1 in $\mathcal{Z}'_n$ we have

$\sum_{\xi_2 \in \mathcal{Z}'_n} \overline{a_{\xi,\xi_2}} a_{\xi_2,\xi_1} = \delta_{\xi,\xi_1}$.

Using a standard argument we see that there is a unique family of elements $c_{\xi,\xi_1} \in \mathbf{Z}[v]$ (defined for $\xi, \xi_1 \in \mathcal{Z}'_n$) such that for any $\xi, \xi_1 \in \mathcal{Z}'_n$ with $\xi \in \mathcal{Z}_n^{\eta}, \xi_1 \in \mathcal{Z}_n^{\eta_1}$ we have

$c_{\xi,\xi_1} = \sum_{\xi_2 \in \mathcal{Z}'_n} \overline{c_{\xi,\xi_2}} a_{\xi_2,\xi_1}$;

$c_{\xi,\xi_1} \neq 0$ implies $\eta_1 \prec \eta$ or $\xi = \xi_1$;

$c_{\xi,\xi_1} \neq 0$, $\xi \neq \xi_1$ implies $c_{\xi,\xi_1} \in v\mathbf{Z}[v]$;

$c_{\xi,\xi_1} = 1$ if $\xi = \xi_1$.

For $\xi \in \mathcal{Z}'_n$ we set $W_n^{\xi} = \sum_{\xi_1 \in \mathcal{Z}'_n} c_{\xi,\xi_1} \xi_1$. Then $\beta(W_n^{\xi}) = W_n^{\xi} \mod \mathcal{R}_G$.

1.14. Until the end of 1.16 we assume that (G, ι) is rigid. For any $\mathcal{F} \in \mathcal{T}_G^{pr}$ such that $S_{\mathcal{F}} = C_G^{\iota}$ we set $\mathcal{S}_{\mathcal{F}} = \mu^{-1}(\mathcal{F})$ which, by [L5, 11.9], is a single G^{ι}-orbit $\mathcal{J}_G$ (μ as in 1.5). Let

$$\mathcal{C}' = \{r_{\mathcal{F}}^{-1} \mathbf{I}_{\mathcal{S}_{\mathcal{F}}}; \mathcal{F} \in \mathcal{T}_G^{pr}, S_{\mathcal{F}} = C_G^{\iota}\} \subset K_G;$$

here $r_{\mathcal{F}} \in \mathcal{A}$ is as in 1.3.

1.15. Let $n \in \Delta$. For any $x \in K_G$ we define $Y_n(x) \in K_G$ by the conditions

$$(Y_n(x) : \mathcal{Z}'_n) = 0, \quad x = Y_n(x) + \sum_{\xi \in \mathcal{Z}'_n} \gamma_\xi \xi$$

with $\gamma_x \in \mathbf{Q}(v)$. The coefficients γ_ξ are determined from the system of linear equations
$\sum_{\xi \in \mathcal{Z}'_n} (\xi : \xi')\gamma_\xi = (x : \xi')$
with $\xi' \in \mathcal{Z}'_n$, whose matrix of coefficients is invertible by 1.12(e). Let

$$J_{-n} = \{\xi_0 \in \mathcal{Z}'_{-n}; Y_n(W_{-n}^{\xi_0}) \notin \mathcal{R}_G\}$$

and let $\mathcal{C}_n$ be the image of the map $J_{-n} \to K_G$, $\xi_0 \mapsto Y_n(W_{-n}^{\xi_0})$. This can be regarded as a surjective map $h_n : J_{-n} \to \mathcal{C}_n$.

Lemma 1.16. *h_n is bijective.*

See 3.10 for a proof.

Lemma 1.17. *For $n \in \Delta$, the union $\mathcal{Z}'_n \cup \mathcal{C}_n \cup \mathcal{C}'$ is disjoint.*

See 3.11 for a proof.

1.18. If (G, ι) is not rigid then $\underline{\mathfrak{P}}_n = \underline{\mathfrak{P}}'_n$ and the definition of the subsets $\mathcal{Z}_n^\eta$ ($\eta \in \underline{\mathfrak{P}}_n$) is complete. If (G, ι) is rigid and $n \in \Delta$ we set $\mathcal{Z}_n^{\{G\}} = \mathcal{C}_n \cup \mathcal{C}'$. By 1.17, this union is disjoint. The definition of the subsets $\mathcal{Z}_n^\eta$ ($\eta \in \underline{\mathfrak{P}}_n$) is complete.

We set $\mathcal{Z}_n = \mathcal{Z}'_n$ if (G, ι) is not rigid and $\mathcal{Z}_n = \mathcal{Z}'_n \cup \mathcal{Z}_n^{\{G\}}$ if (G, ι) is rigid. By 1.17, the last union is disjoint.

1.19. For $n \in \Delta$ and $\xi \in \mathcal{Z}_n$ we define an element W_n^ξ as follows. When $\xi \in \mathcal{Z}'_n$, this is already defined in 1.13. When (G, ι) is rigid we set $W_n^\xi = W_{-n}^{h_n^{-1}(\xi)}$ if $\xi \in \mathcal{C}_n$ and $W_n^\xi = \xi$ if $\xi \in \mathcal{C}'$.

We now define a matrix $(c_{\xi,\xi'})$ with entries in $\mathbf{Q}(v)$ indexed by $\mathcal{Z}_n \times \mathcal{Z}_n$ by the following requirements:

When $\xi, \xi' \in \mathcal{Z}'_n$ then $c_{\xi,\xi'}$ are as in 1.13.

When $\xi \in \mathcal{Z}'_n, \xi' \notin \mathcal{Z}'_n$ then $c_{\xi,\xi'} = 0$.

When $\xi \notin \mathcal{Z}'_n, \xi' \notin \mathcal{Z}'_n$ then $c_{\xi,\xi'} = \delta_{\xi,\xi'}$.

When $\xi \notin \mathcal{Z}'_n$ then $c_{\xi,\xi'}$ for $\xi' \in \mathcal{Z}'_n$ are determined by the system of linear equations $(W_n^\xi : \xi'') = \sum_{\xi' \in \mathcal{Z}'_n} (\xi' : \xi'')c_{\xi,\xi'}$ (with $\xi'' \in \mathcal{Z}'_n$) whose matrix of coefficients has invertible determinant.

Note that for any $\xi \in \mathcal{Z}_n$ we have

(a) $$W_n^\xi = \sum_{\xi' \in \mathcal{Z}_n} c_{\xi,\xi'} \xi'.$$

Lemma 1.20. *Let $n \in \Delta$. Let $\mathcal{S} \in \underline{\mathcal{J}}_G$. There exist $e_{\mathcal{S},\xi} \in \mathcal{A}$ (for $\xi \in \mathcal{Z}_n$) and $r \in \mathcal{R}_G$ such that $\mathbf{I}_\mathcal{S} = \sum_{\xi \in \mathcal{Z}_n} e_{\mathcal{S},\xi}\xi + r$.*

See 3.14 for a proof.

2. Geometric preliminaries

2.1. In this section we assume that $\mathbf{k}$ is an algebraic closure of the finite field F_p with $|F_p| = p$. For $q \in \{p, p^2, \dots\}$ let F_q be the subfield of $\mathbf{k}$ with $|F_q| = q$. If X is an algebraic variety we denote by $\mathcal{D}(X)$ the bounded derived category of (constructible) $\bar{\mathbf{Q}}_l$-sheaves on X. For $K \in \mathcal{D}(X)$ let $\mathcal{H}^i K$ be the i-th cohomology sheaf of K. For $n \in \mathbf{Z}$ let $\bar{\mathbf{Q}}_l(n/2)$ be as in the Introduction to [L8]. We write $K[[n/2]]$ instead of $K[n] \otimes \bar{\mathbf{Q}}_l(n/2)$. We fix a square root $\sqrt{p}$ of p in $\bar{\mathbf{Q}}_l$. If $q = p^e (e \in \mathbf{N})$, we set $\sqrt{q} = (\sqrt{p})^e$. We shall assume that Frobenius relative to F_q acts on $\bar{\mathbf{Q}}_l(n/2)$ as multiplication by $(\sqrt{q})^{-n}$.

For a connected affine algebraic group $\mathcal{G}$ we have

$$\vartheta_{\mathcal{G}}|_{v=-1/\sqrt{q}} = |L\mathcal{G}(F_q)|^{-1} |\mathcal{G}(F_q)|$$

for any F_q-rational structure on $\mathcal{G}$ such that $\underline{\mathcal{G}}$ is F_q-split; here $\vartheta_{\mathcal{G}}$ is as in 1.3.

Define $\omega : \mathbf{k} \longrightarrow \mathbf{k}$ by $x \mapsto x^p - x$. Let $\mathcal{U}$ be a local system of rank 1 on $\mathbf{k}$ such that $\mathcal{U} \oplus \bar{\mathbf{Q}}_l$ is a direct summand of $\omega_! \bar{\mathbf{Q}}_l$. Let E, E' be two $\mathbf{k}$-vector spaces of the same dimension $< \infty$ and let $\sigma : E \times E' \longrightarrow \mathbf{k}$ be a perfect bilinear pairing. Let $s : E \times E' \longrightarrow E$, $s' : E \times E' \longrightarrow E'$ be the projections. Recall that the Fourier-Deligne transform is the functor $\mathcal{D}(E) \longrightarrow \mathcal{D}(E')$ given by $A \mapsto s'_!(s^*(A) \otimes \sigma^* \mathcal{U})[[\dim E/2]]$.

We fix a perfect symmetric bilinear pairing $\langle , \rangle : LG \times LG \longrightarrow \mathbf{k}$ which is invariant under the adjoint action of G.

In this section we fix $n \in \Delta$. For any G^ι-orbit $\mathcal{O}$ on $L_n G$ let $\bar{\mathcal{O}}$ be the closure of $\mathcal{O}$ in $L_n G$. The natural G^ι-action on $L_n G$ has only finitely many orbits [L5, 3.5]. Let $\overset{\circ}{L}_n G$ be the unique open G^ι-orbit on $L_n G$.

2.2. Let V be an algebraic variety with a given family $\mathfrak{f}$ of simple perverse sheaves with the following property: any complex in $\mathfrak{f}$ comes from a mixed complex on V relative to a rational structure on V over some F_q. Let $\mathcal{D}^{\mathfrak{f}}(V)$ be the subcategory of $\mathcal{D}(V)$ whose objects are complexes K such that for any j, any composition factor of ${}^p H^j(K)$ is in $\mathfrak{f}$. Let $\mathcal{K}^{\mathfrak{f}}(V)$ be the free $\mathcal{A}$-module with basis $\mathbf{B}^{\mathfrak{f}}(V)$ given by the isomorphism classes of simple perverse sheaves in $\mathfrak{f}$. Let K be an object of $\mathcal{D}^{\mathfrak{f}}(V)$ with a given mixed structure relative to a rational structure of V over some F_q. We set

$$gr(K) = \sum_A \sum_{j,h \in \mathbf{Z}} (-1)^j (\text{mult. of } A \text{ in } {}^p H^j(K)_h)(-v)^{-h} A \in \mathcal{K}^{\mathfrak{f}}(V),$$

where A runs over a set of representatives for the isomorphism classes in $\mathfrak{f}$ and the subscript h denotes the subquotient of pure weight h of a mixed perverse sheaf. (This agrees with the definition in [L7, 36.8] after the change of variable $v \mapsto (-v)^{-1}$.) Note that $gr(K[[m/2]]) = v^m gr(K)$ for $m \in \mathbf{Z}$.

Now let V_1 be another algebraic variety with a given family $\mathfrak{f}_1$ of simple perverse sheaves like $\mathfrak{f}$ for V. Then $\mathcal{D}^{\mathfrak{f}_1}(V_1)$ is defined. Assume that we are given a functor

$\Theta : \mathcal{D}(V) \to \mathcal{D}(V_1)$ which restricts to a functor $\mathcal{D}^{\mathfrak{f}}(V) \to \mathcal{D}^{\mathfrak{f}_1}(V_1)$. Assume also that Θ is a composition of functors of the form $a_!, a^*$ induced by various maps a between algebraic varieties. In particular, Θ preserves the triangulated structures and makes sense also on the mixed level. Define an $\mathcal{A}$-linear map $gr(\Theta) : \mathcal{K}^{\mathfrak{f}}(V) \to \mathcal{K}^{\mathfrak{f}_1}(V_1)$ by the following requirement: if $A \in \mathfrak{f}$ is regarded as a pure complex of weight 0 (relative to a rational structure of V over some F_q) then $gr(\Theta)(A) = gr(\Theta(A))$ where $\Theta(A)$ is regarded as a mixed complex on V_1 (with mixed structure defined by that of A). Note that $gr(\Theta)(A)$ does not depend on the choice of mixed structure. If $\Theta' : \mathcal{D}(V_1) \to \mathcal{D}(V_2)$ is another functor like Θ then so is $\Theta'\Theta$ and we have

(a) $gr(\Theta'\Theta) = gr(\Theta')gr(\Theta)$.

2.3. For an algebraic variety V with a fixed rational structure over some F_q and a mixed complex K on V, we define a function $\chi_K : V(F_q) \to \bar{\mathbf{Q}}_l$ by

$$\chi_K(x) = \sum_j (-1)^j (\text{trace of Frobenius on } \mathcal{H}^j_x(K)).$$

2.4. Let $\tilde{\mathfrak{I}}_{L_nG}$ be the collection consisting of all irreducible G^ι-equivariant local systems on various G^ι-orbits in L_nG. Let $\mathfrak{I}_{L_nG}$ be the set of all isomorphism classes of irreducible G^ι-equivariant local systems on various G^ι-orbits in L_nG. For $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_nG}, \kappa \in \mathfrak{I}_{L_nG}$ we write $\mathcal{L} \in \kappa$ instead of "κ is the isomorphism class of $\mathcal{L}$".

For $\mathcal{L} \in \kappa$ (as above) we say that $\mathcal{L}$ or κ is *cuspidal* (resp. *semicuspidal*) if (G, ι) is rigid, $S_\kappa = \overset{\circ}{L}_nG$ and there exists $\mathcal{F} \in \mathcal{T}^{cu}_G$ (resp. $\mathcal{F} \in \mathcal{T}^{pr}_G$) such that $\overset{\circ}{L}_nG \subset S_{\mathcal{F}}$, $\mathcal{L} \cong \mathcal{F}|_{S_\kappa}$.

On the other hand, if $\mathcal{F} \in \mathcal{T}^{cu}_G$ and $S_{\mathcal{F}} \cap L_nG \neq \emptyset$ then (G, ι) is rigid and $\mathcal{F}|_{\overset{\circ}{L}_nG}$ is irreducible, cuspidal in $\mathfrak{I}_{L_nG}$. (See [L5, 4.4].)

We write $\mathcal{K}(L_nG)$, $\mathbf{B}(L_nG)$ instead of $\mathcal{K}^{\mathfrak{f}}(L_nG)$, $\mathbf{B}^{\mathfrak{f}}(L_nG)$ (see 2.2) where $\mathfrak{f}$ is the family of simple G^ι-equivariant perverse sheaves on L_nG. The notation $\mathcal{K}(L_nG)$, $\mathbf{B}(L_nG)$ agrees with that in [L5, 3.9].

