18.997 Topics in Combinatorial Optimization	March 4, 2004
Lecture 8	
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This lecture covers the proof of the Bessy-Thomassé Theorem, formerly known as the Gallai Conjecture. Also, we discuss the cyclic stable set polytope, and show that it is totally dual integral (TDI) (see lecture 5 for more on TDI systems of inequalities).

1 Recap and Definitions

In this section we provide a brief recap of some definitions we saw in the previous lecture. Also we answer a question that remained unanswered in the previous lecture regarding the polynomiality of finding a valid ordering given any strongly connected directed graph.

For a strongly connected digraph $D = (V, A)$, with $|V| = n$, we make the following definitions.

1. Given an *enumeration* of the vertices, $\{v_1, \ldots, v_n\}$, an arc $(v_i, v_j) \in A$ is called *backward* if $i > j$ and *forward* if $i < j$.
2. An *ordering* $\mathcal{O}$, is an equivalence class of enumerations of a graph. The equivalence class is defined by the equivalence relations
   (a) $v_1, v_2, \ldots, v_n \sim v_2, v_3, \ldots, v_n, v_1$,
   (b) $v_1, v_2, \ldots, v_n \sim v_2, v_1, v_3, \ldots, v_n$, if there is no arc between v_1 and v_2, i.e., $(v_1, v_2), (v_2, v_1) \notin A$.
3. Given an ordering $\mathcal{O}$, the *index* with respect to $\mathcal{O}$ of a directed cycle C, denoted $i_{\mathcal{O}}(C)$, is the number of backward arcs in C. Recall from the last lecture that the index is well defined, since the index is invariant under the equivalence operations defined above.
4. We say that an ordering $\mathcal{O}$ is *valid* if for any arc $(u, v) \in A$, there exists a cycle C containing that arc, with index 1: $i_{\mathcal{O}}(C) = 1$. We showed in the last lecture that there always exists a valid ordering.
5. A *cyclic stable set* S with respect to a valid ordering $\mathcal{O}$, is such that S is a stable set on the underlying undirected graph, and also there exists some enumeration $\{v_1, \ldots, v_n\}$ of the ordering such that $S = \{v_1, \ldots, v_k\}$, where $k = |S|$.

Last time we proved that any strongly connected digraph has a valid ordering. In fact, given any such graph, a valid ordering can be found in time polynomial in the size of the graph. Recall that the proof of the existence theorem showed that the minimizer of

$$\min_{\mathcal{O}} \sum_{\text{directed cycles } C} i_{\mathcal{O}}(C),$$

must be a valid ordering. Given any ordering $\mathcal{O}$, we showed in the proof that in a polynomial number of steps (essentially, by repeated "local swaps"), if $\mathcal{O}$ is not valid, we can obtain a new ordering $\mathcal{O}_1$, reducing the number of arcs for which there are no cycles of index 1 containing them. Therefore we can find a valid ordering in polynomial time.

2 The Bessy-Thomassé Theorem

Recall the statement of the theorem.

Theorem 1 *Given a strongly connected digraph $D = (V, A)$, and a valid ordering $\mathcal{O}$, if $\alpha_{\mathcal{O}}$ denotes the size of the largest cardinality cyclic stable set, then*

$$\alpha_{\mathcal{O}} = \min \sum_{\{C_1,\ldots,C_p\}} i_{\mathcal{O}}(C_i),$$

where the cycles $\{C_1, \ldots, C_p\}$ cover the vertex set V.

The inequality

$$\alpha_{\mathcal{O}} \leq \min \sum_{\{C_1,\ldots,C_p\}} i_{\mathcal{O}}(C_i),$$

is straightforward (as each vertex of a cycle stable set must be contained in (at least) one directed cycle and the corresponding entering arc must be backward), so we consider only the proof of the reverse inequality.

Before we prove this theorem, we make some remarks. It is important to note that the cyclic stability number, $\alpha_{\mathcal{O}}$, depends on the ordering $\mathcal{O}$ chosen. To illustrate this, recall our digraph on five vertices from last lecture. In Figure 2, we exhibit two different orderings where the cyclic stability number is different.

