18.997 Topics in Combinatorial Optimization April 1st, 2004

Lecture 14

Lecturer: Michel X. Goemans *Scribe: Mohamed Mostagir*

In this lecture, we continue with more results on matroid union, as well as tie together some loose ends from the past couple of lectures.

1 Testing for Independence

Recall from the matroid union theorem (lectures 12 and 13) that if $M_1 = (S_1, \mathcal{I}_1), ..., M_k = (S_k, \mathcal{I}_k)$ are matroids, then their union $M_1 \vee ... \vee M_k = (S_1 \cup ... \cup S_k, \{I_1 \cup ... \cup I_k : I_i \in \mathcal{I}_i\})$ is also a matroid. If I is an independent set in the union, then $I \in \mathcal{I}_1 \vee ... \vee \mathcal{I}_k$. We're interested in determining whether $I + s$ is also in the union, i.e. if we add an element s to I, then does $I + s \in \mathcal{I}_1 \vee ... \vee \mathcal{I}_k$? Is $I + s$ still independent? In order to help answer this question, we give the following construction. For each matroid M_i, define an arc set $D_{M_i}(I_i) = \{(x, y) : x \in I_i, y \notin I_i, I_i - x + y \in \mathcal{I}_i\}$. That is, $D_{M_i}(I_i)$ is the set of pairs of elements, where removing the first element from I and adding the second still leaves us with an independent set. Let $D = \bigcup D_{M_i}(I_i)$ be the superposition of all the $D_{M_i}(I_i)$'s. We will show that *shortest* paths in D can be used to obtain other independent sets in the union. We give one more definition before stating the main result of this section. Let $F_i = \{x : I_i + x \in \mathcal{I}_i\}$, and let $F = \bigcup F_i$.

Theorem 1 *Let I be an independent set in the union. Then $I + s \in \mathcal{I}_1 \vee ... \vee \mathcal{I}_k$ iff $\exists$ an $F - s$ path in D.*

Proof: ($\Downarrow$) We first show that if there is no $F - s$ path in D, then $I + s$ is not independent. Define $T = \{v : \exists \text{ a } v - s \text{ path in } D\}$, and assume that $\nexists$ an $F - s$ path in D, i.e. $T \cap F = \emptyset$. The following claim helps us in proving this direction.

Claim 2 *$|I_i \cap T|$ is a maximal independent subset of T in every matroid.*

From the claim, we can see that $|I_i \cap T| = r_i(T), i = 1, ..., k$. This implies that $(I+s) \cap T = ((I_1 \cup ... \cup I_k) + s) \cap T = (I_1 \cap T) \cup (I_2 \cap T) \cup ... \cup \{s\}$, and therefore, $|(I+s) \cap T| > r_1(T) + r_2(T) + \cdots + r_k(T) \geq r_{M_1 \vee M_2 \vee ... \vee M_k}(T)$, and $I + s$ is not independent. To prove the claim, we proceed as follows. We know that $I_i \cap T$ is an independent set in $\mathcal{I}_i$. Suppose that this set is not maximal, then there exists an element $y \in T \setminus I_i$ that can be added to the set with independence still maintained. We know that $y \notin F$ because T and F are disjoint, so $I_i + y \notin \mathcal{I}_i$, which implies that there exists an x such that $x \in I_i \setminus (I_i \cap T)$ and that we can remove x and add y to have $I_i - x + y \in \mathcal{I}_i$. But by definition of D_i, this means that there is an arc from x to y, which means that x can reach s and is therefore in T, a contradiction. A simpler way to show the existence of x is as follows. Take $I_i \cap T + y$, which is independent, and note that $I_i + y$ is not independent. We can keep adding elements from I_i to $I_i \cap T + y$ until there's one element left in I_i, this element is x.

($\Uparrow$) Take a shortest $F - s$ path P, with $P = \{s_0, s_1, ..., s_p = s\}$, and assume $s_0 \in F_1$ (i.e. $I_1 \cup s_0 \in \mathcal{I}_1$.) Since P is a shortest path, the set of edges (s_i, s_{i+1}) with $i = 0, ..., p-1$, and $s_i \in I_j$ gives a unique perfect matching in $D_{M_j}(I_j)$. Let S_j be the endpoints of these edges. As a consequence of lemma 5 in lecture 11, this implies that $I_j \triangle S_j \in \mathcal{I}_j$, for all j (the fact that we take a shortest path means that the matching is unique). Also, as in lecture 11, we can also argue that $(I_1 \triangle S_1) \cup \{s_0\} \in \mathcal{I}_1$. This implies that $I \cup \{s\} \in \mathcal{I}$.

So we can test membership if we have an independent set of the union expressed in terms of the independent sets of the matroids.

$\square$

2 Some applications based on matroid union

For the remainder of this lecture, we will give various results related to matroid union, as well as some basic properties of the rank function. In the following, $M = (S, \mathcal{I})$ is a matroid with rank function r; and $k \in \mathbb{Z}_+$ is a positive integer.