For $\kappa \in \mathfrak{I}_{L_nG}$ we set $\underline{\kappa}^\bullet = IC(\bar{S}_\kappa, \mathcal{L})[[\dim S_\kappa/2]]$ (extended by 0 on $L_nG - \bar{S}_\kappa$) where $\mathcal{L} \in \kappa$. We have $\mathbf{B}(L_nG) = \{\underline{\kappa}^\bullet; \kappa \in \mathfrak{I}_{L_nG}\}$. We set

$$B(L_nG) = \{\underline{\kappa}; \kappa \in \mathfrak{I}_{L_nG}\}.$$

We define a $\mathbf{Z}$-linear involution $\beta : \mathcal{K}(L_nG) \to \mathcal{K}(L_nG)$ by $\beta(v^m\underline{\kappa}^\bullet) = v^{-m}\underline{\kappa}^\bullet$ for $m \in \mathbf{Z}$, $\kappa \in \mathfrak{I}_{L_nG}$.

We choose a rational structure for G over some F_q with Frobenius map $F : G \to G$ such that $\iota(t^q) = F(\iota(t))$ for any $t \in \mathbf{k}^*$, such that any G^ι-orbit in L_nG or $L_{-n}G$ is defined over F_q and such that any irreducible G^ι-equivariant local system over such an orbit admits a mixed structure. Then G^ι is defined over F_q. We assume as we may that G^ι is F_q-split and any connected component of $\mathcal{P}^\iota$ is defined over F_q.

Let $\kappa \in \mathfrak{I}_{L_nG}$ and let $\mathcal{L} \in \kappa$. Let $i : S_\kappa \longrightarrow L_nG$ be the inclusion. We choose a mixed structure for $\mathcal{L}$ which is pure of weight 0. Then $i_!\mathcal{L}[[\dim S_\kappa/2]]$ is naturally a mixed complex on L_nG and

$$\underline{\kappa} := gr(i_!\mathcal{L}[[\dim S_\kappa/2]]) \in \mathcal{K}(L_nG)$$

is defined as in 2.2. It is independent of the choice of rational/mixed structures. Using the definitions and the purity statement in [L5, 18.2] we see that

(a) $$\underline{\kappa}^\bullet = \sum_{\kappa' \in \mathfrak{I}_{L_nG}} f_{\kappa,\kappa'}\underline{\kappa}'$$

where we have (in $\mathbf{Z}[v]$):

$$f_{\kappa,\kappa'} = \sum_{i'} (\text{mult. of } \kappa' \text{ in the local system } \mathcal{H}^{i'} IC(\bar{S}_\kappa, \mathcal{L})|_{S_{\kappa'}})v^{\dim S_\kappa - \dim S_{\kappa'} - i'}$$

if $S_{\kappa'} \subset \bar{S}_\kappa$,

$$f_{\kappa,\kappa'} = 0 \text{ if } S_{\kappa'} \not\subset \bar{S}_\kappa.$$

In particular,
$f_{\kappa,\kappa} = 1$,
$f_{\kappa,\kappa'} = 0$ if $S_{\kappa'} = S_\kappa, \kappa' \neq \kappa$,
$f_{\kappa,\kappa'} \in v\mathbf{Z}[v]$ if $\kappa' \neq \kappa$.
We see that the $B(L_nG)$ is an $\mathcal{A}$-basis of $\mathcal{K}(L_nG)$.

2.5 Induction. Let $P \in \mathcal{P}^\iota$. Then $\mathcal{K}(L_n\underline{P})$ is defined as in 2.4 (in terms of $\underline{P}, \iota$ instead of G, ι). Now P^ι and its subgroup U_P^ι act freely on $G^\iota \times L_nP$ by $y : (g, x) \mapsto (gy^{-1}, \mathrm{Ad}(y)x)$; we form the quotients $E' = G^\iota \times_{U_P^\iota} L_nP$, $E'' = G^\iota \times_{P^\iota} L_nP$. Let $\pi : L_nP \longrightarrow L_n\underline{P}$ be the canonical map. We have a diagram

$$L_n\underline{P} \xleftarrow{a} E' \xrightarrow{b} E'' \xrightarrow{c} L_nG$$

where $a(g, x) = \pi(x)$, $b(g, x) = (g, x)$, $c(g, x) = \mathrm{Ad}(g)x$.

Note that a is smooth with connected fibres of dimension $s = \dim L_0P + \dim L_nU_P$, b is a principal $\underline{P}^\iota$-bundle and c is proper.

Let A be a simple $\underline{P}^\iota$-equivariant perverse sheaf on $L_n\underline{P}$. There is a well defined simple perverse sheaf $\tilde{A}$ on E'' such that

$$a^*A[[s/2]] = b^*\tilde{A}[[\dim \underline{P}^\iota/2]].$$

Moreover, if we regard A as a pure complex of weight zero (relative to a rational structure over some F_q) then $\tilde{A}$ is naturally pure of weight zero and $c_!\tilde{A}$ is naturally a mixed complex whose perverse cohomology sheaves are G^ι-equivariant. Hence $gr(c_!\tilde{A}) \in \mathcal{K}(L_nG)$ is well defined; it is independent of the choice of mixed structure for A. Now $A \mapsto gr(c_!\tilde{A})$ defines an $\mathcal{A}$-linear map

$$\mathrm{ind}_P^G : \mathcal{K}(L_n\underline{P}) \longrightarrow \mathcal{K}(L_nG).$$

2.6. Now assume that $A = IC(L_n\underline{P}, \mathcal{L})[[\dim L_n\underline{P}/2]]$ where $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_n\underline{P}}$ is cuspidal. Let

$$\dot{L}_nG = \{(gP^\iota, z) \in G^\iota/P^\iota \times L_nG; \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})\}.$$

We have a diagram

$$\overset{\circ}{L}_n\underline{P} \xleftarrow{\tilde{a}} \{(g, z) \in G^\iota \times L_nG; \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})\} \xrightarrow{\tilde{b}} \dot{L}_nG \xrightarrow{\tilde{c}} L_nG$$

with

$$\tilde{a}(g, z) = \pi(\mathrm{Ad}(g^{-1})z), \tilde{b}(g, z) = (gP^\iota, z), \tilde{c}(gP^\iota, z) = z.$$

Let $\dot{\mathcal{L}}$ be the local system on $\dot{L}_nG$ defined by $\tilde{a}^*\mathcal{L} = \tilde{b}^*\dot{\mathcal{L}}$. Using [L5, 4.4(b)], we see as in [L5, 6.6] that $c_!\tilde{A} = \tilde{c}_!\dot{\mathcal{L}}[[\dim L_0U_P/2 + \dim L_nP/2]]$. If $\mathcal{L}$ is regarded as a pure local system of weight zero (relative to a rational structure over some F_q) then $\dot{\mathcal{L}}, A, \tilde{A}$ are naturally mixed of weight zero and in $\mathcal{K}(L_nG)$ we have

$$gr(c_!\tilde{A}) = gr(\tilde{c}_!\dot{\mathcal{L}}[[\frac{\dim L_0U_P}{2} + \frac{\dim L_nP}{2}]]) = v^{\dim L_0U_P + \dim L_nP} gr(c'_!\dot{\mathcal{L}}).$$

2.7. We now fix $P, P' \in \mathcal{P}^\iota$. Let $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_n\underline{P}}$ (resp. $\mathcal{L}' \in \tilde{\mathfrak{I}}_{L_n\underline{P}'}$) be cuspidal. Let

$$A = IC(L_n\underline{P}, \mathcal{L})[[\dim L_n\underline{P}/2]] \in \mathcal{D}(L_n\underline{P}),$$
$$A' = IC(L_n\underline{P}', \mathcal{L}')[[\dim L_n\underline{P}'/2]] \in \mathcal{D}(L_n\underline{P}').$$

Let $\dot{L}_nG, \dot{\mathcal{L}}, \tilde{c}, c, \pi$ be as in 2.5, 2.6, and let $\dot{L}'_nG, \dot{\mathcal{L}}', \tilde{c}', c', \pi'$ be the analogous entities defined in terms of $P', \mathcal{L}'$.

Let $R = \{h \in G^\iota; hPh^{-1}$ and P' have a common Levi$\}$. For $h \in R$ we set $Q = hPh^{-1}$; we have isomorphisms

$$\underline{P} \xrightarrow{d} \underline{Q} \xleftarrow{e} (Q \cap P')/U_{Q\cap P'} \xrightarrow{f} \underline{P}'$$

(d is induced by $\mathrm{Ad}(h)$, e and f are induced by the inclusions $Q \cap P' \subset Q$, $Q \cap P' \subset P'$). Then $fe^{-1}d : \underline{P} \to \underline{P}'$ is an isomorphism compatible with the homomorphisms $\iota : \mathbf{k}^* \to \underline{P}, \iota : \mathbf{k}^* \to \underline{P}'$. It induces a Lie algebra isomorphism $L\underline{P} \xrightarrow{\sim} L\underline{P}'$ compatible with the gradings hence an isomorphism $\overset{\circ}{L}_n\underline{P} \xrightarrow{\sim} \overset{\circ}{L}_n\underline{P}'$. This carries $\mathcal{L}$ to a local system ${}^h\mathcal{L}$ on $\overset{\circ}{L}_n\underline{P}'$. We set

$$\tau(h) = \dim \frac{L_nU_{P'} + \mathrm{Ad}(h)(L_nU_P)}{L_nU_{P'} \cap \mathrm{Ad}(h)(L_nU_P)} - \dim \frac{L_0U_{P'} + \mathrm{Ad}(h)(L_0U_P)}{L_0U_{P'} \cap \mathrm{Ad}(h)(L_0U_P)}.$$

Let R' be the set of all $h \in R$ such that ${}^h\mathcal{L} \cong \mathcal{L}'^*$. Note that R and R' are unions of (P'^ι, P^ι)-double cosets in G^ι and that $\tau(h)$ depends only on the (P'^ι, P^ι)-double coset Ω that contains h; we shall write also τ_Ω instead of $\tau(h)$.

2.8. In the setup of 2.7, we choose a rational structure for G over some F_q with Frobenius map $F: G \to G$ as in 2.4. We assume as we may that $F(P) = P, F(P') = P'$ and that $F^*\mathcal{L} \cong \mathcal{L}$, $F^*\mathcal{L}' \cong \mathcal{L}'$.

For various varieties X connected with G which inherit an F_q-rational structure from G we shall write X^F instead of $X(F_q)$.

We may assume that $\mathcal{L}, \mathcal{L}'$ have mixed structures such that all values of $\chi_{\mathcal{L}} : (\overset{\circ}{L}_n\underline{P})^F \to \bar{\mathbf{Q}}_l$, $\chi_{\mathcal{L}'} : (\overset{\circ}{L}_n\underline{P}')^F \to \bar{\mathbf{Q}}_l$ are roots of 1. Then $\mathcal{L}, \mathcal{L}', A, A'$ are pure of weight 0. Note that $\dot{\mathcal{L}}, \dot{\mathcal{L}}', \tilde{c}_!\dot{\mathcal{L}}, \tilde{c}'_!\tilde{\mathcal{L}}', \tilde{A}, \tilde{A}', c_!\tilde{A}, c'_!\tilde{A}'$ hence also $\tilde{c}_!\dot{\mathcal{L}} \otimes \tilde{c}'_!\tilde{\mathcal{L}}', c_!A \otimes c'_!\tilde{A}'$ are naturally mixed complexes. We have

$$\chi_{c_!\tilde{A}} = (\sqrt{q})^{-\dim L_0U_P - \dim L_nP}\chi_{\tilde{c}_!\dot{\mathcal{L}}}, \quad \chi_{c'_!\tilde{A}'} = (\sqrt{q})^{-\dim L_0U_{P'} - \dim L_nP'}\chi_{\tilde{c}'_!\dot{\mathcal{L}}'}.$$

Lemma 2.9. *We have*

(a) $$\sum_{x \in (L_nG)^F} \chi_{c_!\tilde{A} \otimes c'_!\tilde{A}'}(x) = (\frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}}\sum_{\Omega} \epsilon_\Omega(-v)^{\tau_\Omega})|_{v=-1/\sqrt{q}}.$$

Here Ω runs over the P'^ι, P^ι-double cosets in G^ι such that $\Omega \subset R'$ and ϵ_Ω are roots of 1. *Note that $\frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}} \in \mathbf{Z}[v^2]$.*

Let N be the left hand side of (a). We have

$$N = (\sqrt{q})^{\varpi}\sum_{x \in (L_nG)^F}\chi_{\tilde{c}_!\dot{\mathcal{L}} \otimes \tilde{c}'_!\tilde{\mathcal{L}}'}(x) = (\sqrt{q})^{\varpi}\sum_{x' \in X^F}\chi_{\dot{\mathcal{L}} \boxtimes \dot{\mathcal{L}}'}(x')$$

where

$$\varpi = -\dim L_0U_P - \dim L_nP - \dim L_0U_{P'} - \dim L_nP',$$

$$X = \dot{L}_nG \times_{L_nG} \dot{L}'_nG = \{(gP^\iota, g'P'^\iota, z) \in G^\iota/P^\iota \times G^\iota/P'^\iota \times L_nG;$$
$$\mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n\underline{P}), \mathrm{Ad}(g'^{-1})z \in \pi'^{-1}(\overset{\circ}{L}_n\underline{P}')\}.$$

We have a partition $X = \sqcup_\Omega X_\Omega$ into locally closed subvarieties indexed by the various (P'^ι, P^ι)-double cosets Ω in G^ι, where

$$X_\Omega = \{(gP^\iota, g'P'^\iota, z) \in G^\iota/P^\iota \times G^\iota/P'^\iota \times L_nG; g'^{-1}g \in \Omega,$$
$$\mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n\underline{P}), \mathrm{Ad}(g'^{-1})z \in \pi'^{-1}(\overset{\circ}{L}_n\underline{P}')\}.$$

(There are only finitely many such Ω since P'^ι, P^ι are parabolic subgroups of G^ι.) From our assumptions we see that each Ω is defined over F_q. We have $N = \sum_\Omega N_\Omega$ where

$$N_\Omega = (\sqrt{q})^{\varpi}\sum_{(gP^\iota, g'P'^\iota, z) \in X^F_\Omega}\chi_{\dot{\mathcal{L}}}(gP^\iota, z)\chi_{\dot{\mathcal{L}}'}(g'P'^\iota, z).$$

We fix an Ω as above. Now $G^{\iota F}$ acts on $(\dot{L}_n G)^F$ by $\tilde{g} : (gP^\iota, z) \mapsto (\tilde{g}gP^\iota, \mathrm{Ad}(\tilde{g})z)$ and from the definitions we see that $\chi_{\dot{\mathcal{L}}}$ is constant on the orbits of this action. A similar property holds for $\chi_{\dot{\mathcal{L}}'}$. It follows that the function

$$X_\Omega^F \to \bar{\mathbf{Q}}_l, \quad (gP^\iota, g'P'^\iota, z) \mapsto \chi_{\dot{\mathcal{L}}}(gP^\iota, z)\chi_{\dot{\mathcal{L}}'}(g'P'^\iota, z)$$

is constant on the orbits of the $G^{\iota F}$-action on X_Ω^F given by

$$\tilde{g} : (gP^\iota, g'P'^\iota, z) \mapsto (\tilde{g}gP^\iota, \tilde{g}g'P'^\iota, \mathrm{Ad}(\tilde{g})z).$$

Since $\alpha : X_\Omega^F \to (G^\iota/P'^\iota)^F$, $(gP^\iota, g'P'^\iota, z) \mapsto g'P'^\iota$ is compatible with the obvious actions of $G^{\iota F}$ and since the $G^{\iota F}$-action on $(G^\iota/P'^\iota)^F$ is transitive, we see that for $y \in (G^\iota/P'^\iota)^F$, the sum

$$\sum_{(gP^\iota, g'P'^\iota, z) \in \alpha^{-1}(y)} \chi_{\dot{\mathcal{L}}}(gP^\iota, z)\chi_{\dot{\mathcal{L}}'}(g'P'^\iota, z)$$

is independent of the choice of y. It follows that $N_\Omega = (\sqrt{q})^\varpi |(G^\iota/P'^\iota)^F| N'_\Omega$ where

$$N'_\Omega = \sum_{\substack{(gP^\iota, z) \in (\dot{L}_n G)^F; \\ z \in \pi'^{-1}(\overset{\circ}{L}_n \underline{P}), g \in \Omega}} \chi_{\dot{\mathcal{L}}}(gP^\iota, z)\chi_{\mathcal{L}'}(\pi'(z)).$$

(We have used that $\chi_{\dot{\mathcal{L}}'}(P'^\iota, z) = \chi_{\mathcal{L}'}(\pi'(z))$ which follows from the definitions.) We set

$$Y_\Omega = \{(gP^\iota, z) \in G^\iota/P^\iota \times L_n P'; \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n \underline{P}), g \in \Omega\}.$$

Define $\sigma : Y_\Omega \to L_n \underline{P}'$ by $\sigma(gP^\iota, z) = \pi'(z)$. Then $N'_\Omega = \sum_{\xi \in (\overset{\circ}{L}_n \underline{P}')^F} N''_\Omega(\xi)\chi_{\mathcal{L}'}(\xi)$ where

$$N''_\Omega(\xi) = \sum_{(gP^\iota, z) \in \sigma^{-1}(\xi)^F} \chi_{\dot{\mathcal{L}}}(gP^\iota, z).$$

Let $K_\Omega = \sigma_!(\dot{\mathcal{L}}|_{Y_\Omega})$. This is naturally a mixed complex over $L_n \underline{P}'$ and we have $N''_\Omega(\xi) = \chi_{K_\Omega}(\xi)$ for $\xi \in (\overset{\circ}{L}_n \underline{P}')^F$. If we assume that $hPh^{-1} \cap P'$ contains no Levi of hPh^{-1} for some/any $h \in \Omega$, then we have $K_\Omega = 0$ (see [L5, 8.4(b)]); hence $\chi_{K_\Omega} = 0$ and $N''_\Omega(\xi) = 0$ for any $\xi \in (\overset{\circ}{L}_n \underline{P}')^F$. It follows that $N'_\Omega = 0$ hence $N_\Omega = 0$.