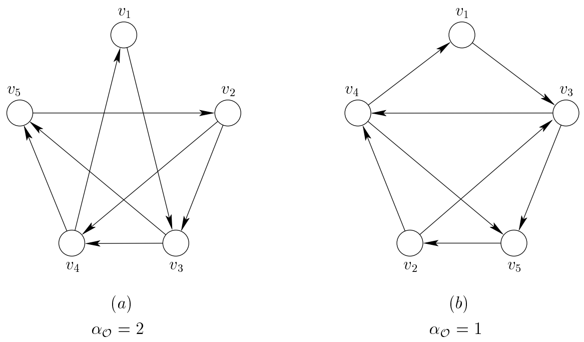

Figure 1: In figure (a) above, the cyclic stability number equals 2, where as in (b), the cyclic stability number equals 1.

Computing the stability number of a general graph is known to be NP-hard. One of the corollaries of the Bessy-Thomassé Theorem is that the cyclic stability number can be computed efficiently. This follows because we can can compute the quantity $\min \sum_{\{C_1,\ldots,C_p\}} i_{\mathcal{O}}(C_i)$ in the right hand side of the theorem above, efficiently. We can do this by formulating a network flow problem that computes the minimization. To do this, fix an enumeration of the ordering. Attach a cost of 0 to every forward arc in the digraph under the given enumeration, and a cost of 1 to every backward arc. Next, split each vertex v into a pair $\{v_{\text{out}}, v_{\text{in}}\}$ with a directed edge $(v_{\text{out}}, v_{\text{in}})$ with flow capacity bounded from below by 1. Then for every arc (u, v) in the original graph, draw an arc $(u_{\text{out}}, v_{\text{in}})$ in the network flow graph. Finding a minimum cost flow in this network can be done efficiently, and it amounts to finding a set of cycles $\{C_1, \ldots, C_p\}$ that cover V, and minimize $\sum_{\{C_1,\ldots,C_p\}} i_{\mathcal{O}}(C_i)$.

A key step in the proof of the Bessy-Thomassé Theorem is a lemma that provides a sufficient condition for a subset S of vertices to be a cyclic stable set.

Lemma 2 *Given a valid ordering $\mathcal{O}$, fix an enumeration, $\{v_1, \ldots, v_n\}$. Let $S \subseteq V$ be a subset of the vertices. If there are no forward paths between any two vertices of S, then S is a cyclic stable set.*

Proof: Suppose, to the contrary, that S has no forward arcs, but S is not a cyclic stable. Let v_i be the first element of the enumeration in S. If we rotate the enumeration so that v_i becomes v_1, no forward paths are either created or destroyed in S, so we may assume, without loss of generality, that $v_1 \in S$. If S is a cyclic stable set with respect to $\mathcal{O}$, then there exists some enumeration of $\mathcal{O}$ for which the elements of S are the first $k = |S|$ elements of the enumeration. Equivalently, there exists a sequence of local steps, or swaps we can make according to the equivalence relations defining an ordering, to move from the current enumeration to one of the correct form. If S is not a cyclic stable set, as we assume, then this is not possible. Consider the enumeration which brings S "as close as possible" to having all its elements at the beginning of the enumeration, as illustrated in Figure 2. By this we mean that as many elements of S as possible are listed first in the enumeration, and furthermore, the first element of S not part of the initial string of elements of S (which we call $S_<$) is as close to $S_<$ as possible. We denote by $S_<$ the elements of S that are at the beginning of

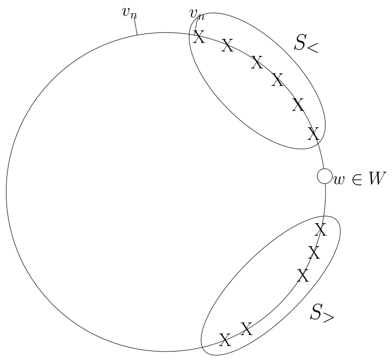

Figure 2: The figure exhibits the enumeration with respect to which as many elements of S as possible are the first elements of the enumeration. Since S is assumed not to be a cyclic stable set, there must be some element w sandwiched by elements of S.

