18.997 Topics in Combinatorial Optimization	March 2, 2004
Lecture 7	
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1 Gallai's Conjecture

In this lecture, we will be concerned with graph coverings by a collection of paths or cycles. The goal will be to cover all the vertices by a small number of either paths or cycles, and this number will be bounded by the independence number $\alpha(G)$. ($\alpha(G)$ is the maximum size of an independent, or stable, set, i.e. a set of vertices inducing no edges.) For a directed graph D, $\alpha(D)$ refers to the corresponding undirected graph. Let's start with the following statement, proved by Gallai and Milgram in 1960.

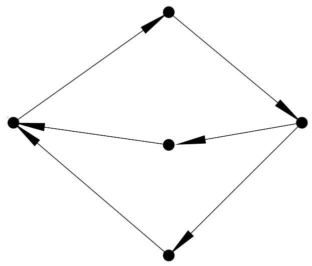
Theorem 1 *(Gallai-Milgram) In every directed graph D, the vertices can be partitioned into $\alpha(D)$ vertex-disjoint directed paths.*

Observe that we may able to partition V with fewer directed paths, as it is the case for example for a directed cycle (one path suffices but $\alpha(D) = \lfloor n/2 \rfloor$).

Consequently, Gallai suggested a related conjecture, which has been recently proved by Bessy and Thomassé.

Theorem 2 *(Bessy-Thomassé) In every strongly connected digraph D, the vertices can be covered by at most $\alpha(D)$ directed cycles.*

In contrast to the Gallai-Milgram theorem, the directed cycles cannot be always chosen to be vertex-disjoint, and this explains the "covering" instead of "partitioning" in teh statement. Below is an example of a directed graph D which has $\alpha(D) = 3$, and it can be covered by 2 cycles, but there are no two disjoint cycles in D.



Before we proceed to the proofs, let's note a few connections to other theorems.

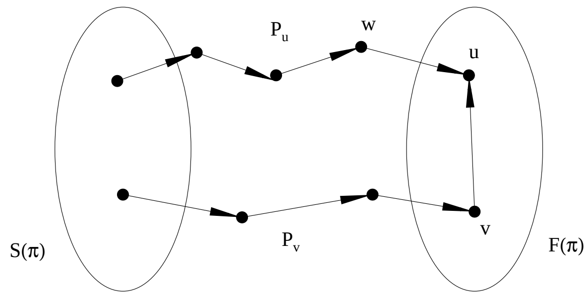
- Gallai-Milgram implies Dilworth's theorem. For any poset, there is a digraph naturally associated with it, where (i, j) is an arc if $i < j$. Paths correspond to chains and stable sets to antichains which implies that the poset can be partitioned into $\alpha(D)$ chains where $\alpha(D)$ is the maximum size of an antichain.

- Bessy-Thomassé implies that every digraph can be covered by at most $\alpha(D)$ directed paths. (Add a new vertex connected to everything in both directions, which makes the graph strongly connected. Then find a cycle covering and turn the cycles into paths, removing the new vertex if necessary.) However, this is weaker than Gallai-Milgram, because the paths are not necessarily disjoint (so this is a cover and not a partition).

- For a tournament, Gallai-Milgram implies the existence of a hamiltonian path. For a strongly connected tournament, Bessy-Thomassé implies the existence of a hamiltonian cycle. These are theorems previously proved by Rédei (1934) and Camion (1960).

Now we turn to the proof of Gallai-Milgram. The theorem is proved by repeated application of the following lemma (starting with a trivial partitioning into n (singleton) paths).

Lemma 3 *Let π be a partitioning of the vertices of D into directed paths $P_1, P_2, \dots P_l$, $l > \alpha(D)$. Let $S(\pi)$ denotes the starting vertices, and $F(\pi)$ the ending vertices of $P_1, \dots P_l$. Then there is a partitioning π' into $l-1$ directed paths such that $S(\pi') \subset S(\pi)$ and $F(\pi') \subset F(\pi)$.*



Proof: We proceed by induction on the size of D. Since $|F(\pi)| > \alpha(D)$, there must be an arc (v, u) with $u, v \in F(\pi)$. Consider the paths P_u, P_v whose endpoints are u and v, respectively.

1. If P_u has length 0, we can just remove it and extend P_v by the arc (v, u), which yields a covering π' of $l-1$ paths.

2. If P_u has length at least 1, let w be the vertex on P_u preceding u. We remove vertex u and apply the inductive hypothesis on the remaining digraph $\hat{D}$. The partition π can be also restricted to a partition $\hat{\pi}$ by removing u from P_u. Note that now $v, w \in F(\hat{\pi})$, since v remains in $\hat{D}$ and w was the last vertex on P_u before u. Since $\alpha(\hat{D}) \leq \alpha(D)$, by induction there is a partitioning of $\hat{D}$ into $l-1$ paths $\hat{\pi}'$, such that $F(\hat{\pi}')$ contains all the vertices of $F(\hat{\pi})$ except one. If $w \in F(\hat{\pi}')$, then we extend the path ending at w by (w, u). Otherwise $v \in F(\hat{\pi}')$, and we extend the path ending at v by (v, u). Either way, we obtain a partitioning π' where $F(\pi') \subset F(\pi)$.