On the other hand, if we assume that $hPh^{-1} \cap P'$ contains no Levi of P' for some/any $h \in \Omega$, then we have again $N_\Omega = 0$. (This follows from the previous paragraph applied to P, P', Ω^{-1} instead of P', P, Ω.)

Assume that Ω is not as in the previous two paragraphs. Thus, setting $Q = hPh^{-1}$ for some $h \in \Omega^F$, the intersection $Q \cap P'$ contains a Levi of Q and also a

Levi of P'; it follows that Q, P' have a common Levi. We have $F(Q) = Q$ and $Q \in \mathcal{P}^\iota$. We have isomorphisms

$$\underline{P} \xrightarrow{d} \underline{Q} \xleftarrow{e} (Q \cap P')/U_{Q\cap P'} \xrightarrow{f} \underline{P'}$$

(d is induced by $\mathrm{Ad}(h)$, e and f are induced by the inclusions $Q \cap P' \subset Q$, $Q \cap P' \subset P'$). Then $fe^{-1}d : \underline{P} \longrightarrow \underline{P'}$ is an isomorphism compatible with the homomorphisms $\iota : \mathbf{k}^* \longrightarrow \underline{P}, \iota : \mathbf{k}^* \longrightarrow \underline{P'}$. It induces a Lie algebra isomorphism $L\underline{P} \xrightarrow{\sim} L\underline{P'}$ compatible with the gradings hence an isomorphism $\overset{\circ}{L}_n\underline{P} \xrightarrow{\sim} \overset{\circ}{L}_n\underline{P'}$. This carries $\mathcal{L}$ to a mixed local system ${}^h\mathcal{L}$ on $\overset{\circ}{L}_n\underline{P'}$. By [L5, 8.4(a)] and its proof [L5, 8.8], in which Tate twists must be also taken in account, we see that $K_\Omega|_{\overset{\circ}{L}_n\underline{P'}} = {}^h\mathcal{L}[[-\delta_\Omega]]$ where

$$\delta_\Omega = \dim L_0U_{P'} - \dim(L_0U_{P'} \cap L_0U_Q) + \dim(L_nU_{P'} \cap L_nU_Q)$$

and $K_\Omega|_{L_n\underline{P'} - \overset{\circ}{L}_n\underline{P'}} = 0$. We see that

$$\text{(b)} \qquad N'_\Omega = q^{\delta_\Omega} \sum_{\xi \in (\overset{\circ}{L}_n\underline{P'})^F} \chi_{{}^h\mathcal{L}}(\xi)\chi_{\mathcal{L}'}(\xi) = q^{\delta_\Omega} \sum_{\xi \in (\overset{\circ}{L}_n\underline{P'})^F} \chi_{{}^h\mathcal{L}\otimes\mathcal{L}'}(\xi).$$

If ${}^h\mathcal{L} \not\cong \mathcal{L}'^*$ then ${}^h\mathcal{L} \otimes \mathcal{L}'$ has no direct summand $\cong \bar{\mathbf{Q}}_l$ hence by an argument as in [L6, 23.5] we have $H^j_c(\overset{\circ}{L}_n\underline{P'}, {}^h\mathcal{L} \otimes \mathcal{L}') = 0$ for any j; it follows that the last sum is 0 so that $N'_\Omega = 0$. (To use [L6, 23.5] we need to know that the transitive action of $\underline{P'}^\iota/Z^0_{\underline{P'}}$ on $\overset{\circ}{L}_n\underline{P'}$ has isotropy groups with unipotent identity components; in fact in our case the isotropy groups are finite as we can see from [L5, 4.4, 2.5(a)].) It follows that $N_\Omega = 0$.

We now assume that ${}^h\mathcal{L} \cong \mathcal{L}'^*$. Then ${}^h\mathcal{L} \otimes \mathcal{L}'$ has a unique direct summand isomorphic to $\bar{\mathbf{Q}}_l$ and Frobenius acts on the stalk of this direct summand at any point in $\overset{\circ}{L}_n\underline{P'}^F$ as multiplication by a root of unity $\epsilon(h)$. By an argument in [L6, 24.14] we see that

$$\sum_{\xi \in \overset{\circ}{L}_n\underline{P'}^F} \chi_{{}^h\mathcal{L}\otimes\mathcal{L}'}(\xi) = \epsilon(h)|\underline{P'}^{\iota F}||Z^{0F}_{\underline{P'}}|^{-1}.$$

Hence $N_\Omega = \epsilon(h)(\sqrt{q})^{\varpi+2\delta_\Omega}|(G^\iota/P'^\iota)^F||\underline{P'}^{\iota F}/Z^{0F}_{\underline{P'}}|$ that is,

$$\text{(c)} \qquad N_\Omega = \epsilon(h)(\sqrt{q})^{\varpi+2\delta_\Omega-2\dim L_0U_{P'}}|G^{\iota F}||Z^{0F}_{\underline{P'}}|^{-1}.$$

We set
$s_0 = \dim L_0U_P = \dim L_0U_Q, s'_0 = \dim L_0U_{P'}$,

$s_n = \dim L_nU_P = \dim L_nU_Q, s'_n = \dim L_nU_{P'}$,
$t'_0 = \dim(L_0U_{P'} \cap L_0U_Q)$, $t'_n = \dim(L_nU_{P'} \cap L_nU_Q)$,
$r_0 = \dim(L_0U_{P'} + L_0U_Q)$, $r_n = \dim(L_nU_{P'} + L_nU_Q)$.
We have $r_0 + t'_0 = s_0 + s'_0$, $r_n + t'_n = s_n + s'_n$.

Since Q^ι, P'^ι are parabolic subgroups of G^ι with a common Levi, we have $s_0 = s'_0$. Let $z = \dim Z^0_{\underline{P}} = \dim Z^0_{\underline{P}'}$. Let $g = \dim G^\iota$. Since the action of $\underline{P}^\iota/Z^0_{\underline{P}}$ on $L_n\underline{P}$ has an open orbit with finite stabilizers, we have $\dim L_n\underline{P} = \dim \underline{P}^\iota - z$. Moreover, $\dim \underline{P}^\iota = \dim G^\iota - 2\dim U^\iota_P = g - 2s_0$. Thus $\dim L_n\underline{P} = g - z - 2s_0$. Similarly, $\dim L_n\underline{P}' = g - z - 2s_0$. We have $\dim L_nP = \dim L_n\underline{P} + \dim L_nU_P = g - z - 2s_0 + s_n$. Similarly, $\dim L_nP' = g - z - 2s_0 + s'_n$. We see that the exponent of $\sqrt{q}$ in (c) is

$$\begin{aligned}
&\varpi + 2\delta_\Omega - 2\dim L_nU_{P'}\\
&= -s_0 - (g - z - 2s_0 + s_n) - s_0 - (g - z - 2s_0 + s'_n) - 2t'_0 + 2t'_n\\
&= 2(s_0 - t'_0) - (s_n + s'_n - 2t'_n) - 2g + 2z\\
&= (r_0 - t'_0) - (r_n - t'_n) - 2g + 2z = -\tau_\Omega - 2g + 2z.
\end{aligned}$$

Thus we have

$$N_\Omega = \epsilon(h)(\sqrt{q})^{-\tau_\Omega}\frac{|G^{\iota F}|q^{-g}}{|Z^{0F}_{\underline{P}'}|q^{-z}}.$$

The lemma follows.

2.10. Let $\mathfrak{f}_2$ be the family $\{\bar{\mathbf{Q}}_l\}$ of simple perverse sheaves on a point. Write $\mathcal{K}(\text{point})$ instead of $\mathcal{K}^{\mathfrak{f}_2}(\text{point})$. We identify $\mathcal{K}(\text{point}) = \mathcal{A}$ in an obvious way. Define an $\mathcal{A}$-bilinear pairing

$$(:) : \mathcal{K}(L_nG) \times \mathcal{K}(L_nG) \longrightarrow \mathcal{A}$$

by the requirement that, if K, K' are simple G^ι-equivariant perverse sheaves on L_nG, we have

(a) $$(K : K') = gr(\rho_! i^*(K \boxtimes K')) \in \mathcal{A}$$

where K, K' are regarded as pure complexes of weight zero (relative to a rational structure over some F_q), $i : L_nG \longrightarrow L_nG \times L_nG$ is the diagonal and $\rho : L_nG \longrightarrow$ point is the obvious map (so that $\rho_! i^*(K \boxtimes K')$ is a mixed complex). Note that $(K : K')$ does not depend on the choices hence it is well defined.

2.11. Let $P, P' \in \mathcal{P}^\iota$. Let $V = (G^\iota \times_{P^\iota} L_nP) \times (G^\iota \times_{P'^\iota} L_nP')$, $V_1 = L_nG \times L_nG$, $V_2 =$ point. Let $\mathfrak{f}$ be the family of simple perverse sheaves on V of the form $\tilde{A} \boxtimes \tilde{A}'$ where $\tilde{A}$ is defined in terms of a simple $\underline{P}^\iota$-equivariant perverse sheaf A on $L_n\underline{P}$ as in 2.5 and $\tilde{A}'$ is defined in a similar way in terms of A', a simple $\underline{P}'^\iota$-equivariant perverse sheaf on $L_n\underline{P}'$. Let $\mathfrak{f}_1$ be the family of simple $G^\iota \times G^\iota$-equivariant perverse sheaves on V_1. Let $\mathfrak{f}_2$ be as in 2.10. Let $\Theta = (c, c')_! : \mathcal{D}(V) \longrightarrow \mathcal{D}(V_1)$ where $c : G^\iota \times_{P^\iota} L_n P \longrightarrow L_n G$ is as in 2.5 and $c' : G^\iota \times_{P'^\iota} L_n P' \longrightarrow L_n G$ is the analogous map. Let $\Theta' = \rho_! i^* : \mathcal{D}(V_1) \longrightarrow \mathcal{D}(V_2)$ with ρ, i as in 2.10. Then 2.2(a) is applicable. Thus we have $gr(\rho_! i^*(c, c')_!) = gr(\rho_! i^*) gr((c, c')_!)$. If $A, A', \tilde{A}, \tilde{A}'$ are as above, then

$$(\mathrm{ind}_P^G(A) : \mathrm{ind}_{P'}^G(A')) = gr(\rho_! i^*) gr((c, c')_!)(\tilde{A} \boxtimes \tilde{A}')$$

hence

(a) $$(\mathrm{ind}_P^G(A) : \mathrm{ind}_{P'}^G(A')) = gr(\rho_! i^*(c, c')_!)(\tilde{A} \boxtimes \tilde{A}').$$

We apply this to $A, A', \tilde{A}, \tilde{A}'$ as in 2.8. We choose an F_q-rational structure on G and mixed structures on A, A' as in 2.8. From (a) we see that

$$(\mathrm{ind}_P^G(A) : \mathrm{ind}_{P'}^G(A')) = \sum_{j,h \in \mathbf{Z}} d_{j,h} (-1)^j (-v)^{-h}$$

where $d_{j,h} = \dim H_c^j(L_n G, c_! \tilde{A} \otimes c'_! \tilde{A}')_h$ and the subscript h denotes the subquotient of pure weight h of a mixed vector space. Let $\{\lambda_{j,h;k}; k \in [1, d_{j,h}]\}$ be the eigenvalues of the Frobenius map on $H_c^j(L_n G, c_! \tilde{A} \otimes c'_! \tilde{A}')_h$. By the Grothendieck trace formula for the s-th power of the Frobenius map ($s \in \mathbf{Z}_{>0}$), we have

$$\sum_{j,h \in \mathbf{Z}} \sum_{k \in [1, d_{j,h}]} (-1)^j \lambda_{j,h,k}^s = \sum_{x \in L_n G(F_{q^s})} \chi_{c_! \tilde{A} \otimes c'_! \tilde{A}'}(x)$$

(in the right hand side χ is taken relative to F_{q^s}). Using Lemma 2.9 and its proof (with q replaced by q^s) we deduce

$$\sum_{j,h \in \mathbf{Z}} \sum_{k \in [1, d_{j,h}]} (-1)^j \lambda_{j,h,k}^s = \sum_{\Omega \in R'} \Big(\sum_{i \in [1,u]} q^{-s a_i} - \sum_{i' \in [1,u']} q^{-s b_{i'}} \Big) \sum_{\Omega} \epsilon_\Omega^s (\sqrt{q})^{-s \tau_\Omega}.$$

Here we write
$$\vartheta_{G^\iota} / \vartheta_{Z_{\underline{P}}^0} = \sum_{i \in [1,u]} v^{2a_i} - \sum_{i' \in [1,u']} v^{2b_{i'}}$$

with $a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_{u'}$ in $\mathbf{N}$. We can find some integer $m \geq 1$ such that $\epsilon_\Omega^m = 1$ for any $\Omega \in R'$. Then for any $s \in m\mathbf{Z}_{>0}$ we have

$$\begin{aligned}
&\sum_{\substack{j,h \in \mathbf{Z} \\ j \text{ even}}} \sum_{k \in [1, d_{j,h}]} \lambda_{j,h,k}^s + \sum_{\Omega \in R'} \sum_{i' \in [1,u']} (\sqrt{q})^{-2s b_{i'} - s\tau_\Omega} \\
&= \sum_{\substack{j,h \in \mathbf{Z} \\ j \text{ odd}}} \sum_{k \in [1, d_{j,h}]} \lambda_{j,h,k}^s + \sum_{\Omega \in R'} \sum_{i \in [1,u]} (\sqrt{q})^{-2s a_i - s\tau_\Omega}.
\end{aligned}$$