the enumeration, by $S_>$ the remaining elements of S, and by W the elements after the last element of $S_<$ and before the first element of $S_>$, as illustrated in Figure 2. Since there are no forward paths joining any two elements of S, for any $w \in W$ there cannot be a forward path from $S_<$ to w, *and* a forward path from w to an element of $S_>$. Consider the first $w \in W$ where there is no forward path from $S_<$ to w (if there is such a vertex). Because w is assumed to be the first such vertex, there can be no forward path from any vertex v coming before w in the enumeration. If there were such a vertex v, then if there were a forward path from $S_<$ to v, we would also have a forward path from $S_<$ to w. If there were no forward path from $S_<$ to v, it would contradict our assumption that w is the *first* vertex in W that has no forward path from $S_<$. In particular, then, there are no arcs from any vertex before w in the enumeration, to w. However, there also can be no arc from w to any vertex before it in the enumeration. This follows because $\mathcal{O}$ was assumed to be a valid ordering. If there were such an arc, say (w, v) for v earlier in the enumeration, because we assume there are no forward arcs from any vertex coming before w, to w, and that w is the first such vertex, then any cycle C containing the arc (w, v) must have $i_{\mathcal{O}}(C) \geq 2$, a contradiction to the validity of the ordering $\mathcal{O}$. Therefore there are no arcs between w and any vertex previous to w in the enumeration. But then using the equivalence relations, we can swap w with each element before it, including then each element of $S_<$. But this contradicts our assumption that the first element of

$S \setminus S_<$ was as close as possible to $S_<$. Therefore there are no elements in W that have no forward paths from S. In particular, this implies that there are no forward paths from any $w \in W$ to $S_>$. Then, let $v_j \in S$ be the first vertex in $S_>$. By assumption, unless W is empty, $v_{j-1} \in W$, and there is no forward path from v_{j-1} to $S_>$, and in particular, $(v_{j-1}, v_j) \notin A$. But then, since $\mathcal{O}$ is a valid ordering, $(v_j, v_{j-1}) \notin A$. In this case, we can swap the two vertices, contradicting our assumption that our enumeration put S "as close as possible" to having its elements at the beginning of the enumeration. Therefore W must be empty, and S is indeed a cyclic stable set. □

We now move to the proof of the Bessy-Thomassé Theorem.

Proof:

The Main Idea: We want to show that the size of the maximum cyclic stable set equals the minimum total index of a family of cycles covering V. Essentially the proof relies on mapping our digraph D to a poset T. At this point, we appeal to Dilworth's Theorem (lecture 6). Recall that the strong version of Dilworth's Theorem tells us that the size of the largest antichain in the poset equals the minimum number of chains needed to partition the elements of the poset. We show that our maximum size cyclic stable set S in D, corresponds naturally to an antichain in the poset T. Thus the size of the largest antichain in T is at least the size of S, i.e., $\alpha_{\mathcal{O}}$. Then we use Lemma 2 to show that any antichain in T corresponds to a cyclic stable set in D. Thus we have that the size of the largest antichain in T is exactly $\alpha_{\mathcal{O}}$.

Dilworth's Theorem now links the number of chains partitioning T to $\alpha_{\mathcal{O}}$. The final part of the proof recovers a covering family of cycles from the chains in T.

The Proof: For the given ordering $\mathcal{O}$, let $S = \{v_1, \ldots, v_k\}$ denote the maximum size cyclic stable set, with corresponding enumeration of $V = \{v_1, \ldots, v_k, \ldots, v_n\}$. We note that since S is, in particular, a stable set, we can permute its elements as we wish within the given ordering.

We form an acyclic digraph $D' = (V', A')$ from D as follows. Let $V' = \{v_1, \ldots, v_n, v_1', \ldots, v_k'\}$ (we duplicate the elements of S) so that $|V'| = n + k$. Next, if $(v, w) \in A$ is a forward arc, then $(v, w) \in A'$. If $(w, v_i) \in A$ is a backward arc into a vertex of S (i.e., if $v_i \in S$) then $(w, v_i') \in A'$. Note that by our choice of enumeration, any arcs *into* v_i, $i \leq k$, must be backward. Therefore the digraph D' is acyclic. It is illustrated in Figure 2.

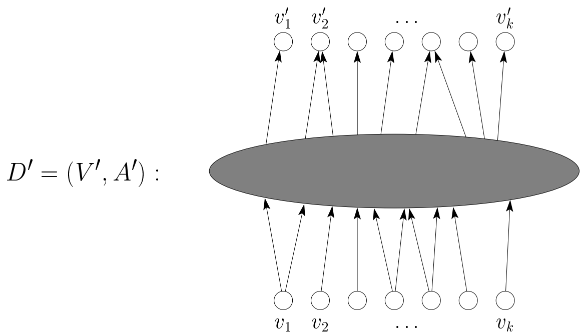

Figure 3: This figure illustrates the directed acyclic graph D' we obtain from splitting the vertices in S and drawing arcs as explained above.

