18.997 Exercises

List of exercises on polytopes updated on April 30, 2003..

1. Prove Radon's theorem wich says that for any set V of $d+2$ points in R^d, there exists a partition of V into V_1 and V_2 such that $conv(V_1) \cap conv(V_2) \neq \emptyset$.

2. Show that if a polytope is k-neighborly then every $(2k-1)$-face is a simplex. Show that a $(\lfloor d/2 \rfloor + 1)$-neighborly polytope is a simplex. Deduce also that all neighborly polytopes for d even are simplicial. (This is not the case for d odd; just consider $d = 3$.)

3. Consider the cyclic polytope $C_d(n)$. Show the following extension to Gale's evenness condition. Let S be a subset of $[n]$ of cardinality k. Then $\{x(t_i) : i \in S\}$ define a $k-1$-face of $C_d(n)$ if and only if the number of $i, j \notin S$ with (i) $k \in S$ for every k with $i < k < j$ and (ii) $j - i$ even is at most $d - k$. Deduce that $C_d(n)$ is neighborly (i.e. $\lfloor d/2 \rfloor$-neighborly).

4. Give one formula (there are many ways of writing it) giving the number of $k-1$-faces of $C_d(n)$.

5. This exercise shows that there are curves other than the moment curve that define the cyclic polytopes. Let $p(t) = (\cos(t), \sin(t), \cos(2t), \sin(2t), \cdots, \cos(dt), \sin(dt)) \in R^{2d}$ and let $P = \text{conv}\{p(t_i) : 1 \leq i \leq n\}$ where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < 2\pi$ and $n \geq 2d+1$. Show that P is combinatorially equivalent to $C_{2d}(n)$.

6. For general polytopes P, show that the orderings of the facets of P given by line shellings are in 1-to-1 correspondence with the orderings of the vertices of $P^\triangle$ induced by linear function $\langle c, v \rangle$ for $v \in vert(P^\triangle)$. Furthermore, for simplical polytopes P, show that an ordering of the facets of P gives a shelling iff the corresponding ordering of the vertices of the simple polytope $P^\triangle$ induces a good orientation.

7. For simplicial d-polytopes, show that the Dehn-Somerville equations and the relationship between the f- and h-vectors imply that

$$f_{k-1} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1},$$

   for $k = 0, \cdots, d$.

8. Given a 3-connected, planar graph G and the existence of a representation of G and its dual dual G^* as a primal-dual circle packing on the sphere, derive Steinitz' theorem (as was stated in lecture). That is, for every circle C_i^* in the packing corresponding to a vertex of the dual graph, we let H_i be the plane whose intersection with the sphere is C_i^* and we let the polytope P be given by the intersection of halfspaces supported by the H_i's (side chosen appropriately). Show that the graph of P is indeed G.

9. Consider a $(0, \frac{1}{2}, 1)$-polytope P in R^d, i.e. this is a d-polytope whose vertices have components taking values only in $\{0, \frac{1}{2}, 1\}$. Show that the diameter of such a polytope is at most $2d-1$. Show that this is tight for d even (take d even and then try to blow this example up for higher d). What upper bound on the diameter can you prove for $\{0, 1, 2, \cdots, k\}$-polytopes?

10. Complete the proof of Rivin's theorem on the inscribibality of 3-polytopes. The missing step was to show that the projection on x of the polytope $P_{\theta,x}$ involving θ and x gives the polytope P_x in x defined by all cutset constraints, and that (i) if $(\theta, x) \in P_{\theta,x}$ with $\theta > 0$ then the cutset constraints are satisfied strictly and (ii) if $x \in Proj_x(P_{\theta,x})$ and $x_e > 0$ for $e \in E$ and the cutset constraints are strict then there exists $\theta > 0$ such that $(\theta, x) \in P_{\theta,x}$.

11. Consider the unit ball B in R^n and let v be a vector of norm $> n$, say $v = ke_1$ with $k > n$. Show that the $conv(B \cup \{v\})$ contains an ellispoid of greater volume than B.