It follows that the multisets

$$\cup_{\substack{j,h\in\mathbf{Z}\\ j\text{ even}}}\cup_{k\in[1,d_{j,h}]}\{\lambda_{j,h,k}\}\cup\cup_{\Omega\in R'}\cup_{i'\in[1,u']}\{(\sqrt{q})^{-2b_{i'}-\tau_\Omega}\},$$

$$\cup_{\substack{j,h\in\mathbf{Z}\\ j\text{ odd}}}\cup_{k\in[1,d_{j,h}]}\{\lambda_{j,h,k}\}\cup\cup_{\Omega\in R'}\cup_{i\in[1,u]}\{(\sqrt{q})^{-2a_i-\tau_\Omega}\}$$

coincide. Hence for any $h\in\mathbf{Z}$ these two multisets contain the same number of elements ξ of weight h (that is such that any complex absolute value of ξ is $(\sqrt{q})^h$). Since $\lambda_{j,h,k}$ has weight h, we see that for any h we have

$$\sum_{j\in 2\mathbf{Z}}d_{j,h}+|\{(\Omega,i')\in R'\times[1,u'];-2b_{i'}-\tau_\Omega=h\}|$$
$$=\sum_{j\in 2\mathbf{Z}+1}d_{j,h}+|\{(\Omega,i)\in R'\times[1,u];-2a_i-\tau_\Omega=h\}|.$$

It follows that

$$\sum_{j,h\in\mathbf{Z};j\text{ even}}d_{j,h}(-v)^{-h}+\sum_{\Omega\in R'}\sum_{i'\in[1,u']}v^{2b_{i'}}(-v)^{\tau_\Omega}$$
$$=\sum_{j,h\in\mathbf{Z};j\text{ odd}}d_{j,h}(-v)^{-h}+\sum_{\Omega\in R'}\sum_{i\in[1,u]}v^{2a_i}(-v)^{\tau_\Omega}.$$

Equivalently,

$$\sum_{j,h\in\mathbf{Z}}d_{j,h}(-1)^j(-v)^{-h}=\frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}}\sum_{\Omega\in R'}(-v)^{\tau_\Omega}$$

that is,

(b) $$(\mathrm{ind}_P^G(A):\mathrm{ind}_{P'}^G(A'))=\frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}}\sum_{\Omega\in R'}(-v)^{\tau_\Omega}.$$

2.12. The pefect pairing $\langle,\rangle:LG\times LG\to\mathbf{k}$ (see 2.1) restricts to a perfect pairing $L_nG\times L_{-n}G\to\mathbf{k}$ denoted again by $\langle,\rangle$. Note that $\langle\mathrm{Ad}(g)x,\mathrm{Ad}(g)x'\rangle=\langle x,x'\rangle$ for $x\in L_nG,x'\in L_{-n}G,g\in G^\iota$. The Fourier-Deligne transform $\mathcal{D}(L_nG)\to\mathcal{D}(L_{-n}G)$ (defined as in 2.1 in terms of $\langle,\rangle$) takes a simple G^ι-equivariant perverse sheaf A on L_nG to a simple G^ι-equivariant perverse sheaf $\Phi_n^G(A)$ (or $\Phi_n(A)$) on $L_{-n}G$. Moreover $A\mapsto\Phi_n(A)$ defines a bijection $\mathbf{B}(L_nG)\xrightarrow{\sim}\mathbf{B}(L_{-n}G)$ and this extends uniquely to an isomorphism $\Phi_n:\mathcal{K}(L_nG)\xrightarrow{\sim}\mathcal{K}(L_{-n}G)$ of $\mathcal{A}$-modules. From [L5, 3.14(a)] we see that this is the inverse of the isomorphism $\Phi_{-n}:\mathcal{K}(L_{-n}G)\xrightarrow{\sim}\mathcal{K}(L_nG)$ defined like Φ_n in terms of $-n$ instead of n. Let

$$\kappa\mapsto\dot\kappa,\quad\mathfrak{I}_{L_nG}\to\mathfrak{I}_{L_{-n}G}$$

be the bijection such that for any $\kappa \in \tilde{\mathfrak{I}}_{L_nG}$ we have

(a) $$\Phi_n(\underline{\kappa}^\bullet) = \underline{\dot{\kappa}}^\bullet.$$

The inverse of this bijection is denoted again by $\kappa \mapsto \dot{\kappa}$.

For any simple G^ι-equivariant perverse sheaf A on L_nG, the restriction of the G^ι-action on LG_n to Z_G (a subgroup of G^ι) is the trivial action of Z_G. Then Z_G acts naturally by automorphisms on A and this action is via scalar multiplication by a character $c_A : Z_G \to \mathbf{k}^*$ (trivial on the identity component of Z_G). From the definitions we see that

(b) $c_{\Phi_n(A)} = c_A$.

Now assume that (G, ι) is rigid and that $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_nG}$ is cuspidal (so that $S_{\mathcal{L}} = \overset{\circ}{L}_nG$). Let $A = IC(L_nG, \mathcal{L})[[\dim L_nG/2]]$. According to [L5, 10.6(e)] we have $\Phi_n(A) = IC(L_{-n}G, \mathcal{L}')[[\dim L_{-n}G/2]]$ where $\mathcal{L}' \in \tilde{\mathfrak{I}}_{L_{-n}G}$ is cuspidal (so that $S_{\mathcal{L}} = \overset{\circ}{L}_{-n}G$). We can find $\mathcal{F} \in \mathcal{T}_G^{cu}, \mathcal{F}' \in \mathcal{T}_G^{cu}$ such that $\mathcal{L} = \mathcal{F}|_{\overset{\circ}{L}_nG}$, $\mathcal{L}' = \mathcal{F}'|_{\overset{\circ}{L}_{-n}G}$. We show:

(c) $\mathcal{F} = \mathcal{F}'$.

From (b) we see that the natural action of Z_G on any stalk of $\mathcal{L}$ and the natural action of Z_G on any stalk of $\mathcal{L}'$ is through the same character of Z_G. Now Z_G also acts naturally on each stalk of $\mathcal{F}$ and on each stalk of $\mathcal{F}'$ though some character of Z_G (in the adjoint action of G on LG, Z_G acts trivially). Since $\mathcal{L} = \mathcal{F}|_{\overset{\circ}{L}_nG}$, $\mathcal{L}' = \mathcal{F}'|_{\overset{\circ}{L}_{-n}G}$, the character of Z_G attached to $\mathcal{F}$ is the same as the character of Z_G attached to $\mathcal{F}'$. But from the classification of cuspidal local systems it is known that an object in $\mathcal{T}_G^{cu}$ is completely determined by the associated character of Z_G. This proves (c).

2.13. Let $P \in \mathcal{P}^\iota$. Define an $\mathcal{A}$-linear map $\widetilde{\mathrm{Ind}}_{LP}^{LG} : \mathcal{K}(L_n\underline{P}) \to \mathcal{K}(L_nG)$ as in [L5, 6.2]. We show:

(a) $$\widetilde{\mathrm{Ind}}_{LP}^{LG}(\xi) = \mathrm{ind}_P^G(\xi) \text{ for any } \xi \in \mathcal{K}(L_n\underline{P}).$$

Let A be a simple $\underline{P}^\iota$-equivariant perverse sheaf on $L_n\underline{P}$. Let $\tilde{A}$ and $c : E'' \to L_nG$ be as in 2.5. We regard A as a pure complex of weight zero (relative to a rational structure over some F_q). Then $\tilde{A}$ is naturally pure of weight zero and, by Deligne [D], $c_!\tilde{A}$ is pure of weight zero. Using [BBD] we deduce that for any j, ${}^pH^j(c_!\tilde{A})$ is pure of weight j. Hence the definition of $\mathrm{ind}_P^G(A)$ becomes

(b) $$\mathrm{ind}_P^G(A) = \sum_{A_1} \sum_{j \in \mathbf{Z}} (\text{mult. of } A_1 \text{ in } {}^pH^j(c_!\tilde{A}))v^{-j}A_1$$

where A_1 runs over the set of simple G^ι-equivariant perverse sheaves on L_nG (up to isomorphism). On the other hand, since $c_!\tilde{A}$ has weight zero, we have (by

[BBD]) $c_!\tilde{A} \cong \oplus_{j\in\mathbf{Z}} {}^pH^j(c_!\tilde{A})[-j]$ in $\mathcal{D}(L_nG)$ hence $\widetilde{\mathrm{Ind}}_{LP}^{LG}(A)$ is equal to the right hand side of (b). We see that $\mathrm{ind}_P^G(A) = \widetilde{\mathrm{Ind}}_{LP}^{LG}(A)$. This proves (a).

From (a) we see that a number of results proved in [L5] for $\widetilde{\mathrm{Ind}}_{LP}^{LG}$ imply corresponding results for ind_P^G (see (c)-(f) below).

(c) *The elements* $\mathrm{ind}_P^G(A) \in \mathcal{K}(L_nG)$ *with* $P \in \mathcal{P}^\iota$ *and* A *as in 2.6 span the* $\mathbf{Q}(v)$*-vector space* $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(L_nG)$.
(We use [L5, 13.3, 17.3].)

(d) *Let* $P, P' \in \mathcal{P}^\iota$. *Assume that* $P \subset P'$. *Let* Q *be the image of* P *under* $P' \to \underline{P}'$. *Note that* Q *is a parabolic subgroup of* $\underline{P}'$ *containing* $\iota(\mathbf{k}^*)$ *and* $\underline{Q} = \underline{P}$.*)* *Then* $\mathrm{ind}_P^G : \mathcal{K}(L_n\underline{P}) \to \mathcal{K}(L_nG)$ *is equal to the composition*

$$\mathcal{K}(L_n\underline{P}) \xrightarrow{\mathrm{ind}_Q^{\underline{P}'}} \mathcal{K}(L_n\underline{P}') \xrightarrow{\mathrm{ind}_{P'}^G} \mathcal{K}(L_nG).$$

(We use [L5, 6.4].)

(e) *Let* $P \in \mathcal{P}^\iota$. *For any* $\xi \in \mathcal{K}(L_n\underline{P})$ *we have* $\mathrm{ind}_P^G(\Phi_n^{\underline{P}}(\xi)) = \Phi_n^G(\mathrm{ind}_P^G(\xi)) \in \mathcal{K}(L_{-n}G)$ *(here* ind_P^G *in the left hand side is defined in terms of* $-n$ *instead of* n*).*
(We use [L5, 10.5].)

(f) *Let* $P \in \mathcal{P}^\iota$, $\xi \in \mathcal{K}(L_n\underline{P})$. *We have* $\mathrm{ind}_P^G(\beta(\xi)) = \beta(\mathrm{ind}_P^G(\xi))$.
The proof is along the lines of the proof of the analogous equality [L7, 36.9(c)] which is based on the relative hard Lefschetz theorem [BBD].

2.14. Let $\mathbf{Z}((v))$ be the ring of power series $\sum_{j\in\mathbf{Z}} a_jv^j$ ($a_j \in \mathbf{Z}$) such that $a_j = 0$ for $j \ll 0$. We have naturally $\mathcal{A} \subset \mathbf{Z}((v))$ and $\mathbf{Z}((v))$ becomes an $\mathcal{A}$-algebra.

Let $\{,\} : \mathcal{K}(L_nG) \times \mathcal{K}(L_nG) \to \mathbf{Z}((v))$ be the $\mathcal{A}$-bilinear pairing defined in [L5, 3.11]. We show:

(a) $$(\xi : \xi') = \vartheta_{G^\iota}\{\xi, \xi'\}|_{v\mapsto -v} \text{ for any } \xi, \xi' \in \mathcal{K}(L_nG).$$

If $\xi = \mathrm{ind}_P^G(A), \xi' = \mathrm{ind}_{P'}^G(A')$ with P, P', A, A' as in 2.7, then (a) follows by comparing 2.11(b) with the analogous formula for $\{\widetilde{\mathrm{Ind}}_{LP}^{LG}(A), \widetilde{\mathrm{Ind}}_{LP'}^{LG}(A')\}$ in [L5, 15.3]. (We use also 2.13(a).) This implies (a) in general, in view of 2.13(c).

For $\kappa \in \mathfrak{I}_{L_nG}, \mathcal{L} \in \kappa$, we define $\kappa^* \in \mathfrak{I}_{L_nG}$ by $\mathcal{L}^* \in \kappa^*$. Then $\underline{\kappa^*}^\bullet$ is the Verdier dual of $\underline{\kappa}^\bullet$. We show that for κ, κ' in $\mathfrak{I}_{L_nG}$ we have

(b) $$(\underline{\kappa}^\bullet : \underline{\kappa'}^\bullet) \in \delta_{\kappa',\kappa^*} + v\mathbf{Z}[v].$$

(c) $$(\underline{\kappa} : \underline{\kappa'}) \in \delta_{\kappa',\kappa^*} + v\mathbf{Z}[v].$$

Using (a) and [L5, 3.11(d)], we see that to prove (b) it is enough to verify the following statement.

If $f \in \mathcal{A}$ and $\vartheta_{G^\iota}^{-1} f \in \delta + v\mathbf{Z}[[v]]$ with $\delta \in \mathbf{Z}$ then $f \in \delta + v\mathbf{Z}[v]$.
This is clear since $\vartheta_{G^\iota}^{-1} \in 1 + v\mathbf{Z}[[v]]$.

Now (c) follows from (b) using the fact that the transition matrix from $(\underline{\kappa})$ to $(\underline{\kappa}^\bullet)$ is uni-triangular with off-diagonal entries in $v\mathbf{Z}[v]$ (see 2.4(a)).

2.15. Let $x \in L_nG$. Let $\mathcal{O}$ be the G^ι-orbit of x in L_nG. Let $P = P(x)$ be the parabolic subgroup associated to x in [L5, 5.2]. Recall that $\iota(\mathbf{k}^*) \subset P$. We show:

(a) *The adjoint action of U_P^ι on $x + L_nU_P$ is transitive.*

Let S be the orbit of x under this action. Since S is an orbit of an action of a unipotent group on the affine space $x + L_nU_P$, it is closed in $x + L_nU_P$. Hence it is enough to show that $\dim S = \dim L_nU_P$ or that $\dim U_P^\iota - \dim(U_P^\iota)_x = \dim L_nU_P$, where $(U_P^\iota)_x$ is the stabilizer of x in U_P^ι. The last equality is proved in the course of the proof of [L5, 5.9]. This proves (a).

Let $\pi : L_nP \to L_n\underline{P}$ be the canonical map. We have $x \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})$, see [L5, 5.3(b)]. We show:

(b) *The adjoint action of P^ι on $\pi^{-1}(\overset{\circ}{L}_n\underline{P})$ is transitive.*

Let $y \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})$. To show that y is in the P^ι-orbit of x we may replace y by a P^ι-conjugate. Hence we may assume that $y \in x + L_nU_P$. In that case we may use (a). This proves (b).

Let $L_n\underline{P} \xleftarrow{a} E' \xrightarrow{b} E'' \xrightarrow{c} L_nG$ be as in 2.5 (defined in terms of the present P). Let $E_1'' = G^\iota \times_{P^\iota} \pi^{-1}(\overset{\circ}{L}_n\underline{P})$, an open subset of E''. We show:

(c) *$c : E'' \to L_nG$ restricts to an isomorphism $E_1'' \to \mathcal{O}$.*

Using (b) we see that G^ι acts transitively on E_1'' hence $c(E_1'')$ is a single G^ι-orbit. It contains x hence it equals $\mathcal{O}$. We see that $E_1'' \subset c^{-1}(\mathcal{O})$. By the proof of [L5, 6.8(b)], c restricts to an isomorphism $c^{-1}(\mathcal{O}) \xrightarrow{\sim} \mathcal{O}$. In particular, $c^{-1}(\mathcal{O})$ is a single G^ι-orbit. Since E_1'' is a G^ι-orbit contained in $c^{-1}(\mathcal{O})$ we must have $E_1'' = c^{-1}(\mathcal{O})$ and (c) follows.