$\square$

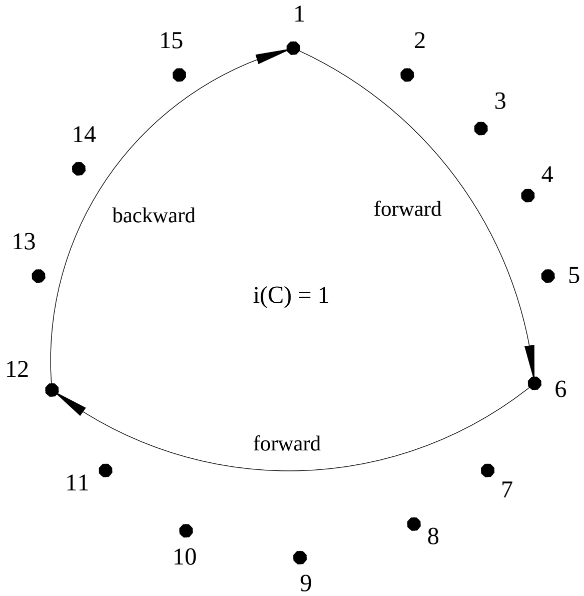
In the rest of the lecture, we prepare the ground for the proof of Bessy-Thomassé.

Definition 1 *For a directed graph $D = (V, A)$, consider enumerations of the vertex set, such as $(v_1, v_2, \dots v_n)$. We define a cyclic ordering as an equivalence class of enumerations with respect to the following equivalence relations:*

- $(v_1, v_2, \dots v_{n-1}, v_n) \sim (v_2, v_3, \dots, v_n, v_1)$
- $(v_1, v_2, \dots v_n) \sim (v_2, v_1, \dots v_n)$ *if* $(v_1, v_2) \notin A$ *and* $(v_2, v_1) \notin A$.

For an enumeration, we call an arc (v_i, v_j) forward if $i < j$, and backward if $i > j$. Also, we can visualize all arcs as going clockwise around the circle, and then backward arcs are those which cross the boundary between v_n and v_1.

For a cyclic ordering $\mathcal{O}$ and a directed cycle C, let $i_{\mathcal{O}}(C)$ denote the number of backward arcs in C. Note that $i_{\mathcal{O}}(C)$ depends on the cyclic ordering, but not on a specific enumeration within the equivalence class; $i_{\mathcal{O}}(C)$ can be also viewed as the number of times the cycle loops around the circle.



We call a cyclic ordering valid, if every arc is contained in a directed cycle of index $i_{\mathcal{O}}(C) = 1$.

S is a cyclic stable set with respect to $\mathcal{O}$, if it is a stable set which is consecutive in some enumeration of $\mathcal{O}$.

Theorem 4 *For every strongly connected directed graph, there exists a valid cyclic ordering.*

Proof: Let $\mathcal{O}$ be a cyclic ordering, minimizing the sum of indices over all directed cycles, $\sum_C i_{\mathcal{O}}(C)$. Suppose that $\mathcal{O}$ is not valid, i.e. some arc is not in any directed cycle of index 1. Consider such an arc (v_j, v_i) and an enumeration such that (v_j, v_i) is a backward arc and $j - i$ is as small as possible. Since there is no cycle of index 1 containing (v_j, v_i), there is no forward path from v_i to v_j. First assume that $j - i > 1$. One implication of this is that (v_{k+1}, v_k) is never an arc. Let k be the maximum smaller than j, such that v_k can be reached by a forward path from v_i.

1. $k = i$: there is no arc between v_i and v_{i+1}, so we can swap v_i with v_{i+1}, which moves v_i closer to v_j (contradiction).

2. $k \neq i$: there are no arcs from v_k to v_l, $k < l \leq j$. Also, there are no arcs from v_l to v_k for the same indices l; otherwise the existence of this arc would contradict the minimality of $j - i$. Therefore we can swap v_k with v_{k+1}, then with v_{k+2}, until we swap v_k with v_j, which creates an enumeration where v_i and v_j are closer to each other (contradiction).

Thus we can assume that $j - i = 1$. In this case, we switch v_i and v_j, which creates a different cyclic ordering. Every directed cycle which contains (v_j, v_i) decreases its index by 1. No other cycle is affected. Therefore we find a cyclic ordering $\mathcal{O}'$ with a smaller value of $\sum_C i_{\mathcal{O}'}(C)$, which is a contradiction. $\square$

A corollary is Camion's theorem for strongly connected tournaments.

Corollary 5 *For a strongly connected tournament, there is a cyclic ordering $(v_1, v_2, \dots v_n)$ such that all the edges (v_i, v_{i+1}) are oriented clockwise, i.e. there is a hamiltonian cycle.*