FINITE ELEMENT MULTIWAVELETS

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Abstract. Finite elements with support on two intervals span the space of piecewise polynomials with degree $2n - 1$ and $n - 1$ continuous derivatives. Function values and $n - 1$ derivatives at each meshpoint determine these “Hermite finite elements”. The $n$ basis functions satisfy a dilation equation with $n$ by $n$ matrix coefficients. Orthogonal to this scaling subspace is a wavelet subspace. It is spanned by the translates of $n$ wavelets $W_i(t)$, each supported on three intervals. The wavelets are orthogonal to all rescalings $W_i(2^j t - k)$, but not to translates at the same level ($j = 0$). These new multiwavelets achieve $2n$ vanishing moments and high regularity with symmetry and short support.

1. Splines and Finite Elements

Spline wavelets are semiorthogonal but they are not short. The B-spline plays the role of the scaling function. The box function on a unit interval is the first example ($p = 1$), leading to the Haar wavelet. The hat function over two intervals is the second B-spline ($p = 2$). The corresponding wavelet (piecewise linear) is in Figure 1. The convolution of $p$ box functions gives the B-spline of degree $p - 1$ with $p - 2$ continuous derivatives. It is nonzero, in fact positive, over $p$ intervals. For $p = 4$ this is the cubic spline, and for $p > 1$ the B-spline is not orthogonal to its translates. Nevertheless it satisfies a dilation equation and allows a multiresolution analysis.
The space $V_1$ of splines with knots at the half-integers clearly contains the space $V_0$ of splines with knots at the integers. The orthogonal complement of $V_0$ is the wavelet subspace $W_0$:

$$V_1 = V_0 \oplus W_0$$

splines at half integers = splines $\oplus$ spline wavelets.

Where $V_0$ is spanned by translates of the B-spline, $W_0$ is spanned by translates of the spline wavelet. For each $p$ this wavelet is nonzero over $2p - 1$ intervals — certainly not short. Chui [1] gives a formula for the wavelets, and he observes that they are semiorthogonal: the spaces $W_0, W_1, W_2, \ldots$ are orthogonal. This is immediate from the recursive multiresolution

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1.$$

Wavelets at different scaling levels are orthogonal, but the translates at the same scaling level are not. This semiorthogonality is more than biorthogonality but less than orthogonality.

$V_0$ contains all polynomials of degree less than $p$. The wavelets are orthogonal to these polynomials, so they have $p$ vanishing moments. The lowpass filter coefficients, which enter the dilation equation, are the binomial coefficients in $(\frac{1}{2} + \frac{1}{2})^p$. The frequency response has a corresponding factor $(e^{-i\omega} + 1)^p/2^p$, with a $p$th order zero at $\omega = \pi$. The iterated lowpass filter, with rescaling, converges quickly to the B-spline of degree $p - 1$.

There is another space of piecewise polynomials, the subject of this paper, with fewer continuous derivatives but shorter basis functions. We call this $V_0$ the finite element space, to distinguish it from the spline space. The finite elements have odd degree $2n - 1$, with $n - 1$ continuous derivatives. Where spline interpolation fits a set of function values by solving a coupled linear system, finite element interpolation is completely uncoupled. The function value and $n-1$ derivative values are given at each node. Between
one node and the next, these \(2n\) values produce a unique polynomial of
degree \(2n - 1\). The finite elements are \textit{locally} determined, while spline
interpolation is global. This local property is particularly valuable at boundary
points and jumps — we want wavelets that are as far as possible also local
(= short).

The piecewise linear case \(n = 1\) is included among splines (the hat function).
What is important is that for all \(n\) the basis functions are nonzero only
over \textit{two intervals}. Also important: There are \(n\) basis functions. \textit{In wavelet
language this means \(n\) scaling functions \(\phi_1(t)\), \ldots, \(\phi_n(t)\). Their translates
span \(V_0\). The functions are alternately symmetric and antisymmetric about
\(t = 1\), if we place them on the interval \([0, 2]\). They are easy to construct.