Now let $\mathcal{L}'$ be an irreducible $\underline{P}^\iota$-equivariant local system on $\overset{\circ}{L}_n\underline{P}$. Define $\kappa' \in \mathfrak{I}_{L_n\underline{P}}$ by $\mathcal{L}' \in \kappa'$. Then $\underline{\kappa}' \in \mathcal{K}(L_n\underline{P})$ is defined as in 2.4. Let $\tilde{\mathcal{L}}'$ be the local system on E_1'' whose inverse image under the obvious map $G^\iota \times_{U_P^\iota} \pi^{-1}(\overset{\circ}{L}_n\underline{P}) \to E_1''$ coincides with the inverse image of $\mathcal{L}'$ under

$G^\iota \times_{U_P^\iota} \pi^{-1}(\overset{\circ}{L}_n\underline{P}) \to \overset{\circ}{L}_n\underline{P}_n$, $(g, x) \mapsto \pi(x)$.

Let $\mathcal{L}$ be the irreducible G^ι-equivariant local system on $\mathcal{O}$ corresponding to $\tilde{\mathcal{L}}'$ under the isomorphism $E_1'' \to \mathcal{O}$ induced by c. (See (c).) Define $\kappa \in \mathfrak{I}_{L_nG}$ by $\mathcal{L} \in \kappa$. Then $\underline{\kappa} \in \mathcal{K}(L_nG)$ is defined as in 2.4. We show:

(d) $\underline{\kappa} = \operatorname{ind}_P^G(\underline{\kappa}')$.

We choose an F_q-rational structure on G as in 2.4 so that x is F_q-rational hence P is defined over F_q, and a mixed structure for $\mathcal{L}'$ which is pure of weight 0. Let $i' : \overset{\circ}{L}_n\underline{P}_n \to L_n\underline{P}$, $i : \mathcal{O} \to L_nG$, $i_1 : E_1'' \to E''$ be the inclusions. Let $(A_r)_{r \in [1,m]}$ be a set of representatives for the simple $\underline{P}^\iota$-equivariant perverse sheaves on $L_n\underline{P}$. Now $A = i'_!\mathcal{L}'[[\dim \overset{\circ}{L}_n\underline{P}/2]])$ is naturally a mixed complex on $L_n\underline{P}$ and we have

$$\underline{\kappa}' = \sum_{j,h} \sum_{r \in [1,m]} (-1)^j (\text{mult. of } A_r \text{ in } {}^pH_j(A)_h)(-v)^{-h} A_r \in \mathcal{K}(L_n\underline{P}).$$

We attach to each A_r a simple perverse sheaf $\tilde{A}_r$ on E'' by
$a^* A_r[[s/2]] = b^*\tilde{A}_r[[\dim \underline{P}^\iota/2]]$
where s is as in 2.5. Let $\tilde{A} = i_{1!}\tilde{\mathcal{L}}'[[\dim E''_1/2]] \in \mathcal{D}(E'')$. From the definitions we have $a^*A[[s/2]] = b^*\tilde{A}[[\dim \underline{P}^\iota/2]]$. Since a, b are smooth morphisms with connected fibres of dimension $s, \dim \underline{P}^\iota$ we deduce that

$$a^*({}^pH^j(A)_h)[[s/2]] = b^*({}^pH^j(\tilde{A})_h)[[\dim \underline{P}^\iota/2]]$$

and

$$\text{mult. of } A_r \text{ in } {}^pH_j(A)_h = \text{mult. of } \tilde{A}_r \text{ in } {}^pH_j(\tilde{A})_h$$

for any r, j, h. Hence

$$\underline{\kappa}' = \sum_{j,h} \sum_{r\in[1,m]} (-1)^j (\text{mult. of } \tilde{A}_r \text{ in } {}^pH_j(\tilde{A})_h)(-v)^{-h} A_r.$$

By the definition of ind_P^G we have (in $\mathcal{K}(L_nG)$):

$$\mathrm{ind}_P^G(\underline{\kappa}') = \sum_{j,h} \sum_{r\in[1,m]} (-1)^j (\text{mult. of } \tilde{A}_r \text{ in } {}^pH_j(\tilde{A})_h)(-v)^{-h} gr(c_!(\tilde{A}_r)) = gr(c_!\xi)$$

where

$$\xi = \sum_{j,h} \sum_{r\in[1,m]} (-1)^j (\text{mult. of } \tilde{A}_r \text{ in } {}^pH_j(\tilde{A})_h)(-v)^{-h} \tilde{A}_r = gr(\tilde{A}) \in \mathcal{K}^{\mathfrak{f}}(E'')$$

and $\mathfrak{f}$ is the family of simple perverse sheaves on E'' of the form $\tilde{A}_r (r \in [1, m])$. Thus,

$$\mathrm{ind}_P^G(\underline{\kappa}') = gr(c_!(gr(i_{1!}\tilde{\mathcal{L}}'[[\dim E''_1/2]]))).$$

Let $\mathfrak{f}_1$ be the family of simple G^ι-equivariant perverse sheaves on E''_1. Let $\mathfrak{f}_0$ be the family of simple G^ι-equivariant perverse sheaves on L_nG. Applying 2.2(a) to $\Theta = i_{1!} : \mathcal{D}^{\mathfrak{f}_1}(E''_1) \to \mathcal{D}^{\mathfrak{f}}(E'')$, $\Theta' = c_! : \mathcal{D}^{\mathfrak{f}}(E'') \to \mathcal{D}^{\mathfrak{f}_0}(L_nG)$, we obtain $gr(c_!i_{1!}) = gr(c_!)gr(i_{1!})$. We see that

$$\mathrm{ind}_P^G(\underline{\kappa}') = gr(c_!i_{1!}\tilde{\mathcal{L}}'[[\dim \mathcal{O}/2]]).$$

(Recall that $\dim E''_1 = \dim \mathcal{O}$.) From the definitions we have $c_!i_{1!}\tilde{\mathcal{L}}' = i_!\mathcal{L}$. Hence

$$\mathrm{ind}_P^G(\underline{\kappa}') = gr(i_!\mathcal{L}[[\dim \mathcal{O}/2]]) = \underline{\kappa}.$$

This proves (d).

2.16. We preserve the setup of 2.15. Let $\mathcal{O}_2$ be a $\underline{P}^\iota$-orbit in $L_n\underline{P} - \overset{\circ}{L}_n\underline{P}$. Let $E''_2 = G^\iota \times_{P^\iota} \pi^{-1}(\mathcal{O}_2)$, a subset of E''. We show:

(a) *The image of E''_2 under $c : E'' \to L_nG$ is contained in $\bar{\mathcal{O}} - \mathcal{O}$.*

Let $y \in c(E'')$. We show that $y \in \bar{\mathcal{O}}$. We have $y = \mathrm{Ad}(g)\eta$ for some $g \in G^\iota$ and $\eta \in L_nP$. Replacing y by $\mathrm{Ad}(g^{-1})y$ we may assume that $y \in L_nP$. By [L5, 5.9], L_nP is contained in the closure of the P^ι-orbit of x in L_nP which is clearly contained in $\bar{\mathcal{O}}$. Thus $y \in \bar{\mathcal{O}}$. We see that $c(E'') \subset \bar{\mathcal{O}}$. In particular, $c(E''_2) \subset \bar{\mathcal{O}}$. By the proof of 2.15(c) we have $E''_1 = c^{-1}(\mathcal{O})$. Since $E''_1 \cap E''_2 = \emptyset$, we have $c^{-1}(\mathcal{O}) \cap E''_2 = \emptyset$ hence $c(E''_2) \cap \mathcal{O} = \emptyset$. Thus, $c(E''_2) \subset \bar{\mathcal{O}} - \mathcal{O}$ and (a) is proved.

Now let $\kappa'' \in \mathfrak{I}_{L_n\underline{P}}$ be such that $S_{\kappa''} = \mathcal{O}_2$. Then $\underline{\kappa}'' \in \mathcal{K}(L_n\underline{P})$ is defined as in 2.4. From the definitions we see that $\mathrm{ind}_P^G(\underline{\kappa}'') \in \sum_\kappa \mathcal{A}\underline{\kappa}$ where κ runs over the elements of $\mathfrak{I}_{L_nG}$ such that S_κ is contained in the closure of $c(E''_2)$. Using this and (a) we see that

(b) $\mathrm{ind}_P^G(\underline{\kappa}'') \in \sum_{\kappa; S_\kappa \subset \bar{\mathcal{O}} - \mathcal{O}} \mathcal{A}\underline{\kappa}$.

2.17. Let $\mathcal{L}', \mathcal{L}'' \in \tilde{\mathfrak{I}}_{L_nG}$. Define $\kappa', \kappa'' \in \mathfrak{I}_{L_nG}$ by $\mathcal{L}' \in \kappa'$, $\mathcal{L}'' \in \kappa''$. Assume that $S_{\mathcal{L}'} \cap S_{\mathcal{L}''} = \emptyset$. We show:

(a) $$(\underline{\kappa}' : \underline{\kappa}'') = 0.$$

We choose an F_q-rational structure on G as in 2.4 and mixed structures on $\mathcal{L}', \mathcal{L}''$ which makes them pure of weight 0. Let $i' : S_{\mathcal{L}'} \to L_nG$, $i'' : S_{\mathcal{L}''} \to L_nG$. Let $V = S_{\mathcal{L}'} \times S_{\mathcal{L}''}, V_1 = L_nG \times L_nG, V_2 =$ point. Let $\mathfrak{f}_1, \mathfrak{f}_2$ be as in 2.11. Let $\mathfrak{f}$ be the family of simple perverse sheaves on V consisting of $\mathcal{L}' \boxtimes \mathcal{L}''$. Let $\Theta = (i', i'')_! : \mathcal{D}(V) \to \mathcal{D}(V_1)$. Let $\Theta' = \rho_! i^* : \mathcal{D}(V_1) \to \mathcal{D}(V_2)$ where ρ, i are as in 2.10. Then 2.2(a) is applicable. Thus we have $gr(\rho_! i^*(i', i'')_!) = gr(\rho_! i^*) gr((i', i'')_!)$. Hence

$$gr(\rho_! i^*) gr((i', i'')_!)(\mathcal{L}' \boxtimes \mathcal{L}'') = gr(\rho_! i^*(i', i'')_!)(\mathcal{L}' \boxtimes \mathcal{L}'').$$

The right hand side is zero since $i^*(i', i'')_!(\mathcal{L}' \boxtimes \mathcal{L}'') = 0$ (by our assumption $S_{\mathcal{L}'} \cap S_{\mathcal{L}''} = \emptyset$). Thus, $gr(\rho_! i^*) gr((i', i'')_!)(\mathcal{L}' \boxtimes \mathcal{L}'') = 0$. Using this and 2.10(a) we see that (a) holds.

3. Computation of multiplicities

3.1. For any $n \in \Delta$ let $G^\iota \backslash L_nG$ be the (finite) set of G^ι-orbits on L_nG. We define a map

(a) $\mathfrak{P}_n \to G^\iota \backslash L_nG$

as follows. Let $P \in \mathfrak{P}_n$. Let $M, L_t^r G$ be as in 1.9. We have $L_0^0 = LG'$ for a well defined connected reductive subgroup G' of G. Now G' acts on $L_n^n G$ (by restriction of the adjoint action of G on LG) and there is a unique open G'-orbit $\mathcal{O}_0$ for this action. Since $G' \subset G^\iota$, there is a unique G^ι-orbit $\mathcal{O}$ on L_nG that contains $\mathcal{O}_0$. Now (a) associates $\mathcal{O}$ to P. Clearly (a) factors through a map

(b) $\underline{\mathfrak{P}}_n \to G^\iota \backslash L_n G$.
This is a bijection. Its inverse associates to the G^ι-orbit of $x \in L_n G$ the G^ι-orbit of the parabolic subgroup associated to x in [L5, 5.2].

Another parametrization of $G^\iota \backslash L_n G$ was given by Vinberg [V] (see also Kawanaka [K]).

For any $\eta \in \underline{\mathfrak{P}}_n$ let $\mathcal{O}_\eta$ be the G^ι-orbit in $L_n G$ corresponding to η under (b). Note that

(c) $\dim \mathcal{O}_\eta = d_\eta$
where d_η is as in 1.10. This follows easily from [L5, 5.4(a), 5.9].

3.2. In the remainder of this section we assume that $\mathbf{k}$ is as in 2.1. From the bijection 3.1(b) we see that $B(L_n G) = \sqcup_{\eta \in \underline{\mathfrak{P}}_n} B_n^\eta$ where

$$B_n^\eta := \{\underline{\kappa}; \kappa \in \mathfrak{I}_{L_n G}, S_\kappa = \mathcal{O}_\eta\}.$$

3.3. We set ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G) = \mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(L_n G)$. For $n \in \Delta$ we define a $\mathbf{Q}(v)$-linear map

$$t_n = t_n^G : K_G \to {}^{\mathbf{Q}(v)}\mathcal{K}(L_n G)$$

by sending the basis element $\mathbf{I}_{\mathcal{S}}$ to $\operatorname{ind}_P^G(A)$ where $(P, \mathcal{E}) \in \mathcal{S}$ and
$A = IC(L_n \underline{P}, \mathcal{E}|_{\overset{\circ}{L_n \underline{P}}})[[\dim L_n \underline{P}/2]]$.

The pairing ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G) \times {}^{\mathbf{Q}(v)}\mathcal{K}(L_n G) \to \mathbf{Q}(v)$ obtained from the pairing (:) in 2.10 by linear extension will be denoted again by (:). From 2.11(b) we see that the equality

(a) $(t_n(\xi) : t_n(\xi')) = (\xi : \xi')$
holds when ξ, ξ' run through a basis of K_G; hence it holds for any ξ, ξ' in K_G. Since t_n is surjective (see 2.13(c)) and the pairing (:) on ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G)$ is non-degenerate (see 2.14(b)) we see that $\ker t_n = \mathcal{R}_G$ so that t_n induces an isomorphism

(b) $$\tilde{t}_n : K_G/\mathcal{R}_G \to {}^{\mathbf{Q}(v)}\mathcal{K}(L_n G).$$

3.4. Let $n \in \Delta$. We extend $\beta : \mathcal{K}(L_n G) \to \mathcal{K}(L_n G)$ to a $\mathbf{Q}$-linear endomorphism of ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G)$ (denoted again by β) by $\rho \otimes \xi \mapsto \bar{\rho} \otimes \beta(\xi)$. We show that

(a) $\beta(t_n(\xi)) = t_n(\beta(\xi))$
for any $\xi \in K_G$. We may assume that $\xi = \mathbf{I}_{\mathcal{S}}$ for some $\mathcal{S} \in \underline{\mathcal{J}}_G$. Then we have $\beta(\xi) = \xi$. It is enough to show that $\beta(\operatorname{ind}_P^G(A)) = \operatorname{ind}_P^G(A)$ where P, A are as in 3.3. By 2.13(f) we have $\beta(\operatorname{ind}_P^G(A)) = \operatorname{ind}_P^G(\beta(A))$ and it remains to note that $\beta(A) = A$.

