| **18.997 Topics in Combinatorial Optimization** | February 10th, 2004 |
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| Lecture 3 | |
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In this lecture we will cover:

1. Topics related to Edmonds-Gallai decompositions ([Sch03], Chapter 24).
2. Factor critical-graphs and ear-decompositions ([Sch03], Chapter 24).

Topics mentioned but covered during subsequent lectures are:

1. The matching polytope ([Sch03], Chapter 25).
2. Total Dual Integrality (TDI) and the Cunningham-Marsh formula ([Sch03], Chapter 25).

A detailed reference on matchings is the book *Matching Theory* by Lovasz and Plummer, [LP86].

# 1 Petersen's Theorem

Before stating Petersen's theorem, we recall that a graph is called *cubic* if each of its vertices has degree exactly 3, and *bridgeless* if it cannot be disconnected by deleting any one edge (in other words any pair of vertices has edge connectivity at least 2).

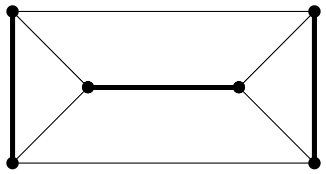

Figure 1: A bridgeless cubic graph and a perfect matching on it. Edges in the matching are bold.

**Theorem 1 (Petersen)** *Any bridgeless cubic graph has a perfect matching.*

**Proof:** We will show that for any $V \subseteq U$, we have $c_o(G-U) \le |U|$ (here $c_o(G)$ is the number of odd components of the graph $G$). The theorem will then follow from the Tutte-Berge formula.

Consider an arbitrary $U \subset V$. Each odd component of $G-U$ is left by an odd number of edges, since $G$ is cubic. Since $G$ is also bridgeless each component is left by at least 2 edges, hence by at least 3 edges. On the other hand, the set of edges leaving all odd components of $G-U$ is a subset of the edges leaving $U$, and there are at most $3|U|$ edges

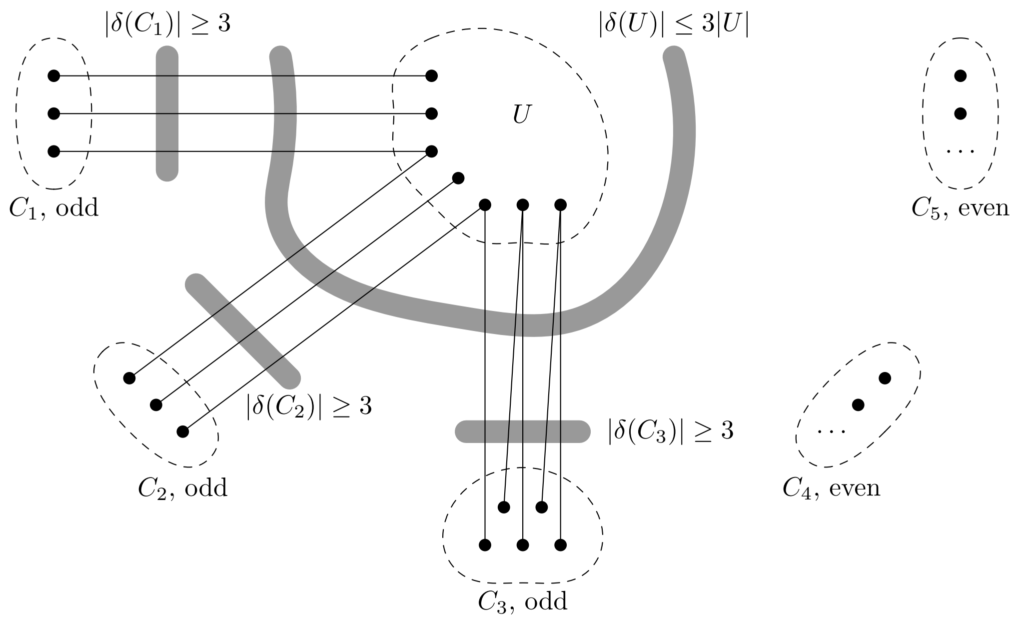


Figure 2: Illustration of the proof of Petersen's theorem. Edges inside $U$ and $C_i$, as well as between $C_4, C_5$ and $U$ are omitted.

leaving $U$, since $G$ is cubic. Among these $3|U|$ edges, there are at least 3 edges per each odd component, therefore there are at most $|U|$ odd components. (See Figure 2.) $\square$

A bridgeless cubic graph and a perfect matching for it are shown in Figure 1.