Rescaled by \(t \rightarrow 2t\), the space \(V_1\) of fine-mesh elements clearly contains
the space \(V_0\) of coarse-mesh elements. Therefore each basis function \(\phi(t)\) in
\(V_0\) is a combination of translates of the \(n\) fine-mesh basis functions \(\phi(2t)\).
This is the \textit{dilation equation}. It is a vector equation

\[
\begin{bmatrix}
\phi_1(t) \\
\vdots \\
\phi_n(t)
\end{bmatrix} = \sum_{k=0}^{2} C_k \begin{bmatrix}
\phi_1(2t - k) \\
\vdots \\
\phi_n(2t - k)
\end{bmatrix}
\]

(1)

with matrix coefficients \(C_0, C_1, C_2\). Our first step is to find these coefficients.
They enter the \textit{lowpass multifilter}, which has only three taps; the filter
coefficients are \(n\) by \(n\) matrices.

The literature on multifilters and multiwavelets is small but growing [2–
7]. An earlier paper of Auscher and thesis of Herve [10] have been mentioned
to us. The corresponding filter bank has \(2n\) channels, \(n\) low and \(n\) high.
With matrix coefficients it becomes possible to combine orthogonality with
symmetry (linear phase) and short support and simplicity. We believe that
short support will be especially valuable near boundaries.

The wavelets are basis functions for the orthogonal complement \(W_0\),
in the multiresolution \(V_1 = V_0 \oplus W_0\). The support of these finite element
multiwavelets \(W_1(t), \ldots, W_n(t)\) turns out to be \textit{three intervals}. They are
alternately symmetric and antisymmetric about \(t = 3/2\), the center of \([0,3]\).
They lie in \(V_1\) (half-integer nodes) and satisfy the wavelet equation

\[
\begin{bmatrix}
W_1(t) \\
\vdots \\
W_n(t)
\end{bmatrix} = \sum_{k=0}^{4} D_k \begin{bmatrix}
\phi_1(2t - k) \\
\vdots \\
\phi_n(2t - k)
\end{bmatrix}.
\]

(2)

We compute the \(D\)'s from the \(C\)'s, which have a particularly nice formula.
Figure 2 shows the cubic scaling functions \(\phi_1\) and \(\phi_2\) — these are “Hermite
cubics” in the finite element literature [8–9]. Figure 3 displays the cubic
multiwavelets \(W_1\) and \(W_2\). We can provide a Mathematica code to construct
and draw the finite element scaling functions and multiwavelets for each $n$. The wavelets have $p = 2n$ vanishing moments, because they are orthogonal to the space $V_0$ containing all polynomials of degree less than $2n$.

These multiwavelets were first constructed by Goodman and Lee [6]. Their method (not using the dilation equation) leads to spline wavelets of smoothness $C^{r-1}$ for varying $r$, and they find an efficient recursive formula in the special finite element case. It is this case, with support on $[0,3]$, that we develop by an explicit formula for the $C$'s and an implicit algorithm for the $D$'s.

New piecewise polynomial multiwavelets have recently been constructed by Donovan, Geronimo, and Hardin, with orthogonality to translates. They have support $[0,2]$ with a larger multiplicity $n$. The translates of $n = 3$ piecewise linear wavelets are an orthogonal basis for their space $W_0$. It is remarkable that orthogonal piecewise polynomials are still waiting to be discovered.

![Figure 2](image1.png)

*Figure 2. Cubic finite element scaling functions $\phi_1$ and $\phi_2$."

![Figure 3](image2.png)

*Figure 3. Cubic finite element wavelets $w_1$ and $w_2$."

2. Dilation Equation for the Finite Elements

We emphasize above all the distinction between splines and finite elements. The splines are smoother, with larger support. To determine an interpolating spline, we give one condition at each node, namely the value to be interpolated. There is one B-spline centered at that node (but neighboring B-splines are nonzero there, so we solve a linear system for the interpolating spline). The cubic B-spline is supported on [0, 4] and is nonzero at the nodes t = 1, 2, 3.

Finite elements are less smooth, with minimal support [0, 2]. They are piecewise polynomials (two pieces) of degree 2n − 1. We can interpolate n values at each node, namely the function and its first n − 1 derivatives. There are n different elements centered at this node, to match these n conditions. (Neighboring finite elements are zero at this node, so there is no linear system to be solved.)