We extend $\Phi_n : \mathcal{K}(L_n G) \to \mathcal{K}(L_{-n} G)$ to a $\mathbf{Q}(v)$-linear isomorphism ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G) \to {}^{\mathbf{Q}(v)}\mathcal{K}(L_{-n} G)$ (denoted again by Φ_n). We show that

(b) $\Phi_n(t_n(\xi)) = t_{-n}(\xi)$
for any $\xi \in K_G$. We may assume that $\xi = \mathbf{I}_{\mathcal{S}}$ for some $\mathcal{S} \in \underline{\mathcal{J}}_G$. It is then enough to show that if $(P, \mathcal{E}) \in \mathcal{S}$ and

$A=IC(L_n\underline{P},\mathcal{E}|_{\overset{\circ}{L}_n\underline{P}})[[\dim L_n\underline{P}/2]]$, $A'=IC(L_{-n}\underline{P},\mathcal{E}|_{\overset{\circ}{L}_{-n}\underline{P}})[[\dim L_{-n}\underline{P}/2]]$, then $\Phi_n^G(\mathrm{ind}_P^G(A))=\mathrm{ind}_P^G(A')$. Using 2.13(e) we see that it is enough to show that $\mathrm{ind}_P^G(\Phi_n^{\underline{P}}(A))=\mathrm{ind}_P^G(A')$. Hence it is enough to show that $\Phi_n^{\underline{P}}(A)=A'$. This follows from 2.12(c) (applied to $\underline{P}$ instead of G).

3.5. Let $n\in\Delta$. Let $Q\in\mathcal{P}^\iota$. For any $\xi\in K_{\underline{Q}}$ we have

(a) $t_n^G(f_Q^G(\xi))=\mathrm{ind}_Q^G(t_n^{\underline{Q}}(\xi))$

where f_Q^G is as in 1.4 and ind_Q^G is extended by $\mathbf{Q}(v)$-linearity. We may assume that $\xi=\mathbf{I}_{\mathcal{S}'}$ for some $\mathcal{S}'\in\underline{\mathcal{J}_Q}$; then the result follows from 2.13(d).

We show that 1.8(c) holds. Using 3.3(b) we see that it is enough to show that $t_n(f_Q^G(\mathcal{R}_{\underline{Q}}))=0$. Using (a) it is enough to show that $\mathrm{ind}_Q^G(t_n^{\underline{Q}}(\mathcal{R}_{\underline{Q}}))=0$. This follows from $t_n^{\underline{Q}}(\mathcal{R}_{\underline{Q}})=0$ (see 3.3).

3.6. In Section 1 we tried to associate to any $n\in\Delta$ and $\eta\in\underline{\mathfrak{P}}_n$ a subset $\mathcal{Z}_n^\eta$ of K_G. We will go again through the definitions (with the help of results in Section 2) and we will add the requirement that

(a) *for any $\eta\in\underline{\mathfrak{P}}_n$, t_n restricts to a bijection $\mathcal{Z}_n^\eta\xrightarrow{\sim}B_n^\eta$.*

We may assume that G is not a torus and that the subsets $\mathcal{Z}_n^\eta$ are already defined when G is replaced by any $\underline{P}$ with $P\in\mathfrak{P}'_n$. (If G is a torus then η must be $\{G\}$ and we define $\mathcal{Z}_n^\eta$ to be the subset consisting of the unique basis element of K_G.)

Assume first that $n\in\Delta$ and $\eta\in\underline{\mathfrak{P}}'_n$ (see 1.9). We define $\mathcal{Z}_n^\eta$ as in 1.11. We show that (a) holds for our η. Let $\xi\in\mathcal{Z}_n^\eta$. With notation in 1.11(i) we have $\xi=f_P^G(\xi')$ for some $\xi'\in\mathcal{Z}_n^{\{\underline{P}\}}$ where $P\in\eta$. Using 3.5(a), $t_n^G(f_P^G(\xi'))$ is equal to $\mathrm{ind}_P^G(t_n^{\underline{P}}(\xi'))$ which by the induction hypothesis belongs to $\mathrm{ind}_P^G(B_n^{\{\underline{P}\}})$. Thus, $t_n(f_P^G(\xi'))\in\mathrm{ind}_P^G(B_n^{\{\underline{P}\}})$. By 2.15(d) and its proof, ind_P^G maps $B_n^{\{\underline{P}\}}$ into B_n^η and in fact defines a bijection $a:B_n^{\{\underline{P}\}}\xrightarrow{\sim}B_n^\eta$ (we use the isomorphism in [L5, 5.8]). Thus we have $t_n(f_P^G(\xi'))\in B_n^\eta$. We consider the diagram

$$\begin{tikzcd}
\mathcal{Z}_n^{\{\underline{P}\}} \arrow[r, "a_1"] \arrow[d, "a_2"'] & \mathcal{Z}_n^\eta \arrow[d, "a_3"'] \\
B_n^{\{\underline{P}\}} \arrow[r, "a"] & B_n^\eta
\end{tikzcd}$$

where a_1 defined by ind_P^G, a_2 is defined by $t_n^{\underline{P}}$ and a_3 is defined by t_n^G. This diagram is commutative by 3.5(a). By the induction hypothesis, a_2 is a bijection. We have just seen that a is a bijection. It follows that a_3a_1 is a bijection. Hence a_1 is injective. By the definition of $\mathcal{Z}_n^\eta$, a_1 is surjective. Thus, a_1 is a bijection. In particular, 1.12(a) holds. Since a_3a_1 is a bijection we see that a_3 is a bijection. Thus (a) holds in our case.

3.7. Let $n\in\Delta$. Define $\mathcal{Z}'_n$ as in 1.11. Let $L'_nG=\{x\in L_nG;P(x)\ne G\}$. Let

$$I_n=\{\kappa\in\mathfrak{I}_{L_nG};S_\kappa\subset L'_nG\},\quad \underline{I}_n=\{\underline{\kappa};\kappa\in I_n\}.$$

Now t_n defines a bijection $\mathcal{Z}'_n \xrightarrow{\sim} \underline{I}_n$.

We show that 1.12(b) holds. Let η, η' be two distinct elements of $\underline{\mathfrak{P}}'_n$. Then $\mathcal{Z}^\eta_n, \mathcal{Z}^{\eta'}_n$ are disjoint since their images B^η_n, $B^{\eta'}_n$ under t_n are disjoint. (A local system in B^η_n has a support different from that of a local system in $B^{\eta'}_n$ since 3.1(b) is a bijection.)

We show that 1.12(c) holds. Using 3.5(a) and 3.6(a) we see that this follows from 2.16(b). (We use also 3.1(c).)

We show that 1.12(d) holds. Using 3.4(a) and 3.6(a) we see that it is enough to prove the following statement (for G instead of $\underline{P}$). If (G, ι) is rigid and $\kappa \in \mathfrak{I}_{L_nG} - I_n$ then $\beta(\kappa) - \kappa \in \sum_{\kappa' \in I_n} \mathcal{A}\kappa'$. This is immediate from the definitions.

We show that 1.12(e) holds. Using 3.3(a) we see that it is enough to show that the matrix with entries $(t_n(\xi) : t_n(\xi'))$ indexed by $\mathcal{Z}'_n \times \mathcal{Z}'_n$ is non-singular. It is also enough to show that the matrix with entries $(\underline{\kappa} : \underline{\kappa}')$ indexed by $I_n \times I_n$ is non-singular. This follows from 2.14(c) since I_n is stable under $\kappa \mapsto \kappa^*$.

3.8. Let $[\underline{I}_n]$ be the $\mathcal{A}$-submodule of $\mathcal{K}(L_nG)$ with basis $\underline{I}_n$. Now $L_nG - L'_nG$ is empty (resp. is $\overset{\circ}{L}_nG$) if (G, ι) is not rigid (resp. rigid). Hence L'_nG is a closed subset of L_nG. This, together with 2.4(a) shows that $\{\underline{\kappa}^\bullet; \kappa \in I_n\}$ is an $\mathcal{A}$-basis of $[\underline{I}_n]$.

3.9. For $\xi \in \mathcal{Z}'_n$ we define W^ξ_n as in 1.13. We have $t_n(\xi) = \underline{\kappa}$ where $\kappa \in I_n$. We show:

(a) $t_n(W^\xi_n) = \underline{\kappa}^\bullet$.

Let $y = t_n(W^\xi_n)$. Applying t_n to the equality $\beta(W^\xi_n) = W^\xi_n \mod \mathcal{R}_G$ in 1.13 and using 3.4(a) we see that $\beta(y) = y$. Applying t_n to the equality $W^\xi_n = \sum_{\xi_1 \in \mathcal{Z}'_n} c_{\xi,\xi_1}\xi_1$ in 1.13 we obtain $y = \sum_{\xi_1 \in \mathcal{Z}'_n} c_{\xi,\xi_1} t_n(\xi_1)$. We see that $y = \sum_{\kappa'} \tilde{f}_{\kappa,\kappa'}\underline{\kappa}'$ where κ' runs over the elements in $\mathfrak{I}_{L_nG}$ and

$\tilde{f}_{\kappa,\kappa'} \neq 0$ implies $\dim S_{\kappa'} < \dim S_\kappa$ or $\xi = \xi'$;

$\tilde{f}_{\kappa,\kappa'} \neq 0$, $\kappa \neq \kappa'$ implies $\tilde{f}_{\kappa,\kappa'} \in v\mathbf{Z}[v]$;

$\tilde{f}_{\kappa,\kappa'} = 1$ if $\kappa = \kappa'$.

These conditions together with $\beta(y) = y$ determine y uniquely. Since $\underline{\kappa}^\bullet$ satisfies the same conditions as y (see 2.4) we see that $y = \underline{\kappa}^\bullet$. This proves (a).

3.10. Until the end of 3.11 we assume that (G, ι) is rigid. For $n \in \Delta$ we define J_{-n} as in 1.15. Let $C = \mathbf{Q}(v) \otimes_{\mathcal{A}} [\underline{I}_n]$. Let $\xi_0 \in \mathcal{Z}'_{-n}$. Then $t_{-n}(\xi_0) = \underline{\kappa}$ where $\kappa \in I_{-n}$. We have $\xi_0 \in J_{-n}$ if and only if $t_n(W^{\xi_0}_{-n}) \notin C$ that is, if and only if $\Phi_n(t_n(W^{\xi_0}_{-n})) \notin \Phi_n(C)$ that is (using 3.4(b)), if and only if $t_{-n}(W^{\xi_0}_{-n})) \notin \Phi_n(C)$ that is (using 3.9(a) with n replaced by $-n$), if and only if $\underline{\kappa}^\bullet \notin \Phi_n(C)$. Now $\{\underline{\kappa_1}^\bullet; \kappa_1 \in I_n\}$ is a $\mathbf{Q}(v)$-basis of C. By 2.12(a), $\{\underline{\dot{\kappa}_1}^\bullet; \kappa_1 \in I_n\}$ is a $\mathbf{Q}(v)$-basis of $\Phi_n(C)$. Hence the condition that $\underline{\kappa}^\bullet \in \Phi_n(C)$ is equivalent to the condition that $\kappa = \dot{\kappa}_1$ for some $\kappa \in I_n$. We see that $\xi_0 \in J_{-n}$ if and only if $\dot{\kappa} \notin I_n$.

Assssume now that $\xi_0 \in J_{-n}$. Since $\dot{\kappa} \notin I_n$, we have $S_{\dot{\kappa}} = \overset{\circ}{L}_nG$. Define h_n as in 1.15. Let $z = t_nh_n(\xi_0)$. From the definitions we have $t_n(W^{\xi_0}_{-n}) - z \in C$

and $(z : C) = 0$. As we have seen earlier we have $\Phi_n(t_n(W^{\xi_0}_{-n})) = \underline{\kappa}^\bullet$. Hence $t_n(W^{\xi_0}_{-n}) = \Phi_{-n}(\underline{\kappa}^\bullet) = \underline{\dot\kappa}^\bullet$. Thus we have $\underline{\dot\kappa}^\bullet - z \in C$. Since $S_{\dot\kappa} = \overset{\circ}{L}_nG$ we have $\underline{\dot\kappa}^\bullet = \underline{\dot\kappa} \mod C$. It follows that $\underline{\dot\kappa} - z \in C$. Using $S_{\dot\kappa} = \overset{\circ}{L}_nG$ and 2.17(a) we see that $(\underline{\dot\kappa} : C) = 0$. Since $(z : C) = 0$ we see that $(\underline{\dot\kappa} - z : C) = 0$. Since $(:)$ is non-degenerate on C (see 3.7) and $\underline{\dot\kappa} - z \in C$ we see that $\underline{\dot\kappa} - z = 0$. Thus

(a) *t_nh_n is the map $\xi_0 \mapsto \underline{\dot\kappa}$ where $\kappa \in I_{-n}$ is given by $t_{-n}(\xi_0) = \underline{\kappa}$.*

We show that 1.16 holds. It is enough to show that h_n is injective. It is also enough to show that t_nh_n is injective. Let $\xi'_0 \in J_{-n}$. Define $\kappa' \in I_{-n}$ by $t_{-n}(\xi'_0) = \underline{\kappa}'$. Assume that $\underline{\dot\kappa} = \underline{\dot\kappa}'$. Then $\dot\kappa = \dot\kappa'$ and $\kappa = \kappa'$. Since $t_{-n} : \mathcal{Z}'_{-n} \longrightarrow I_{-n}$ is bijective it follows that $\xi_0 = \xi'_0$. Thus 1.16 is proved.

Let $\mathcal{C}_n = h_n(J_{-n})$ (see 1.15). The previous proof shows that the map $\mathcal{C}_n \longrightarrow t_n(\mathcal{C}_n)$ (restriction of t_n) is a bijection. We see that

$t_n(\mathcal{C}_n) = \{\underline{\kappa}'; \kappa' \in \mathfrak{I}_{L_nG}, S_{\kappa'} = \overset{\circ}{L}_nG, \dot\kappa' \in I_{-n}\}$.

3.11. Define $\mathcal{C}'$ as in 1.14. If $\mathcal{F} \in \mathcal{T}^{pr}_G, S_{\mathcal{F}} = C^\iota_G$ and $r_{\mathcal{F}}, \mathcal{S}_{\mathcal{F}}$ are as in 1.14, then for $n \in \Delta$ we have $t_n(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}}) = \underline{\kappa} = \underline{\kappa}^\bullet$ where $\kappa \in \mathcal{K}(L_nG)$ is $\mathcal{F}|_{\overset{\circ}{L}_nG}$. (The last two equalities follow from [L5, 11.13].) Replacing n by $-n$ we have similarly $t_{-n}(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}}) = \underline{\kappa}' = \underline{\kappa}'^\bullet$ where $\kappa' \in \mathcal{K}(L_{-n}G)$ is $\mathcal{F}|_{\overset{\circ}{L}_{-n}G}$. Using 3.4(b) we have

$$\Phi_n(\underline{\kappa}^\bullet) = \Phi_n(t_n(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}})) = t_{-n}(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}})) = \underline{\kappa}'^\bullet.$$

Thus $\kappa' = \dot\kappa$ so that $\dot\kappa \notin I_{-n}$.

We show that the map $\mathcal{C}' \longrightarrow t_n(\mathcal{C}')$ (restriction of t_n) is a bijection. It is enough to note that the map $\mathcal{F} \mapsto \mathcal{F}|_{\overset{\circ}{L}_nG}$ is a bijection from $\mathcal{T}^{pr}_G$ to the set of semicuspidal objects in $\mathfrak{I}_{L_nG}$. This follows from the fact that, if $x \in \overset{\circ}{L}_nG$, the centralizer of x in G and the centralizer of x in G^ι have the same group of components.

