Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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Handout 15
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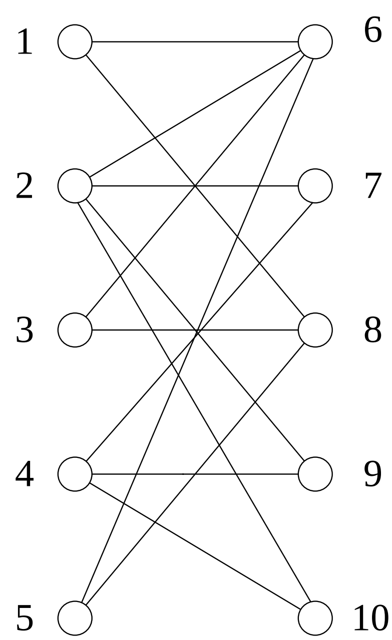
QUIZ 1

You can have one single-sided handwritten sheet of paper with anything you want on it. You can write your solutions on this exam. The last question is most likely the most difficult.

1. Find a minimum vertex cover C in the following graph (list the vertices in C). Argue why it is optimum.



2. Suppose you are given a (not necessarily bipartite) graph $G = (V, E)$ and a matching M. Explain what would be a short proof of (i) M is maximum, and (ii) M is *not* maximum.

3. Consider the partition matroid $M = (E, \mathcal{I})$ defined by $E = E_1 \cup E_2 \cup E_3$, $E_1 = \{1, 2, 3\}$, $E_2 = \{4, 5, 6, 7\}$, $E_3 = \{8, 9, 10\}$, $\mathcal{I} = \{S \subseteq E : |S \cap E_i| \leq 2 \text{ for } i = 1, 2, 3\}$.

   (a) Give 2 different bases of M.

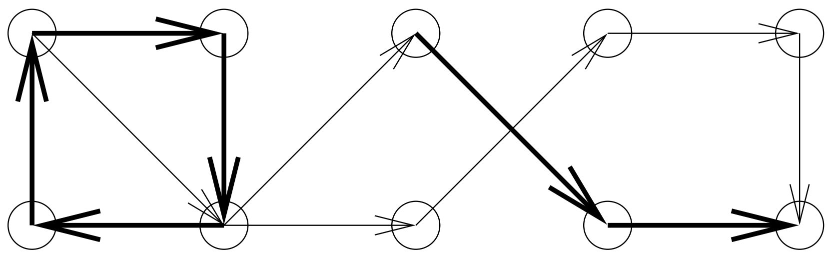
   (b) Give 2 different circuits of M.

   (c) Given the following weights for all elements of E ($c_i = i - 5$), find among all independent sets I of cardinality precisely 5 one of maximum total weight.

i	1	2	3	4	5	6	7	8	9	10
c_i	-4	-3	-2	-1	0	1	2	3	4	5

4. The matching polytope $P \subseteq \mathbb{R}^{|E|}$ of a (not necessarily bipartite) graph $G = (V, E)$ is the convex hull of incidence vectors of all (not necessarily perfect) matchings in G. Argue that the dimension of P is $|E|$.

5. Consider a directed graph $G = (V, A)$. A *1-factor* is a subset of arcs (i.e. directed edges) of G such that the indegree (the number of incoming edges) and outdegree (the number of outgoing edges) of every vertex is at most 1. Here is an example of a 1-factor (the thick edges).



   (a) Give a directed graph G (a few vertices are enough) for which its 1-factors do not form the collection of independent sets of a matroid, and explain why they don't.

   (b) Show that the 1-factors of a directed graph G can be seen as the common independent sets to two matroids M_1 and M_2 on the ground set E.

6. Given a *bipartite* graph $G = (V, E)$ with bipartition $V = A \cup B$ and given an integer k, consider the set of all matchings of cardinality at most k. We know that if there was no constraint on the cardinality (for example, if $k \geq |V|/2$) then the convex hull P of all (incidence vectors of) matchings would be given by

$$P = \{x \in \mathbb{R}^{|E|} : \begin{array}{ll} \sum_{j \in B:(i,j) \in E} x_{ij} \leq 1 & i \in A \\ \sum_{i \in A:(i,j) \in E} x_{ij} \leq 1 & j \in B \\ x_{ij} \geq 0 & (i,j) \in E\} \end{array}$$

In this exercise, you will show that the convex hull P_k of all matchings of cardinality at most k is given by

$$P_k = \{x \in \mathbb{R}^{|E|} : \begin{array}{ll} \sum_{j \in B:(i,j) \in E} x_{ij} \leq 1 & i \in A \\ \sum_{i \in A:(i,j) \in E} x_{ij} \leq 1 & j \in B \\ \sum_{(i,j) \in E} x_{ij} \leq k & \\ x_{ij} \geq 0 & (i,j) \in E\} \end{array}$$

Here are two ways to prove it. Use *either* way for full credit (or both ways for a smiley face).

(a) Show that the underlying matrix A is totally unimodular, where $P_k = \{x : Ax \leq b, x \geq 0\}$. If you use this way, first define what a totally unimodular matrix is, and specify what the matrix A look like. Once you have shown that A is totally unimodular, you do *not* need to continue (saying how this implies that the vertices correspond to matchings of cardinality at most k).

(b) Consider any vertex x^* of P_k. First argue that x^* is in an edge (a face of dimension 1) of P (the matching polytope without restriction on the cardinality), i.e. x^* can be seen as a convex combination of incidence vectors of two *adjacent* matchings M_1 and M_2 of G. Then state (without proof) the condition for two matchings M_1 and M_2 to be adjacent on P. Finally conclude that x^* must have been the incidence vector of either M_1 or M_2.