Figure 2 shows the two “Hermite cubic” elements, which have \( \phi_1(1) = 1 \) and \( \phi_2(1) = 1 \). To interpolate a value \( y \) and a velocity \( v \) at \( t = 1 \), we take \( y \phi_1 + v \phi_2 \). To see in another way how the nodal values determine the interpolating cubic: On a typical interval [0, 1], the cubic has four free coefficients and satisfies four conditions. The value and slope are given at 0 and 1. It is this totally local interpolation that distinguishes finite elements. They are popular as trial and test functions in Galerkin’s method for differential equations (the finite element method).

Like splines, finite elements satisfy a two-scale dilation equation:

\[
\Phi(t) = \sum C_k \Phi(2t - k).
\]

Since the support is [0, 2], the only coefficients are \( C_0, C_1, C_2 \). Since there are \( n \) basis functions at each node, those coefficients \( C_k \) are \( n \) by \( n \) matrices. \( \Phi(t) \) represents the column vector of basis functions \( [\phi_1(t) \ldots \phi_n(t)]^T \). These are polynomials of degree \( 2n - 1 \) on the pieces [0, 1] and [1, 2] determined by

\[
\frac{d}{dt} \phi_j(1) = \delta_{ij} \tag{4}
\]

\[
\frac{d}{dt} \phi_j(0) = (\frac{d}{dt})^{i-1} \phi_j(2) = 0 \tag{5}
\]

for \( i, j = 1, \ldots, n \). There is symmetry for even \( j \) and antisymmetry for odd \( j \):

\[
\phi_j(2 - t) = (-1)^j \phi_j(t). \tag{6}
\]

Restricted to [0, 1], any polynomial \( p(t) \) of degree less than \( 2n \) is a combination of \( \phi_1(t), \ldots, \phi_n(t), \phi_1(t + 1), \ldots, \phi_n(t + 1) \). These functions form a basis; the dimension is \( 2n \). Each function interpolates one of the \( 2n \) nodal
values, given by \( p(t) \) and its \( n - 1 \) derivatives at the endpoints \( t = 0 \) and \( t = 1 \).

We come back to the dilation equation, to determine \( C_0, C_1, C_2 \):

\[
\Phi(t) = C_0 \Phi(2t) + C_1 \Phi(2t - 1) + C_2 \Phi(2t - 2).
\]

(7)

At \( t = 1 \), this equation immediately gives \( C_1 \) as a diagonal matrix with entries \( 1, \frac{1}{2}, \ldots, \left( \frac{1}{2} \right)^{n-1} \). Since \( C_0 \Phi(2) \) and \( C_2 \Phi(0) \) give no contribution, by (5), the derivatives of (7) at \( t = 1 \) are

\[
\left( \frac{d}{dt} \right)^{i-1} \Phi(1) = C_1 2^{i-1} \left( \frac{d}{dt} \right)^{i-1} \Phi(1).
\]

(8)

The left side is the \( i \)th column of the identity matrix, by (4). Dividing by \( 2^{i-1} \) therefore gives the \( i \)th column of \( C_1 \). We now know the diagonal matrix \( C_1 \).

The reader will not be surprised that there is a close relation between \( C_0 \) and \( C_2 \), coming from the symmetry-antisymmetry of the functions \( \phi_j(t) \). Property (7) says that \( \Phi(2 - t) = S \Phi(t) \), where \( S \) is the “alternating sign matrix”

\[
S = \begin{bmatrix}
1 & -1 & \cdots & (-1)^{i-1}
\end{bmatrix} = S^{-1}.
\]

Compare the dilation equation (7) for \( \Phi(t) \) with the same equation for \( \Phi(2 - t) = S \Phi(t) \) in each term. The conclusion (using linear independence of the polynomials) is that

\[
C_0 = SC_2 S^{-1} \quad \text{or} \quad [C_0]_{ij} = (-1)^{i+j}[C_2]_{ij}.
\]

(9)

It remains to determine \( C_2 \).

There are two approaches to this main step. The direct approach is to compute the \( n \) functions \( \phi_j(t) \) explicitly, and substitute into the dilation equation (7). Knowing the diagonal \( C_1 \) and knowing \( C_0 \) from \( C_2 \) (above), we could determine \( C_2 \). We preferred an indirect approach that gives more information about \( C_2 \) — its eigenvalues and eigenvectors.

