18.997 Topics in Combinatorial Optimization	April 27, 2004
Lecture 20	
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1 k-arc-connected orientations

We continue the discussion of how a $2k$-edge-connected graph can be oriented so that the resulting digraph is k-arc-connected. Last time we have seen that this can be achieved using submodular flows. Today we present a different approach, which relates the problem to matroid intersection.

Let $G = (V, E)$ be a $2k$-edge-connected graph and let $D = (V, A)$ denote the bidirected version of G, with two arcs (u, v) and (v, u) for each edge $\{u, v\}$. (All graphs in this lecture can be multigraphs.) We define two matroids on the ground set of arcs A. The first one is a partition matroid:

$$\mathcal{M}_1 = (A, \{B \subseteq A : \forall \text{edge } (u, v); B \text{ contains at most one of the arcs } (u, v), (v, u)\}).$$

The bases of $\mathcal{M}_1$ are exactly the orientations of G. The second matroid, which will force the orientation to be k-arc-connected, is more involved. Define

- $H(U) = \{(v, u) \in A : u \in U\}$
- $\mathcal{C} = \{H(U) : \emptyset \subset U \subset V\}$
- $f(H(U)) = |E(U)| + |\delta(U)| - k = |E| - |E(V \setminus U)| - k$

In other words, $H(U)$ is the set of arcs with their "head" in U (either crossing the cut into U or contained inside U), and $f(H(U))$ is the maximum number of edges oriented like this, so that k arcs leaving U are still available. Note that $\mathcal{C}$ forms a *crossing family*: $\forall H_1, H_2 \in \mathcal{C}; H_1 \cap H_2 \neq \emptyset, H_1 \cup H_2 \neq A \Rightarrow H_1 \cap H_2 \in \mathcal{C}, H_1 \cup H_2 \in \mathcal{C}$. This is simply because $H(U_1) \cap H(U_2) = H(U_1 \cap U_2)$ and $H(U_1) \cup H(U_2) = H(U_1 \cup U_2)$. Also, $f(H(U)) = |E| - |E(V \setminus U)| - k$ is a *crossing submodular function* on $\mathcal{C}$: since $|E(V \setminus U_1)| + |E(V \setminus U_2)| \leq |E(V \setminus (U_1 \cap U_2))| + E(V \setminus (U_1 \cup U_2))|$, $f(H_1 \cap H_2) + f(H_1 \cup H_2) \leq f(H_1) + f(H_2)$. Given these properties, we shall prove that

$$\mathcal{M}_2 = (A, \{B \subseteq A : |B| \leq |E| \ \& \ \forall H \in \mathcal{C}; |B \cap H| \leq f(H)\})$$

is a matroid. Then, k-arc-connected orientations correspond exactly to common bases of $\mathcal{M}_1 \cap \mathcal{M}_2$: bases of $\mathcal{M}_1$ are orientations of G, and an orientation is a base of $\mathcal{M}_2$ if and only if it has at most $\delta(U) - k$ arcs across any directed cut $\delta^{in}(U)$, i.e. it must have at least k arcs across $\delta^{out}(U)$. Therefore a k-arc-connected orientation can be found using matroid intersection. [1]

It remains to prove that $\mathcal{M}_2$ is a matroid. This is implied by the following lemma.

Lemma 1 *Let $\mathcal{C} \subseteq 2^A$ be a crossing family and $f : \mathcal{C} \to \mathbf{Z}$ a crossing submodular function. Then for any $k \in \mathbf{Z}_+$,*

$$\mathcal{B} = \{B \subseteq A : |B| = k \ \& \ \forall H \in \mathcal{C}; |B \cap H| \leq f(H)\}$$

are the bases of a matroid.

Proof: We have to prove the exchange property for $\mathcal{B}$. Let $B_1, B_2 \in \mathcal{B}$, $i \in B_1 \setminus B_2$ and $j \in B_2 \setminus B_1$. If $B_1 - i + j \notin \mathcal{B}$, it means that for some $H \in \mathcal{C}$, $|B_1 \cap H| = f(H)$, $i \notin H$ and $j \in H$, so that we violate the condition by exchanging j for i. Assume that this holds for every $j \in B_2 \setminus B_1$.