Corollary 3 *The maximum size of the union of k independent sets in M is*

$$C = \min_{U \subseteq S} \quad [|S \setminus U| + k \cdot r(U)]$$

Since S can be covered by k bases iff $C = |S|$, we derive:

Corollary 4 (Matroid base covering) *S can be covered by k bases iff $\forall U : k \cdot r(U) \geq |U|$.*

Since there exist k disjoint bases iff $C = kr(S)$, we get:

Corollary 5 (Matroid base packing) *There exist k disjoint bases in M iff*

$$\forall U : |S \setminus U| \geq k(r(S) - r(U))$$

Thinking about these corollaries in terms of graphic matroids, we get

Theorem 6 (Nash-Williams) *G can be covered by k forests iff $\forall T \subseteq V : |E(T)| \leq k(|T| - 1)$.*

Proof: The only if direction is obvious. To see the other direction, consider any set $U \subseteq E$. Assume (V, U) has l connected components. Let $T_1, T_2, \cdots, T_l$ be these l connected components. By assumption, we have that $|E(T_i)| \leq k(|T_i| - 1)$. Summing over i, we get that

$$|U| \leq \sum_i |E(T_i)| \sum_i k(|T_i| - 1) = kr(U),$$

and thus the claim follows from Corollary 4. □

Similarly, we derive:

Theorem 7 (Tutte, Nash-Williams) *G contains k edge-disjoint spanning trees iff $\forall$ partitions ρ of V, with $\rho = (V_1, ..., V_l)$, we have $|\delta(\rho)| \geq (l-1)k$, where $|\delta(\rho)| = \{(u, v) : u \in V_i, v \in V_j, i \neq j\}$.*

We now turn our attention to some of the properties of the rank function. For a matroid $M = (S, \mathcal{I})$ with rank function $r : 2^S \to \mathbb{R}$, we have the following lemma

Lemma 8 *The rank function is submodular: $\forall A$ and $B \subseteq S$, $r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$.*

Proof: Let $I \subseteq A \cap B$ be an inclusion-wise maximal set in $\mathcal{I}$, so $r(A \cap B) = |I|$, and let J be such that $I \subseteq J \subseteq A \cup B$ and $r(A \cup B) = |J|$. Note that both $J \cap A$ and $J \cap B \in \mathcal{I}$ and therefore we have:

$$r(A) \geq |J \cap A|$$

$$r(B) \geq |J \cap B|$$

and $r(A)+r(B) \geq |J \cap A|+|J \cap B| = |J \cap (A \cup B)|+|J \cap (A \cap B)| = |J|+|I| = r(A \cap B)+r(A \cup B)$ □

In fact, the converse also applies, in the sense that if $r : 2^S \to \mathbb{Z}_+$ is such that

i) $r(A) \leq r(B)$ if $A \subseteq B$,

ii) $r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$ for all A, B,

iii) $r(A) \leq |A|$.

Then $(S, \{I : |I| = r(I)\})$ is a matroid.

As a consequence of submodularity, note that if we have a matroid polytope defined by

$$x(U) \leq r(U) \qquad \forall U \subseteq S$$
$$x_s \geq 0 \qquad \forall s \in S$$

or a matroid intersection polytope defined by

$$x(U) \leq r_1(U) \qquad \forall U \subseteq S$$
$$x(U) \leq r_2(U) \qquad \forall U \subseteq S$$
$$x_s \geq 0 \qquad \forall s \in S.$$

Then we can solve the separation problem over these polytopes by solving

$$\min_{U \subseteq S} [r(U) - x(U)].$$

Where $x(U) \leq r(U)$ $\forall U$, i.e. we can solve the separation problem by minimizing a submodular function. Cunningham gives a pseudo-polynomial time algorithm to do this. This was later improved to strongly polynomial time by Schrijver and by Fleischer, Fujishige and Iwata. Schrijver's algorithm will be covered in a few lectures. Submodular function minimization has many interesting applications.

We finish this lecture by returning to matroid unions and proving an interesting lemma about the exchange property for matroid bases (remember that we can switch elements and maintain bases).

Lemma 9 *If B_1 and B_2 are bases of M, and B_1 is partitioned into $X_1 \cup Y_1$, then there exists a partition of B_2 into $X_2 \cup Y_2$ such that $X_1 \cup Y_2$ and $X_2 \cup Y_1$ are bases of M.*

Proof: Let's define two matroids, $M_1 = M/Y_1$ and $M_2 = M/X_1$. We want to show that $B_2 \in \mathcal{I}_1 \vee \mathcal{I}_2$. This would mean that B_2 can be expressed as $X_2 \cup Y_2$ with $X_2 \in \mathcal{I}_1$ and $Y_2 \in \mathcal{Y}_2$, i.e. $X_2 \cup Y_1$ and $X_1 \cup Y_2$ are bases of M. We have from matroid union

$$\begin{aligned} r_{M_1 \vee M_2}(B_2) &= \min_{U \subseteq B_2} |B_2 \setminus U| + r_{M_1}(U \cap (S \setminus Y_1)) + r_{M_2}(U \cap (S \setminus X_1)) \\ &= \min_{U \subseteq B_2} |B_2 \setminus U| + r(U \cup Y_1) - r(Y_1) + r(U \cup X_1) - r(X_1) \\ &\geq |B_2 \setminus U| + r(U \cup Y_1 \cup X_1) + r(U) - |X_1| - |Y_1| \\ &= |B_2| - |U| + |U| = |B_2| \end{aligned}$$

Where the inequality in the second to last equation follows from submodularity. Note that in this equation, the second term cancels with the two last terms, and $r(U) = |U|$ (as U is a subset of a basis) leading to the result. $\square$

We conclude our discussion by giving the following open problem. Suppose S can be partitioned into k bases. Is there a way to express S as $S = \{e_0, e_1, ..., e_{p-1}\}$ where $p = k \cdot r(S)$ such that for all i, $\{e_{i+1}, ..., e_{i+r(S)}\}$ (where the indices are taken modulo p) is a base?