**Theorem** The eigenvalues of \( C_2 \) are \( \left( \frac{1}{2} \right)^n, \left( \frac{1}{2} \right)^{n+1}, \ldots, \left( \frac{1}{2} \right)^{2n-1} \). They enter a diagonal matrix \( \Lambda \). The left eigenvectors of \( C_2 \) are the rows of \( U \):

\[
[U]_{ij} = (-1)^{n+i-j} \frac{(n + i)!}{(n + i - j)!}.
\]

(10)

Then \( C_2 = U^{-1} \Lambda U \).
Example  The piecewise cubic case \( n = 2 \) leads to
\[
\Lambda = \begin{bmatrix}
\frac{1}{4} & 0 \\
0 & \frac{1}{4}
\end{bmatrix}
\quad \text{and} \quad
U = \begin{bmatrix}
\frac{3}{4} & -\frac{3}{4} \\
\frac{1}{4} & \frac{1}{4}
\end{bmatrix}.
\]

Then the matrices \( C_2 = U^{-1} \Lambda U \) and \( C_1 \) and \( C_0 \) are
\[
C_2 = \begin{bmatrix}
\frac{1}{8} & -\frac{3}{8} \\
\frac{1}{8} & -\frac{3}{8}
\end{bmatrix}, \quad
C_1 = \begin{bmatrix}
1 & 0 \\
0 & \frac{1}{2}
\end{bmatrix}, \quad
C_0 = \begin{bmatrix}
-\frac{1}{8} & -\frac{3}{8} \\
-\frac{1}{8} & -\frac{3}{8}
\end{bmatrix}.
\]

These are the filter coefficients in the “lowpass filter with downsampling”, which lies behind the scaling functions and wavelets. They are matrix coefficients because we have a multifilter. In the time domain, a filter is represented by a constant-diagonal matrix (a Toeplitz matrix) containing the filter coefficients. Downsampling removes every other row of this matrix to leave the key operator in this theory, the downsampled lowpass filter matrix \( L \):
\[
L = \begin{bmatrix}
\cdots & C_2 & C_1 & C_0 \\
C_2 & C_1 & C_0 \\
C_2 & C_1 & C_0 \\
\cdots
\end{bmatrix}.
\]

We proved elsewhere \([5]\) that the eigenvalues of \( L \) determine the degree \( p - 1 \) of the polynomials that lie in the scaling space \( V_0 \), and therefore the number \( p \) of vanishing moments for the wavelets in the orthogonal complement \( W_0 \). For the finite element multiwavelets we know \textit{a priori} that \( p = 2n \). Then our condition in \([5]\) says that \( L \) must have eigenvalues \( 1, \frac{1}{2}, (\frac{1}{2})^2, \ldots, (\frac{1}{2})^{2n-1} \).
The first \( n \) of these eigenvalues are accounted for by \( C_1 \). The last \( n \) must be produced by \( C_0 \) and \( C_2 \). This is our indirect approach: We determine the eigenvalues in \( \Lambda = \text{diag}[\left(\frac{1}{2}\right)^n, \ldots, (\frac{1}{2})^{2n-1}] \) by knowing that they are needed as eigenvalues of \( L \).

It is the eigenvectors of \( L \) that will lead us to \( U \). Then from \( \Lambda \) and \( U \) we have found \( C_2 \). The full argument will be given in \([9]\); here we continue with the cubic example \( n = 2 \).

The key to \([5]\) was the role of the left eigenvectors \( y^{(0)}, y^{(1)}, \ldots \) of the infinite matrix \( L \). The components of \( y^{(m)} \) give the combinations of the basis functions and their translates that produce the polynomial \( t^m \):
\[
t^m = \sum_{m=-\infty}^{\infty} y^{(m)}_k \Phi(t - k + 1), \quad m < p.
\]  

(For proof, substitute for \( \Phi \) using the dilation equation. The result simplifies eventually to \( y^{(m)} L = y^{(m)} / 2^m \).)
The components $y_k^{(m)}$ are themselves vectors with $n$ components, conforming to the length of $\Phi$ and the block form of $L$. The simplest example is $m = 0$, where the constant polynomial 1 is produced by adding the finite elements $\phi_1$ with weight 1 at each node. The sum equals 1 with slope zero at each node, so it must be identically 1. The corresponding eigenvector is $y^{(0)} = \ldots (1 \ 0)(1 \ 0) \ldots$:

$$
\begin{bmatrix}
\ldots \\
C_2 & C_1 & C_0 \\
C_2 & C_1 & C_0 \\
\ldots 
\end{bmatrix}
= \begin{bmatrix}
\ldots \\
(1 \ 0)(1 \ 0) \\
\ldots 
\end{bmatrix}.
$$

This requires $[1 \ 0]C_1 = [1 \ 0]$ and also $[1 \ 0][C_0 + C_2] = [1 \ 0]$. The first is clear for $C_1 = \text{diag}(1, \ 1)$. The second produces the upper left entry $\frac{1}{2}$ in $C_0$ and $C_2$ for our cubic example.