Although any bridgeless cubic graph has a perfect matching, it is not true that any such graph can be decomposed into 3 perfect matchings. An example of this is the Petersen graph, depicted in Figure 3.

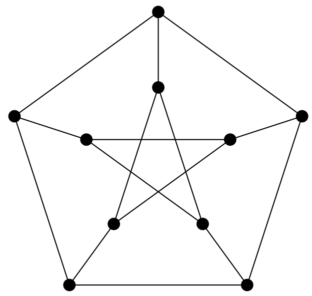

Figure 3: The Petersen graph.

## 1.1 Colorings and matchings

However, we can cover all edges of any bridgeless cubic graph with 4 matchings, as shown by the following theorem. (Note that a coloring is an assignment of colors to edges such that edges sharing a vertex have different colors. Thus, a $k$-coloring is the same as covering all edges with $k$, not necessarily perfect, matchings.)

**Theorem 2 (Vizing, 1964)** *For any graph, there is an edge coloring with at most $\Delta + 1$ colors, where $\Delta := \max_{v \in V} \deg(v)$ is the maximum degree of any vertex in $G$.*

In fact, Holyer (1981) has shown that it is NP-complete to decide whether a given cubic graph is 3-colorable. It is also NP-complete to find the edge-coloring number of a $k$-regular graph, for each $k \geq 3$ (Leven and Galil, 1983).

The following theorem is a particularly appealing result relating matchings and colorings.

**Theorem 3 (Tait, 1878)** *Each planar cubic bridgeless can be decomposed into 3 matchings if and only if the 4-color conjecture holds.*

Since the 4-color conjecture is now a theorem with a complicated proof, an easy proof of Tait's theorem is of interest.

**Conjecture 1 (Fulkerson)** *For any bridgeless cubic graph there is exist 6 perfect matchings that cover each edge exactly twice.*

More conjectures can be found in Chapter 28 of [Sch03], entirely devoted to edge-colorings.

## 2 Ear decompositions

Before proceeding to describe results about ear decompositions, we review a result on factor-critical graphs.

**Definition 1** *A graph $G$ is* factor-critical *if for any vertex $v \in V$, $G - v$ has a perfect matching.*

As before, let $D(G)$ be the set of vertices missed by some maximum-size matching, let $A(G) := N(D(G)) = \{v : \exists w \in U, \{v, w\} \in E\}$ be the set of all vertices neighboring vertices in $D(G)$, and let $C(G) := V \setminus (D(G) \cup A(G))$ contain all other vertices. Recall from Lecture 1 that $U := A(G)$ attains the minimum in the Tutte-Berge formula, $D(G)$ is the union of the odd components of $G - U$, and $C(G)$ is the union of even components of $G - U$.

**Claim 4** *Each odd connected component of $G - A(G)$ is factor-critical.*

**Proof:** We will give a proof that relies on Edmond's algorithm. First, recall from Lecture 2 that $D(G)$ is the set of even vertices of the final forest, hence $A(G)$ is the set of odd vertices. Since there are no edges between even vertices in the final forest, each odd component of $G - A(G)$ is represented in the final graph by an even vertex.

So it suffices to show that any graph obtained by a series of blossom operations starting from a single vertex is factor-critical, and we do this by induction. Clearly, the original vertex is factor-critical (the first blossom, being an odd cycle is also factor-critical).

Now, assume that $G/B$, obtained from $G$ by shrinking $B$, is factor-critical. If $v \notin B$, then $G$ has a maximum matching that missing $v$, because $G/B$ has one and it can be completed by appropriately ading edges of $B$. If $v \in B$, then we can obtain a maximum matching in $G$ that misses $v$ by taking a maximum matching in $G/B$ that misses $B$ (such a matching exists since $G/B$ is factor-critical), and then taking a maximum matching on $B$ that misses $v$. Therefore $G$ is factor-critical. $\square$

An *ear decomposition* $G_0, G_1, \ldots, G_k = G$ of a graph $G$ is a sequence of graphs with the first graph being simple (e.g. a vertex, edge, even cycle, or odd cycle), and each graph $G_{i+1}$ obtained from $G_i$ by *adding an ear.* Adding an ear is done as follows: take two vertices $a$ and $b$ of $G_i$ and add a path $P_i$ from $a$ to $b$ such that all vertices on the path except $a$ and $b$ are new vertices (present in $G_{i+1}$ but not in $G_i$). An ear with $a \neq b$ is called proper (or open), and an ear with $P_i$ having an odd (even) number of edges is called odd (even). (See Figure 4.) Several basic properties of graphs can be translated into the existence of an ear decomposition of a certain kind. Here are some examples.