We now show that 1.17 holds. First we show that $\mathcal{C}_n \cap \mathcal{C}' = \emptyset$. It is enough to show that if $\underline{\kappa} \in t_n(\mathcal{C}_n)$ and $\underline{\tilde\kappa} \in t_n(\mathcal{C}')$ then $\kappa \neq \tilde\kappa$. From our assumption we have $\dot\kappa \in I_{-n}$, $\dot{\tilde\kappa} \notin I_{-n}$ (see above). Thus, $\kappa \neq \tilde\kappa$, as required.

Next we show that $\mathcal{Z}'_n \cap (\mathcal{C}_n \cup \mathcal{C}') = \emptyset$. It is enough to show that if $\underline{\kappa} \in t_n(\mathcal{Z}'_n)$ and $\underline{\tilde\kappa} \in t_n(\mathcal{C}_n \cup \mathcal{C}')$ then $\kappa \neq \tilde\kappa$. From our assumption we have $S_{\underline{\kappa}} \neq \overset{\circ}{L}_nG$, $S_{\underline{\tilde\kappa}} = \overset{\circ}{L}_nG$, Thus, $\kappa \neq \tilde\kappa$, as required. This proves 1.17. We see also that the map $\mathcal{Z}'_n \cup \mathcal{C}_n \cup \mathcal{C}' \longrightarrow t_n(\mathcal{Z}'_n \cup \mathcal{C}_n \cup \mathcal{C}')$ (restriction of t_n) is a bijection.

3.12. If (G, ι) is not rigid then the definition of the subsets $\mathcal{Z}^\eta_n$ $(\eta \in \underline{\mathfrak{P}}_n)$ is complete. If (G, ι) is rigid then $\underline{\mathfrak{P}}_n - \underline{\mathfrak{P}}'_n = \{G\}$. For $n \in \Delta$ we set $\mathcal{Z}^{\{G\}}_n = \mathcal{C}_n \cup \mathcal{C}'$. By 1.17, this union is disjoint. The definition of the subsets $\mathcal{Z}^\eta_n$ $(\eta \in \underline{\mathfrak{P}}_n)$ is complete. Define $\mathcal{Z}_n$ as in 1.18. Note that the map $\mathcal{Z}_n \longrightarrow t_n(\mathcal{Z}_n)$ (restriction of t_n) is a bijection.

We show that $t_n(\mathcal{Z}_n)=B(L_nG)$. Let $\kappa\in\mathfrak{I}_{L_nG}$. If $\kappa\in I_n$ then $\underline{\kappa}\in t_n(\mathcal{Z}'_n)$. If $\kappa\in\mathfrak{I}_{L_nG}-I_n$ and $\dot\kappa\in I_{-n}$ then $\underline{\kappa}\in t_n(\mathcal{C}_n)$. If $\kappa\in\mathfrak{I}_{L_nG}-I_n$ and $\dot\kappa\in\mathfrak{I}_{L_{-n}G}-I_{-n}$ then by [L5, 12.3], we have $\underline{\kappa}\in t_n(\mathcal{C}')$. We see that

(a) *t_n restricts to a bijection $\mathcal{Z}_n\xrightarrow{\sim}B(L_nG)$.*

3.13. For $n\in\Delta$ and $\xi\in\mathcal{Z}_n$ we define an element W_n^ξ as in 1.19. We show:

(a) *If $t_n(\xi)=\underline{\kappa}$ with $\kappa\in\mathfrak{I}_{L_nG}$ then $t_n(W_n^\xi)=\underline{\kappa}^\bullet$.*

When $\xi\in\mathcal{Z}'_n$ this follows from 3.9(a). Next assume that $\xi\in\mathcal{C}_n$. Define $\xi_0\in J_{-n}$ by $h_n(\xi_0)=\xi$. Define $\kappa_0\in\mathfrak{I}_{L_{-n}G}$ by $t_{-n}(\xi_0)=\underline{\kappa_0}$. Using the definition, 3.4(b) and 3.9(a) (for $-n$ instead of n) we have

$t_n(W_n^\xi)=t_n(W_{-n}^{\xi_0})=\Phi_{-n}(t_{-n}(W_{-n}^{\xi_0}))=\Phi_{-n}\underline{\kappa_0}^\bullet=\underline{\dot\kappa_0}^\bullet$.

By 3.10(a) we have $\underline{\kappa}=t_n(\xi)=\underline{\dot\kappa_0}$. Thus (a) holds in our case.

Finally, assume that $\xi\in\mathcal{C}'$. In this case we have $t_n(W_n^\xi)=t_n(\xi)=\underline{\kappa}=\underline{\kappa}^\bullet$, see 3.11. This proves (a).

Let $\kappa\in\mathfrak{I}_{L_nG}$. Let $\xi\in\mathcal{Z}_n$ be such that $t_n(\xi)=\underline{\kappa}$. Let $c_{\xi,\xi'}$ be as in 1.19 ($\xi'\in\mathcal{Z}_n$). Applying t_n to both sides of 1.19(a) and using (a) we obtain

$\underline{\kappa}^\bullet=\sum_{\xi'\in\mathcal{Z}_n}c_{\xi,\xi'}t_n(\xi')$.

Comparing this with 2.4(a) we see that for any κ,κ' in $\mathfrak{I}_{L_nG}$ we have

$$\text{(b)}\qquad f_{\kappa,\kappa'}=c_{\xi,\xi'}$$

where $\xi,\xi'\in\mathcal{Z}_n$ are defined by $t_n\xi=\kappa, t_n\xi'=\kappa'$. Note that (b) provides a method to compute explicitly the matrix of multiplicities $(f_{\kappa,\kappa'})$.

3.14. We prove 1.20. Let $\mathcal{S}\in\underline{\mathcal{J}}^D$. Let $(P,\mathcal{E})\in\mathcal{S}$, $\mathcal{L}=\mathcal{E}|_{\overset{\circ}{L_n\underline{P}}}$. Let $\tilde c,\dot{\mathcal{L}},A$ be as in 2.6. We regard $\dot{\mathcal{L}}$ as a pure local system of weight 0. By [L5, 21.1(b)] and its proof, $\mathcal{H}^i\tilde c_!\dot{\mathcal{L}}$ is pure of weight i and is 0 unless $i\in2\mathbf{N}$. It follows that in $\mathcal{K}(L_nG)$ we have $\mathrm{ind}_P^G(A)=\sum_{\kappa\in\mathfrak{I}_{L_nG}}\tilde e_{\mathcal{S},\kappa}\underline{\kappa}$ where $\tilde e_{\mathcal{S},\kappa}\in\mathcal{A}$ is equal to a power of v times $\sum_i(\text{mult. of }\mathcal{L}'\text{ in }\mathcal{H}^i\tilde c_!\dot{\mathcal{L}})v^{-i}$. (Here $\mathcal{L}'\in\kappa$.) From the definitions we have for any $\xi\in\mathcal{Z}_n$

(a) $e_{\mathcal{S},\xi}=\tilde e_{\mathcal{S},\kappa}$

where $\kappa=t_n(\xi)$. Hence 1.20 holds.

4. Further results

4.1. In this section we assume that $\mathbf{k}$ is as in 2.1.

To any $n\in\Delta$ and any $\kappa\in\mathfrak{I}_{L_nG}$ we shall associate a G^ι-orbit $\mathcal{S}_\kappa\in\underline{\mathcal{J}}_G$, an element $r_\kappa\in\mathcal{A}-\{0\}$ and an element $L_\kappa\in\mathcal{K}(L_nG)$ such that $L_\kappa=t_n(r_\kappa^{-1}\mathbf{I}_{\mathcal{S}_\kappa})$. We may assume that these objects are already defined when G is replaced by $\underline{P}$ with $P\in\mathfrak{P}'_n$.

(i) Assume first that κ is semicuspidal. There is a unique $\mathcal{F}\in\mathcal{J}_G^{pr}$ such that $S_\mathcal{F}\cap L_n\Gamma=\overset{\circ}{L}_nG$ and $\mathcal{F}|_{\overset{\circ}{L}_nG}\in\kappa$. Let $\mathcal{S}_k=\mathcal{S}_\mathcal{F}$ (see 1.18), $r_\kappa=r_\mathcal{F}$ (see 1.3), $L_\kappa=\underline{\kappa}=\underline{\kappa}^\bullet$. These elements satisfy the required condition, see 3.11.

(ii) Next we assume that $\kappa \in I_n$. We have $\underline{\kappa} \in B_n^\eta$ for a unique $\eta \in \underline{\mathfrak{P}}'_n$. Let $P \in \eta$. Then $P \neq G$. Let $a : B_n^{\{P\}} \xrightarrow{\sim} B_n^\eta$ be the bijection in 3.6. Let $\underline{\kappa}_1 = a^{-1}(\underline{\kappa})$. Now $\mathcal{S}_{\kappa_1}$, r_{κ_1}, $L_{\kappa_1} \in \mathcal{K}(L_n\underline{P})$ are already defined from the induction hypothesis. Let $r_\kappa = r_{\kappa_1}$, $L_k = \mathrm{ind}_P^G(L_{\kappa_1})$, $\mathcal{S}_\kappa = a_P^G(\mathcal{S}_{\kappa_1})$. These elements satisfy the required condition.

(iii) Next we assume that $\kappa \notin I_n$ and κ is not semicuspidal. By 3.12 we have $\dot\kappa \in I_{-n}$. Now $\mathcal{S}_{\dot\kappa}$, $r_{\dot\kappa}$, $L_{\dot\kappa} \in \mathcal{K}(L_{-n}G)$ are defined as in (ii). Let $\mathcal{S}_\kappa = \mathcal{S}_{\dot\kappa}$, $r_\kappa = r_{\dot\kappa}$, $L_\kappa = \Phi_{-n}(L_{\dot\kappa})$. These elements satisfy the required condition.

This completes the definition of $\mathcal{S}_\kappa, r_\kappa, L_\kappa$.

4.2. For $n \in \Delta$, we shall define a partial order $\le$ on $\mathfrak{I}_{L_nG}$. We may assume that $\le$ is already defined when G is replaced by $\underline{P}$ with $P \in \mathfrak{P}'_n$.

(i) Assume that at least one of κ, κ' is semicuspidal. Then $\kappa \le \kappa'$ if and only if $\kappa = \kappa'$.

(ii) Assume that $S_\kappa \neq S_{\kappa'}$ and neither κ nor κ' is semicuspidal. Then $\kappa \le \kappa'$ if and only if $S_\kappa \subset \bar S_{\kappa'} - S_{\kappa'}$.

(iii) Assume that $S_\kappa = S_{\kappa'}$ and $\kappa \in I_n$ (hence also $\kappa' \in I_n$). We have $\underline{\kappa}, \underline{\kappa}' \in B_n^\eta$ for a unique $\eta \in \underline{\mathfrak{P}}'_n$. Let $P \in \eta$; then $P \neq G$. Let $a : B_n^{\{P\}} \xrightarrow{\sim} B_n^\eta$ be the bijection in 3.6. Let $\underline{\kappa}_1 = a^{-1}(\underline{\kappa}), \underline{\kappa}'_1 = a^{-1}(\underline{\kappa}')$. We say that $\kappa \le \kappa'$ if and only if $\kappa_1 \le \kappa'_1$ (which is known by the inductive assumption applied to $\underline{P}$).

(iv) Assume that $S_\kappa = S_{\kappa'}$, $\kappa \notin I_n$ (hence $\kappa' \notin I_n$) and neither κ nor κ' is semicuspidal. By 3.12 we have $\dot\kappa \in I_{-n}$, $\dot\kappa' \in I_{-n}$. We say that $\kappa \le \kappa'$ if and only if $\dot\kappa \le \dot\kappa'$ which is known from (ii) (if $S_{\dot\kappa} \neq S_{\dot\kappa'}$) or (iii) (if $S_{\dot\kappa} = S_{\dot\kappa'}$).

This completes the definition of $\le$. We write $\kappa' < \kappa$ instead of $\kappa' \le \kappa, \kappa' \neq \kappa$.

4.3. Example. In this subsection we assume that G is the group of automorphisms of a 4-dimensional $\mathbf{k}$-vector space V preserving a fixed non-degenerate symplectic form. We fix a direct sum decomposition $V = V_{-1} \oplus V_1$ where V_1, V_{-1} are Lagrangian subspaces. For any $t \in \mathbf{k}^*$ define $\iota(t) \in G$ by $\iota(t)x = tx$ for $x \in V_1$, $\iota(t)x = t^{-1}x$ for $x \in V_{-1}$. Let $\Delta = \{2, -2\}$. Note that (G, ι) is rigid. We identify G^ι with $GL(V_1)$ by $g \mapsto g|_{V_1}$. The grading of LG defined by ι has non-zero components in degrees $-2, 0, 2$ and L_2G (resp. $L_{-2}G$) may be identified as a representation of G^ι with S^2V_1 (resp. $S^2V_1^*$) where S^2 stands for the second symmetric power. For $n \in \Delta$, the set $\mathfrak{I}_{L_nG}$ consists of five objects $\kappa_{0,n}, \kappa_{2,n}, \tilde\kappa_{2,n}, \kappa_{3,n}, \tilde\kappa_{3,n}$ where $\kappa_{i,n}$ represents the local system $\bar{\mathbf{Q}}_l$ on the G^ι-orbit of dimension $i (i = 0, 2, 3)$ and $\tilde\kappa_{i,n}$ represents a non-trivial local system of rank 1 on the G^ι-orbit of dimension $i (i = 2, 3)$. The effect of the Fourier-Deligne transform is as follows.

$\Phi_n(\kappa^\bullet_{0,n}) = \kappa^\bullet_{3,-n}$, $\Phi_n(\kappa^\bullet_{2,n}) = \tilde\kappa^\bullet_{3,-n}$, $\Phi_n(\kappa^\bullet_{3,n}) = \kappa^\bullet_{0,-n}$, $\Phi_n(\tilde\kappa^\bullet_{3,n}) = \kappa^\bullet_{2,-n}$,
$\Phi_n(\tilde\kappa^\bullet_{2,n}) = \tilde\kappa^\bullet_{2,n}$.

The partial order in 4.2 is $\kappa_{0,n} < \kappa_{2,n} < \kappa_{3,n} < \tilde\kappa_{3,n}$; $\tilde\kappa_{2,n}$ is isolated. We have

$L_{\kappa_{0,n}} = \kappa^\bullet_{0,n}$, $L_{\kappa_{2,n}} = \kappa^\bullet_{2,n} + \kappa^\bullet_{0,n}$, $L_{\kappa_{3,n}} = \kappa^\bullet_{3,n}$, $L_{\tilde\kappa_{3,n}} = \tilde\kappa^\bullet_{3,n} + \kappa^\bullet_{3,n}$,
$L_{\tilde\kappa_{2,n}} = \tilde\kappa^\bullet_{2,n}$.

4.4. We show that for any $n \in \Delta$ and any $\kappa \in \mathfrak{I}_{L_nG}$ we have

(a) $L_\kappa \in \underline{\kappa} + \sum_{\kappa' \in \mathfrak{I}_{L_nG};\kappa'<\kappa} \mathcal{A}\ \underline{\kappa}'$.

We may assume that (a) is already known when G is replaced by $\underline{P}$ with $P \in \mathfrak{P}'_n$.

(i) Assume that κ is semicuspidal. Then $L_\kappa = \kappa$ and (a) is clear.