The other entries of $C_0$ and $C_2$ come from the way in which the polynomials $t, t^2, t^3$ are produced from the cubics. At a typical node $t = k$, we know the function values $k, k^2, k^3$ and the slopes $1, 2k, 3k^2$. These multiply $\phi_1(t - k + 1)$ and $\phi_2(t - k + 1)$, respectively. Therefore the eigenvectors containing these values and slopes must be

$$
y^{(1)} = \ldots (0 \ 1)(-1 \ 1)(-2 \ 1) \ldots
$$

$$
y^{(2)} = \ldots (0 \ 0)(1 \ -2)(4 \ -4) \ldots
$$

$$
y^{(3)} = \ldots (0 \ 0)(-1 \ 3)(-8 \ 12) \ldots
$$

The equations $y^{(m)}L = y^{(m)}/2^m$ complete our knowledge of $C_0$ and $C_2$.

We can state the whole argument briefly, omitting details: The eigenvalue equation $y^{(m)}L = y^{(m)}/2^m$ leads us to

$$
y_1^{(m)}C_2 = y_1^{(m)}/2^m, \quad n \leq m < 2n.
$$

This gives the eigenvalue matrix for $C_2$. The fact that $L$ is a downsampling Toeplitz matrix leads to a special form (seen above) for its left eigenvectors $y^{(m)}$. This form gives the matrix $U$ in our theorem.

3. The Coefficients $D_k$ and the Wavelets

The wavelets $w_1(t), \ldots, w_n(t)$ are orthogonal to $\phi_1(t), \ldots, \phi_n(t)$. At the same time the wavelets are combinations of $\phi_1(2t - k), \ldots, \phi_n(2t - k)$. This is the wavelet equation with matrix coefficients $D_k$:

$$
\begin{bmatrix}
w_1(t) \\
\vdots \\
w_n(t)
\end{bmatrix} = \sum_{k=0}^4 D_k \begin{bmatrix}
\phi_1(2t - k) \\
\vdots \\
\phi_n(2t - k)
\end{bmatrix}.
$$

(13)
The first question is, why five \( D \)'s?

The support of \( \Phi(2t-k) \) is \([\frac{k}{2}, \frac{k}{2} + 1]\). With coefficients \( D_0, \ldots, D_4 \), the support of the wavelets in (13) will be \([0, 3]\). Orthogonality against \( \Phi(t) \) and its translates will give four equations for the \( D \)'s:

\[
\begin{align*}
W(t) \cdot \Phi^T(t+1) &= 0 \\
W(t) \cdot \Phi^T(t) &= 0 \\
W(t) \cdot \Phi^T(t-1) &= 0 \\
W(t) \cdot \Phi^T(t-2) &= 0. 
\end{align*}
\]  

(14)

There is a nontrivial solution which we want. If we try for a shorter support with fewer \( D \)'s the homogeneous system becomes square and leads to \( D_k = 0 \).

The equations (14) involve integrals of \( w_j(t)\phi_j(t-k) \). Use the wavelet equation (13) to substitute for the \( w \)'s. Use the dilation equation (7) to substitute for the \( \phi \)'s. Then we are integrating \( \phi_0(2t-k)\phi_j(2t-m) \). A change of variables brings these inner product integrals back to

\[
\int \phi_0(2t-k)\phi_j(2t-m)dt = \frac{1}{2} \int \phi_0(t)\phi_j(t - m + k)dt. 
\]  

(15)

The \( \phi_i \) are supported on \([0, 2]\), so the only inner products we need are in the two matrices

\[
X = \Phi(t) \cdot \Phi^T(t) \quad \text{and} \quad Y = \Phi(t) \cdot \Phi^T(t-1) = \Phi(t+1) \cdot \Phi^T(t). 
\]  