In order to use Dilworth's Theorem, we need to have a poset T. We obtain a poset T from the acyclic digraph D' by considering the transitive closure of D'. Since the sets $\{v_1, \ldots, v_k\}$ and

$\{v'_1, \ldots, v'_k\}$ have no incoming and outgoing arcs, respectively, they are both antichains in T. This is also evident from Figure 2. We show that they are in fact maximum size antichains. Consider any antichain I. As the ordering is valid, for any vertex, there exists a directed cycle of index 1 going through it. This translates into a chain in the poset going from v_i to v'_i for any $1 \leq i \leq k$. This means that an antichain I cannot contain both v_i and v'_i. Let I_D be the elements of the original digraph D corresponding to I.

By renumbering the elements of S (recall that we can permute the elements of S within the given ordering) we can assume that $v'_1, \ldots, v'_l \in I$, and $v'_{l+1}, \ldots, v'_k \notin I$. Now rotate the enumeration to obtain $\{\tilde{v_1}, \ldots, \tilde{v_n}\}$ so that $\tilde{v_1} = v_{l+1}$. Since I is an antichain, and since the digraph vertices $\{v_1, \ldots, v_l\}$ corresponding to the poset elements $\{v'_1, \ldots, v'_l\}$ at the "top" of the poset T have been rotated to be the last elements of the enumeration, there are no forward paths between any two elements of I_D. Therefore, by Lemma 2, I_D is a cyclic stable set. Therefore I_D, and consequently I, can have size at most equal to the size of S, that is, $\alpha_{\mathcal{O}}$. We have thus shown that the size of the largest antichain in T is equal to the cyclic stability number $\alpha_{\mathcal{O}}$ of D.

Now consider the minimal partitioning set of chains in the poset T, call these $P_1, \ldots, P_k$ (where $k = |S| = \alpha_{\mathcal{O}}$). Each chain P_i is a chain from v_i to $v'_{\sigma(i)}$, for some permutation σ of $\{1, \ldots, k\}$. By a slight abuse of notation, we also use P_i to refer to the directed path in D from v_i to $v_{\sigma(i)}$ (or cycle if $\sigma(i) = i$). We note that by construction of T, there is exactly one backward arc in each path P_i, namely, the last arc to $v_{\sigma(i)}$. These paths cover the vertex set V. Now, the cycles in the permutation σ correspond to cycles in D. For example, if (12) is a cycle in σ, i.e., if $\sigma(1) = 2$ and $\sigma(2) = 1$, then joining the paths P_1 and P_2 we have a cycle from v_1 to v_1. We note that these cycles may in fact intersect. Since the cycles merely need to cover the vertex set V, distinct cycles can intersect. We need to take care that the same cycle does not intersect itself. If σ happens to be the identity permutation, $\sigma(i) = i$, then each path is a cycle and cannot intersect itself, and hence the proof is complete. If this is not the case, then a cycle in D obtained by joining together the paths P_i that correspond to a cycle of σ may in fact intersect itself. Suppose that i and j are in the same cycle of σ and the paths P_i and P_j intersect, in say v. We can then replace the paths P_i and P_j by two other paths P'_i and P'_j (obtained by switching from one to the other at v) which together cover the same vertices and which corresponds to a new permutation σ' with $\sigma'(i) = \sigma(j)$ and $\sigma'(j) = \sigma(i)$. Now the number of cycles in the permutation has increased by one, and we can repeat this process until no cycle in D (corresponding to each cycle of teh permutation σ) intersects itself.

Since the cycle splitting procedure does not change the total index of the cycles, we know that the total index equals the minimal number of chains required to partition T. But by above, this is exactly the size of the maximum cyclic stable set, and therefore

$$\alpha_{\mathcal{O}} = \min \sum_{\{C_1, \ldots, C_p\}} i_{\mathcal{O}}(C_i),$$

which is what we wanted to prove. □

3 Cyclic Stable Set Polytope

In this section, we follow some recent (unpublished) work of A. Sebö, and define the cyclic stable set polytope of a strongly connected graph D, with a given valid ordering $\mathcal{O}$. Define the polytope $\mathcal{P}$ as follows.

$$\mathcal{P} \triangleq \left\{ x \left| \begin{array}{ll} x(C) \leq i_{\mathcal{O}}(C), & \forall \text{ directed cycles } C \\ x_v \geq 0, & \forall v \in V \end{array} \right. \right\}.$$

We show in this section that the polytope $\mathcal{P}$ is totally dual integral (TDI) (see lecture 5 for more on TDI system of inequalities).