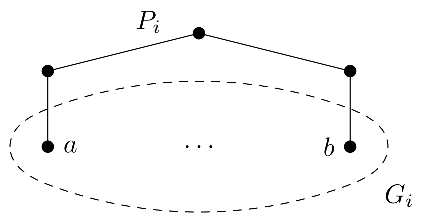


Figure 4: An even proper ear added to $G_i$.

**Theorem 5 (Robbins, 1939 (implicit))** *$G$ is 2-connected if and only if $G$ has a proper ear decomposition starting from a cycle.*

**Proof:** Obviously, any graph that has a proper ear decomposition starting from a cycle is 2-connected.

Conversely, we assume $G$ is 2-connected, and will show by induction how to construct it starting from a cycle. First, since $G$ is 2-connected, it contains at least one cycle, which we can take as the initial cycle.

Now, suppose we have constructed a subgraph $G'$ of $G$. If $V(G') = V(G)$ and we are only missing edges, then we can add these edges as proper ears of length one. If $V(G') \subset V(G)$, then pick a vertex $v \in V(G) \setminus V(G')$. Since $G$ is connected, there is a path $P$ from some $a \in V(G)$ to $v$; since $G$ is 2-connected, there is a path $Q$ distinct from $P$ from $v$ back to some vertex $b \in V(G'), b \neq a$. Hence the paths $P$ and $Q$ form a proper ear from $a$ to $b$ containing at least one new vertex. $\square$

**Theorem 6** *$G$ is factor-critical if and only if $G$ has an odd ear decomposition starting from an odd cycle.*

**Proof:** If $G$ has an odd ear decomposition, then it is factor critical, since blossoming yields a factor critical graph.

Conversely, suppose $G$ is factor-critical. First, we establish the existence of an initial odd cycle. For any $v$, fix a near-perfect matching $M_v$ that misses $v$. Then for an edge $(u, v)$

the existence of $M_u$ and $M_v$ implies there is an alternating even path from $v$ to $u$. By adding $(u, v)$ to it we obtain an odd cycle.

Fix a vertex $v$. We proceed by induction; let $H$ be the vertex set already covered by the odd ear decomposition such that no edge in $M_v$ crosses $H$. Since $G$ is connected, there is an edge $(a, b), a \in H, b \notin H, (a, b) \notin M_v$. Moreover, $M_b \triangle M_v$ contains an alternating path $Q$ from $b$ back to $v$. The first edge $(w, u)$ to cross back into $H$ on $Q$ is not in $M_v$, by the construction of $H$. Therefore, we obtain an odd path from $b$ to $u$, and can increase the size of $H$. $\square$

The two results can be combined. One can show that $G$ is factor-critical and 2-connected if and only it has a proper ear decomposition starting from an odd cycle.

Here is another ear decomposition result. A bipartite ear decomposition starts from an even cycle, and adds an odd length path between vertices of different color. As a result, the graph stays bipartite. **Question:** $G$ is ___ if and only if it has a bipartite ear decomposition. What is ___? (Answer at end of lecture.)

Here is a result on factor-critical graphs which can be used to characterize the facets of teh matching polytope.

**Theorem 7** *Let $G$ be a 2-connected factor-critical graph. Then the number of near-perfect matchings is at least $|E(G)|$.*

**Proof:** We proceed by induction on the number of odd ears. Consider a graph $G'$, and $G$ obtained from $G'$ by adding an odd ear $P = (u_0, \ldots, u_k)$ of $k$ edges. Then $|V(G)| = |V(G')| + k - 1, |E(G)| = |E(G')| + k$.

We can obtain $|E(G')|$ near-perfect matchings by taking $(u_1, u_2), \ldots, (u_{k-2}, u_{k-1})$ into the matching, and then generating $|E(G')|$ near perfect matchings in $G'$. Moreover, we can obtain $k - 1$ by matching all vertices on $P$ except $u_j, j = 1, \ldots, k$, and then taking a near-perfect matching on $G'$ that misses either $u_0$ (if $j$ is odd) or $u_k$ (if $j$ is even). The final matching is obtained by taking the matching missing $u_k$, but not $u_0$, removing the edge matching $u_k$ in $G'$ and adding the edge matching $u_k$ in $P$. $\square$

We note without further discussion that the number of *affinely independent* near-perfect matchings is *equal* to $|E(G)|$.

**Answer:** ___ is that every edge is in a perfect matching.

## References


[LP86] L. Lovász and M. D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Annals of Discrete Mathematics, 29.

[Sch03] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Paths, flows, matchings, Chapters 1–38.