(16)

As a point of honor, we want to avoid an explicit computation of the polynomials \( \phi_i(t) \) and their inner products. Therefore we substitute the dilation equation into (16), and use (15) to bring all arguments involving \( 2t \) back to \( t \). The result is two matrix equations for the matrices \( X \) and \( Y \):

\[
\begin{align*}
2X &= C_0XC_0^T + C_1Y^T C_1^T + C_1YC_1^T + C_1XC_1^T + C_1 Y^T C_1^T + C_1 YC_1^T + C_1 XC_1^T \\
2Y &= C_1Y C_0^T + C_2XC_0^T + C_2 YC_0^T.
\end{align*}
\]  

(17)

This determines \( X \) and \( Y \) up to a scalar factor. Then \( X \) and \( Y \) enter the orthogonality equation (14). Remember that substituting in (14) for \( W(t) \) introduced the unknown \( D \)'s, and substituting for \( \Phi(t) \) introduced the known \( C \)'s. After these substitutions, the four orthogonality equations (14) become

\[
\begin{align*}
D_0 (Y^T C_0^T + XC_0^T) + D_1 Y^T C_0^T &= 0 \\
D_0 (Y C_0^T + YC_0^T) + D_1 (Y^T C_0^T + XC_0^T) + D_2 (Y^T C_0^T + YC_0^T) + D_3 (Y^T C_0^T + XC_0^T) + D_0 Y^T C_0^T &= 0 \\
D_2 YC_0^T + D_2 (Y C_0^T + YC_0^T) + D_3 (Y^T C_0^T + XC_0^T) + D_4 (Y^T C_0^T + YC_0^T) + D_0 (Y^T C_0^T + XC_0^T) &= 0 \\
D_3 YC_0^T + D_4 (Y C_0^T + YC_0^T) &= 0.
\end{align*}
\]  

(18)
It is this system of $4n^2$ homogeneous equations that we solve for $5n^2$ entries in $D_0, \ldots, D_4$. We pick out the solution that has $D_2 = I$. Then the symmetry-antisymmetry of the $\phi$'s is true also for the wavelets.

The property $C_0 = SC_2S$ of the $C$'s extends to the $D$'s with the sign matrix $S = \text{diag}(1, -1, 1, \ldots, (-1)^n)$:

$$D_0 = SD_4S \quad \text{and} \quad D_1 = SD_3S. \quad (19)$$

The first two equations (18) become identical to the last two, using this pattern for the $D$'s and verifying the corresponding pattern $X = SXS$ and $Y = SY^T S$:

$$[X]_{ij} = \int \phi_i(t) \phi_j(t) dt = \int \phi_i(2 - t) \phi_j(2 - t) dt =$$

$$(-1)^{i+j} \int \phi_i(t) \phi_j(t) dt = (-1)^{i+j}[X]_{ij}$$

$$[Y]_{ij} = \int \phi_i(t) \phi_j(t - 1) dt = \int \phi_i(3 - t) \phi_j(2 - t) dt =$$

$$(-1)^{i+j} \int \phi_i(t - 1) \phi_j(t) dt = (-1)^{i+j}[Y]_{ij}$$

Thus (14) reduces to two matrix equations for two unknowns $D_3$ and $D_4$. Our Mathematica subroutine solves these equations, applies the wavelet equations (13), and draws the wavelets. Figure 4 shows the quintic finite elements $\phi_1, \phi_2, \phi_3$ and Figure 5 shows the corresponding wavelets $w_1, w_2, w_3$. 

![Graphs of wavelets](image-url)
Figure 4. Quintic finite element scaling functions $\phi_1, \phi_2, \phi_3$.

Figure 5. Quintic finite element wavelets $w_1, w_2, w_3$.

By construction, the finite elements are orthogonal to the wavelets and their translates. The two spaces are orthogonal complements in $V_1 = V_0 \oplus W_0$. This is invariant under dilation so $V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$. Therefore the wavelets in $W_0$ are orthogonal to their dilates and translates in $W_1$. This is true at all scaling levels, $W_i \perp W_j$, except at the same level $i = j$. The wavelets are therefore called “semiorthogonal”, or
“pre-wavelets”. They have $2n$ vanishing moments and $n - 1$ continuous derivatives, with symmetry and support $[0,3]$.

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References