[1] provided that membership in $\mathcal{M}_2$ can be tested efficiently, which is not explained here.

For each $j \in B_2 \setminus B_1$, let $H_j \in \mathcal{C}$ be the maximal set such that $i \notin H_j, j \in H_j$ and $|B_1 \cap H_j| = f(H_j)$. These sets are disjoint; if $H_j \cap H_{j'} \neq \emptyset$ and $|B_1 \cap H_j| = f(H_j)$, $|B_1 \cap H_{j'}| = f(H_{j'})$, then by crossing submodularity $|B_1 \cap (H_j \cup H_{j'})| = f(H_j \cup H_{j'})$ which contradicts the maximality of H_j and $H_{j'}$. Let $\mathcal{P} = \{H_j : j \in B_2 \setminus B_1\}$ denote the collection of these disjoint sets, and $W = A \setminus \bigcup \mathcal{P}$ the set of remaining uncovered elements. For each $H_j \in \mathcal{P}$, we have $|B_2 \cap H_j| \leq f(H_j) = |B_1 \cap H_j|$. All the elements of $B_2 \setminus B_1$ are covered by $\mathcal{P}$, so $B_2 \cap W \subseteq B_1 \cap W$, and there is an element $i \in W$ which belongs to B_1 but not B_2. Therefore $|B_2 \cap W| < |B_1 \cap W|$ and $|B_2| < |B_1|$ which is a contradiction. $\square$

2 Splitting off

Now we turn to a technique developed by László Lovász, which is very useful for *connectivity augmentation* and other questions concerning edge connectivity.

Theorem 2 *Let $G = (V + s, E)$ be a graph, such that the degree of s is even, and*

$$\forall U; \emptyset \subset U \subset V \Rightarrow |\delta(U)| \geq k \tag{1}$$

Then there are edges $(s, u), (s, t)$ such that

$$G' = (V + s, E \setminus \{(s, u), (s, t)\} \cup \{(t, u)\})$$

satisfies Condition 1.

In other words, we can "split off" a vertex s of even degree, by replacing pairs of edges incident with s by other edges in the graph, and we preserve k-edge-connectivity between all vertices different than s in the remaining graph. We prove the theorem later. Now let's demonstrate its application to the construction of all $2k$-edge-connected graphs. We first need a lemma.

Lemma 3 *Every edge-minimal k-edge connected graph has a vertex of degree k.*

Proof: In a k-edge-connected graph, every cut contains at least k edges. If it's edge-minimal, every edge is contained in a cut of size exactly k (otherwise we can remove the edge without decresing connectivity). Let $S \subset V$ be minimal such that $|\delta(S)| = k$. If $|S| = 1$, we get a vertex of degree k. We prove that $|S| > 1$ leads to a contradiction. $G[S]$ is connected (otherwise S is not minimal), and so $G[S]$ contains an edge e. Let $\delta(T)$ be a cut of size k, cutting e (therefore $S \cap T \neq \emptyset$). If $S \cup T \neq V$, by submodularity, $\delta(S \cap T)$ and $\delta(S \cup T)$ are also cuts of size k. If $S \cup T = V$, then $\delta(S \setminus T) = \delta(T)$ would be a cut of size k. In any case, we get a contradiction with the minimality of S. $\square$

Theorem 4 *Let M_{2k} denote a multigraph of $2k$ parallel edges between two vertices. Any $2k$-edge-connected graph can be built from M_{2k} by*

- *adding edges*
- pinching k *edges: taking k edges $(u_1, v_1), \ldots (u_k, v_k)$, adding a new vertex s, and replacing each (u_i, v_i) by (s, u_i) and (s, v_i).*

Proof: Start with a $2k$-edge-connected graph. Remove edges, until there is a vertex s of degree $2k$ (whose existence follows from the previous lemma). Apply the splitting-off lemma k times, and remove vertex s while preserving k-edge-connectivity. Then continue, until G shrinks to a 2-vertex graph, which must be a multigraph of at least $2k$ parallel edges. We remove some edge to obtain M_{2k}. The reverse procedure consists of repeatedly adding edges and pinching collections of k edges. $\square$

Note: This gives another proof that any $2k$-edge-connected graph G has a k-arc-connected orientation. We start from M_{2k}, where k edges are oriented each way. We build G by adding edges (with arbitraty orientation) and pinching edges, replacing an arc by two arcs oriented the same way. This procedure preserves k-arc-connectivity.