(ii) Assume that $\kappa \in I_n$. Let P, a, κ_1 be as in 4.1(ii). By the induction hypothesis we have $L_{\kappa_1} \in \underline{\kappa}_1 + \sum_{\kappa'_1;\kappa'_1<\kappa_1} \mathcal{A}\ \underline{\kappa}'_1$. Applying ind_P^G and using 2.15(d) we obtain

$L_\kappa \in \underline{\kappa} + \sum_{\kappa'_1 \in X} \mathcal{A}\ \underline{a(\kappa'_1)} + \sum_{\kappa'_1 \in Y} \mathcal{A}\ \mathrm{ind}_P^G(\underline{\kappa}'_1)$

where

$X = \{\kappa'_1; \kappa'_1 < \kappa_1, S_{\kappa'_1} = S_{\kappa_1}\}$, $Y = \{\kappa'_1; \kappa'_1 < \kappa_1, S_{\kappa'_1} \ne S_{\kappa_1}\}$.

For $\kappa'_1 \in X$ we have $S_{a(\kappa'_1)} = S_\kappa$ and $a(k'_1) < \kappa$. For $\kappa'_1 \in Y$ we have $S_{\kappa'_1} \subset \bar{S}_{\kappa_1} - S_{\kappa_1}$. By 2.16(b), for any $\kappa'_1 \in Y$, $\mathrm{ind}_P^G(\underline{\kappa}'_1)$ is an $\mathcal{A}$-linear combination of elements $\underline{\kappa}'$ where $S_{\kappa'} \subset \bar{S}_\kappa - S_\kappa$ (hence $\kappa' < \kappa$). Hence L_κ satisfies (a).

(iii) Assume that $\kappa \notin I_n$ and κ is not semicuspidal. By 3.12 we have $\dot{\kappa} \in I_{-n}$. By (ii), we have

$L_{\dot{\kappa}} \in \underline{\dot{\kappa}} + \sum_{\kappa';\kappa'<\dot{\kappa}} \mathcal{A}\ \underline{\kappa}'$.

Using $\underline{\kappa}_0 \in \underline{\kappa}_0^\bullet + \sum_{\kappa'_0<\kappa_0} \mathcal{A}\ \underline{\kappa}'^\bullet_0$ for $\kappa_0 \in I_{-n}$ we deduce $L_{\dot{\kappa}} \in \underline{\dot{\kappa}}^\bullet + \sum_{\kappa';\kappa'<\dot{\kappa}} \mathcal{A}\ \underline{\kappa}'^\bullet$. Applying Φ_{-n}^G we obtain

$L_\kappa \in \underline{\kappa}^\bullet + \sum_{\kappa';\kappa'<\dot{\kappa}} \mathcal{A}\ \underline{\dot{\kappa}}'^\bullet$.

Using $\underline{\kappa}_1^\bullet \in \underline{\kappa}_1 + \sum_{\kappa'_1<\kappa_1} \mathcal{A}\ \underline{\kappa}'_1$ for $\kappa_1 \in I_n$ we see that it is enough to show that for any κ' such that $\kappa' < \dot{\kappa}$ we have $\dot{\kappa}' < \kappa$. If $S_\kappa = S_{\dot{\kappa}'}$ then this follows from 4.2(iv). If $S_\kappa \ne S_{\dot{\kappa}'}$ then $S_{\dot{\kappa}'} \subset \bar{S}_\kappa - S_\kappa$ (we have $S_\kappa = \overset{\circ}{L}_nG$) hence again $\dot{\kappa}' < \kappa$ (using 4.2(ii)).

This completes the proof of (a).

From (a) we deduce

(b) *The set* $\{L_\kappa; \kappa \in \mathfrak{I}_{L_nG}\}$ *is an* $\mathcal{A}$*-basis of* $\mathcal{K}(L_nG)$.

4.5. We show that:

(a) *the map* $\mathfrak{I}_{L_nG} \longrightarrow \underline{\mathcal{J}}_G, \kappa \mapsto \mathcal{S}_\kappa$ *is injective.*

Let $\kappa, \kappa' \in \mathfrak{I}_{L_nG}$ be such that $\mathcal{S}_\kappa = \mathcal{S}_{\kappa'}$. Then $t_n(\mathcal{S}_\kappa) = t_n(\mathcal{S}_{\kappa'})$ hence $r_\kappa L_\kappa = r_{\kappa'} L_{\kappa'}$. Using now 4.4(b) we deduce that $\kappa = \kappa'$ as desired.

4.6. Let $\xi \in \mathcal{Z}_n$ and let $\kappa = t_n(\xi)$. By [L3, 3.36], the numbers $\tilde{e}_{\mathcal{S},\kappa}|_{v=1}$ (for various $\mathcal{S}$, see 3.14) are the dimensions of the various "weight spaces" of a standard module over an affine Hecke algebra. Using 3.14(a) we see that the dimensions of these weight spaces are given by the numbers $e_{\mathcal{S},\xi}|_{v=1}$ (for various $\mathcal{S}$) hence are computable from the algorithm in Section 1.

4.7. In this subsection we shall summarize some of the results of this paper in terms of the vector space $\bar{K}_G = K_G/\mathcal{R}_G$. (Any text marked as ♠…♠ applies only in the case where (G, ι) is rigid.) Note that the pairing (:) on K_G induces a pairing $\bar{K}_G \times \bar{K}_G \longrightarrow \mathbf{Q}(v)$ denoted again by (:). Also $\beta : K_G \longrightarrow K_G$ induces an involution $\bar{K}_G \longrightarrow \bar{K}_G$ denoted again by β. Let $\pi : K_G \longrightarrow \bar{K}_G$ be the obvious map.

Since for $n \in \Delta$, $B(L_nG)$ is a $\mathbf{Q}(v)$-basis of ${}^{\mathbf{Q}(v)}\mathcal{K}(L_nG)$, we see (using 3.12(a) and 3.3(b)) that π restricts to a bijection of $\mathcal{Z}_n$ onto a basis $\overline{\mathcal{Z}}_n$ of $\bar{K}_G$. We say that $\overline{\mathcal{Z}}_n$ is a *PBW-basis* of $\bar{K}_G$. ♠ The last bijection restricts to bijections of $\mathcal{Z}'_n, \mathcal{C}_n, \mathcal{C}'$ onto subsets $\overline{\mathcal{Z}}'_n, \overline{\mathcal{C}}_n, \overline{\mathcal{C}}'$ of $\overline{\mathcal{Z}}_n$. ♠ In the case where (G,ι) is not rigid we set $\overline{\mathcal{Z}}'_n = \overline{\mathcal{Z}}_n$.

Let $\mathcal{M}_n$ be the $\mathbf{Z}[v]$-submodule of $\bar{K}_G$ with basis $\overline{\mathcal{Z}}_n$. For any $u \in \overline{\mathcal{Z}}_n$ let $\bar{W}^u_n = \pi(W^\xi_n)$ where $\xi \in \mathcal{Z}_n$ is given by $\pi(\xi) = u$. From 3.13(a) and 2.4(a) we see that $\{\bar{W}^n_u; u \in \overline{\mathcal{Z}}_n\}$ is a $\mathbf{Z}[v]$-basis of $\mathcal{M}_n$ and that for any $u \in \overline{\mathcal{Z}}_n$ we have $\bar{W}^u_n - u \in v\mathcal{M}_n$. Define a bijection $u \mapsto \dot{u}$ of $\overline{\mathcal{Z}}_n$ onto $\overline{\mathcal{Z}}_{-n}$ as follows. Let $\xi \in \mathcal{Z}_n$ be such that $\pi(\xi) = u$; let $\kappa \in \mathfrak{I}_{L_nG}$ be such that $t_n(\xi) = \underline{\kappa}$ (see 3.12(a)). Let $\xi' \in \mathcal{Z}_{-n}$ be such that $t_{-n}\xi' = \dot{\underline{\kappa}}$ (see 2.12). Then $\dot{u} = \pi(\xi')$. The inverse of the bijection $u \mapsto \dot{u}$ is denoted again by $u \mapsto \dot{u}$.

For u, ξ, κ, ξ' as above we have $\bar{W}^u_n = \pi(W^\xi_n), t_n(W^\xi_n) = \underline{\kappa}^\bullet$, $\bar{W}^{\dot{u}}_{-n} = \pi(W^{\xi'}_{-n})$, $t_{-n}(W^{\xi'}_{-n}) = \dot{\underline{\kappa}}^\bullet$. By 2.12(a) we have $\Phi_n(\underline{\kappa}^\bullet) = \dot{\underline{\kappa}}^\bullet$ hence $t_{-n}(W^{\xi'}_{-n}) = \Phi_n(t_n(W^\xi_n))$ and this equals $t_{-n}(W^\xi_n))$ (see 3.4(b)). Thus, $t_{-n}(W^{\xi'}_{-n} - W^\xi_n) = 0$. Since $\ker t_{-n} = \mathcal{R}_G$ we see that $W^{\xi'}_{-n} - W^\xi_n \in \mathcal{R}_G$. Applying π we deduce

(a) $\bar{W}^{\dot{u}}_{-n} = \bar{W}^u_n$.

Moreover, from the proof in 3.10, we see that

(b) ♠ $u \in \overline{\mathcal{C}}_n \implies \dot{u} \in \overline{\mathcal{Z}}_{-n}$. ♠

From (a) we see that the basis $(\bar{W}^u_n)$ of $\bar{K}_G$ coincides with the basis $(\bar{W}^u_{-n})$. We call this the *canonical basis* of $\bar{K}_G$. It follows that $\mathcal{M}_n = \mathcal{M}_{-n}$. We shall write $\mathcal{M}$ instead of $\mathcal{M}_n = \mathcal{M}_{-n}$. We have

(c) $\bar{W}^u_n - u \in v\mathcal{M}$, $\bar{W}^{\dot{u}}_{-n} - \dot{u} \in v\mathcal{M}$.

Combining with (a) we see that

(d) $u - \dot{u} \in v\mathcal{M}$ for any $u \in \overline{\mathcal{Z}}_n$.

Let $\pi' : \mathcal{M} \to \mathcal{M}/v\mathcal{M}$ be the obvious map. From (a),(c) we see that there exists a $\mathbf{Z}$-basis X of $\mathcal{M}/v\mathcal{M}$ such that π' restricts to bijections $(\bar{W}^u_n) = (\bar{W}^u_{-n}) \xrightarrow{\sim} X$, $\overline{\mathcal{Z}}_n \xrightarrow{\sim} X$, $\overline{\mathcal{Z}}_{-n} \xrightarrow{\sim} X$. ♠ Moreover, X can be partitioned as $X = X_0 \sqcup X_n \sqcup X_{-n} \sqcup X'$ so that π' restricts to bijections $\overline{\mathcal{C}}_n \xrightarrow{\sim} X_n$, $\overline{\mathcal{C}}_{-n} \xrightarrow{\sim} X_{-n}$, $\overline{\mathcal{C}}' \xrightarrow{\sim} X'$, $\overline{\mathcal{Z}}'_n \xrightarrow{\sim} X_0 \cup X_{-n}$, $\overline{\mathcal{Z}}'_{-n} \xrightarrow{\sim} X_0 \cup X_n$. ♠ We set

(e) $\tilde{X} = \overline{\mathcal{Z}}'_n \cup \overline{\mathcal{Z}}'_{-n}$ (if (G,ι) is not rigid), $\tilde{X} = \overline{\mathcal{Z}}'_n \cup \overline{\mathcal{Z}}'_{-n} \cup \overline{\mathcal{C}}'$ (if (G,ι) is rigid).

We have

(f) $X = \pi'(\tilde{X})$.

We show that

(g) $\tilde{X}$ generates the $\mathbf{Z}[v]$-module $\mathcal{M}$.

If (G,ι) is not rigid, this is clear. ♠ Assume now that (G,ι) is rigid. Let $\mathcal{M}'$ be the $\mathbf{Z}[v]$-submodule of $\mathcal{M}$ generated by $\tilde{X}$. If $u \in \overline{\mathcal{Z}}'_n$ then by the arguments in 1.13, we have $\bar{W}^u_n \in \sum_{u' \in \overline{\mathcal{Z}}'_n} \mathbf{Z}[v]u'$ hence $\bar{W}^u_n \in \mathcal{M}'$. If $u \in \overline{\mathcal{C}}_n$ then $\bar{W}^u_n = \bar{W}^{\dot{u}}_{-n}$ and this is in $\mathcal{M}'$ since $\dot{u} \in \overline{\mathcal{Z}}'_{-n}$ (we use the previous sentence with u, n replaced by $\dot{u}, -n$). If $u \in \overline{\mathcal{C}}'$ then $\bar{W}^u_n = u$ is again in $\mathcal{M}'$. Since $(\bar{W}^u_n)_{u \in \overline{\mathcal{Z}}_n}$ is a $\mathbf{Z}[v]$-basis of $\mathcal{M}$ we see that $\mathcal{M} = \mathcal{M}'$. ♠ This proves (g).

We show how the canonical basis and the PBW-bases are determined in terms of the subsets $\overline{\mathcal{Z}}'_n, \overline{\mathcal{Z}}'_{-n}$ (which are defined by the inductive construction in 1.11(i)) and (in the rigid case) by the set $\overline{\mathcal{C}}'$ which is defined as in 1.14.

We first define $\tilde{X}$ as in (e). Note that $\mathcal{M}$ is defined in terms of $\tilde{X}$ as in (g) and then the basis X of $\mathcal{M}/v\mathcal{M}$ is defined in terms of $\tilde{X}$ as in (f).

Now the canonical basis can be reconstructed in terms of $\mathcal{M}$ and X: for any $x \in X$ there is a unique element $\hat{x} \in \mathcal{M}$ such that $\pi'(\hat{x}) = x$ and $\beta(\hat{x}) = \hat{x}$. The elements $\{\hat{x}; x \in X\}$ form the canonical basis. Now let $n \in \Delta$. We show how to reconstruct the PBW-basis $\overline{\mathcal{Z}}_n$. If (G, ι) is not rigid then $\overline{\mathcal{Z}}_n = \overline{\mathcal{Z}}'_n$ is already known. ♠ Assume now that (G, ι) is rigid. Then the part $\overline{\mathcal{Z}}'_n \cup \overline{\mathcal{C}}'$ of $\overline{\mathcal{Z}}_n$ is already known. It remains to characterize the subset $\overline{\mathcal{C}}_n$ of $\overline{\mathcal{Z}}_n$. For $n \in \Delta$ let X_n be the set of all $x \in \pi'(\overline{\mathcal{Z}}'_{-n})$ such that $x \notin \pi'(\overline{\mathcal{Z}}'_n)$. For any $x \in X_n$ we can write uniquely $\hat{x} = x' + x''$ where x'' is in the subspace of $\bar{K}_G$ spanned by $\overline{\mathcal{Z}}'_n$ and x' is orthogonal under (:) to that subspace. Then $\overline{\mathcal{C}}_n$ consists of the elements x' for various $x \in X_n$. ♠

4.8. Let $n \in \Delta$. Assume that (G, ι) is rigid. Let Ξ_n be the set of all $\kappa \in \mathfrak{I}_{L_nG}$ such that $S_\kappa \neq \overset{\circ}{L}_nG$ and $S_{\dot{\kappa}} = \overset{\circ}{L}_{-n}G$. Here $\kappa \mapsto \dot{\kappa}$ is as in 2.12. It would be interesting to find a simple description of the set of local systems Ξ_n (without using Fourier-Deligne transform). In particular, we would like to know which G^ι-orbits in L_nG are of the form S_κ for some $\kappa \in \Xi_n$. (Our results answer this question only in terms of an algorithm, not in closed form.) For example, if $G = GL_n(\mathbf{k})$ then Ξ_n has only one object: the local system $\bar{\mathbf{Q}}_l$ on the 0-dimensional orbit. In the case studied in 4.3, Ξ_n has two objects: the local system $\bar{\mathbf{Q}}_l$ on the 0 or 2 dimensional orbit.

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Department of Mathematics, M.I.T., Cambridge, MA 02139