Given a cyclic stable set S (cyclic stable with respect to the given ordering), let x^S denote its incidence vector, i.e., $x^S_v = 1$ if $v \in S$, and 0 otherwise. Then in fact $x^S \in \mathcal{P}$. Indeed, consider any directed cycle C. Since S is cyclic stable, C always enters S via a backward arc, and therefore the number of backward arcs of C is at least the cardinality of its intersection with S:

$$(\#\text{ backward arcs in } C) = i_{\mathcal{O}}(C) \geq |C \cap S|,$$

or, equivalently, $x^S(C) \leq i_{\mathcal{O}}(C)$.

Since we have shown that the incidence vector of every cyclic stable set belongs to $\mathcal{P}$, we have:

$$\begin{array}{rlll} \alpha_{\mathcal{O}} & \leq & \max: & \sum_{v\in V} x_v \\ & & \text{s.t.}: & x(C) \leq i_{\mathcal{O}}(C), \quad \forall C \\ & & & x_v \geq 0, \quad \forall v \in V \end{array}$$

By linear programming duality, and then by observing that the optimum value of a minimization can only increase if we add constraints, we have

$$\begin{array}{rlll} \alpha_{\mathcal{O}} & \leq & \max: & \sum_{v\in V} x_v \\ & & \text{s.t.}: & x(C) \leq i_{\mathcal{O}}(C), \quad \forall C \\ & & & x_v \geq 0, \quad \forall v \in V \\ & & & \\ & = & \min: & \sum_{C} i_{\mathcal{O}}(C) y_C \\ & & \text{s.t.}: & \sum_{C\,:\,v\in C} y_C \geq 1, \quad \forall v \in V \\ & & & y_C \geq 0, \quad \forall C \\ & & & \\ & \leq & \min: & \sum_{C} i_{\mathcal{O}}(C) y_C \\ & & \text{s.t.}: & \sum_{C\,:\,v\in C} y_C \geq 1, \quad \forall v \in V \\ & & & y_C \geq 0, \quad \forall C \\ & & & y_C \in \{0,1\}. \end{array}$$

But this last quantity is exactly the minimum total index of a cycle cover of V, and thus by the Bessy-Thomassé Theorem, the final quantity equals $\alpha_{\mathcal{O}}$. Therefore equality must hold throughout.

Recall that in order to prove that the description of $\mathcal{P}$ is TDI, we must show that for all integral objective functions w ($w_v \in \mathbb{Z}$), the dual linear program

$$\begin{array}{ll} \min: & \sum_{C} i_{\mathcal{O}}(C) y_C \\ \text{s.t.}: & \sum_{C\,:\,v\in C} y_C \geq 1, \quad \forall v \in V \\ & y_C \geq 0, \quad \forall C \end{array}$$

has an integral solution whenever its value is finite. We note that we have just proved this statement for the special case $w_v = 1$. We note also that if we have $w_v \leq 0$, we can replace this w_v by 0 without affecting the feasible region of the dual linear program. Therefore, we can assume that we have $w_v \in \mathbb{Z}_+$.

We now construct a strongly connected digraph $D' = (V', A')$, with valid ordering $\mathcal{O}'$ as follows. Let V' consist of w_v copies of each x_v, $\{x_{v,1}, \ldots, x_{v,w_v}\}$ (recall that w_v is a positive integer). If $(v,u) \in A$, then $(x_{v,i}, x_{u,j}) \in A'$ for every $i \leq w_v$ and $j \leq w_u$. From our reasoning above, we know that the linear program associated to the digraph D' (now we have $w_v = 1$ for every $v \in V'$) produces an integral solution that corresponds to a maximum size cyclic stable set in D'. Note that if $x_{v,i}$ is in the stable set S' for D', then we can also take $x_{v,j}$ to be in S' for any $j \leq w_v$. Therefore any maximum size cyclic stable set S' in D' naturally corresponds to a cyclic stable set S in D. Moreover, $|S'| = w'x^S$. Conversely, if S is a cyclic stable set in D, then the set S' of all copies of the vertices in S, is a cyclic stable set in D', with $|S'| = w'x^S$. Therefore given any vector w with $w_v \in \mathbb{Z}_+$, the linear program with objective function $w'x$ has an integral optimal solution. Therefore $\mathcal{P}$ is totally dual integral, as we wished to show.