3 Connectivity augmentation

In this section, we use splitting-off to solve the problem of augmenting a graph by adding some edges, so that the graph becomes k-edge-connected. Let $U \subset V$ and $x : V \to \mathbf{Z}$. We denote $d_E(U) = |\delta(U) \cap E|$ and $x(U) = \sum_{v \in U} x(v)$.

Lemma 5 *Given $G = (V, E)$, there exists of set of edges F such that $(V, E \cup F)$ is k-edge-connected and F has prescribed degrees $d_F(v) = x(v)$, if and only if*

- *$x(V)$ is even, and*
- *$\forall U; \emptyset \subset U \subset V \Rightarrow \; d_E(U) + x(U) \geq k$.*

Proof: These conditions are clearly necessary; we'll now show their sufficiency. For $G = (V, E)$ and $x : V \to \mathbf{Z}$, add a new vertex s, connecting it to each $v \in V$ by $x(v)$ parallel edges. If $x(V)$ is even, the degree of s is even. Due to the second condition, we have augmented all cuts $\delta(U), \emptyset \subset U \subset V$, to size at least k, so we can apply splitting off. It follows that edges incident with s can be replaced by a set of edges F with prescribed degrees $x(v)$, while preserving k-edge-connectivity. □

This yields an approach to finding the smallest augmenting set F. Find $x(v)$ such that $\forall U, \emptyset \subset U \subset V; d_E(U) + x(U) \geq k$ and $x(V)$ is minimal. If $x(V)$ turns out odd, we increase some $x(v)$ by 1 (arbitrarily). In any case, we can augment G to a k-edge-connected subgraph by adding $\lceil x(V)/2 \rceil$ edges, which is optimal.

Theorem 6 *G can be augmented to a k-edge-connected graph by adding γ edges, if and only if for any collection of disjoint subsets of vertices $\mathcal{P}$:*

$$\sum_{U \in \mathcal{P}} (k - d_E(U)) \leq 2\gamma.$$

Proof: Again the condition is clearly necessary; we now show sufficiency. Assume that γ satisfies the condition of the lemma. Start with $x(v) = k$. Decrease the $x(v)$ values arbitrarily, maintaining

$$\forall U; \emptyset \subset U \subset V \Rightarrow x(U) \geq k - d_E(U).$$

If we cannot decrease any $x(v)$ anymore, each v with $x(v) \geq 1$ must be contained in a subset U for which equality $x(U) = k - d_E(U)$ holds. Let $\mathcal{P}$ denote the collection of maximal subsets $U \subset V$ such that $x(U) = k - d_E(U)$. Consider any $S, T \in \mathcal{P}$; if $S \cup T = V$, then $x(V) \leq x(S) + x(T) = (k - d_E(V \setminus S)) + (k - d_E(V \setminus T)) \leq 2\gamma$.

If $S \cup T \neq V$ for any $S, T \in \mathcal{P}$, then $\mathcal{P}$ must be a collection of disjoint sets. Assume $S \cap T \neq \emptyset$: then $x(S) + x(T) = (k - d_E(S)) + (k - d_E(T)) \leq (k - d_E(S \cap T)) + (k - d_E(S \cup T)) \leq x(S \cap T) + x(S \cup T) = x(S) + x(T)$, i.e. all inequalities are equalities and $x(S \cup T) = k - d_E(S \cup T)$ which contradicts the maximality of S, T. Therefore, $\mathcal{P}$ is a partition of $\{v \in V : x(v) \geq 1\}$ and

$$x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (k - d_E(U)) \leq 2\gamma.$$

Finally, we increment some $x(v)$ to make $x(V)$ even, if necessary. Consequently, x satisfies the conditions of Lemma 5, $x(V) \leq 2\gamma$, and therefore we can augment G to a k-edge-connected subgraph by adding at most γ edges. □

The condition on $x(v)$ in the proof can be checked efficiently (by min-cut computations). Therefore we can find the minimum set of γ edges which augment edge connectivity to k, in polynomial time. In contrast, the connectivity augmentation problem with edge weights is NP-hard.