Massachusetts Institute of Technology — Handout 8
18.433: Combinatorial Optimization — March 4th, 2009
Michel X. Goemans

Problem set 3

This problem set is due in class on Friday March 13th, 2009.

1. Exercise 3-16 from the notes on polyhedral theory.
2. Exercise 3-17 from the notes on polyhedral theory.
3. (Optional for undergraduates.) Consider the bipartite matching polytope. Show that any edge of this polytope (a face of dimension 1) connects two matchings M_1 and M_2 such that $M_1 \Delta M_2$ has only one connected component (it is either an alternating path or an alternating cycle).
4. Given a bipartite graph G and given an integer k, let S_k be the set of all incidence vectors of matchings with at most k edges. We are interested in finding a description of $P_k = conv(S_k)$ as a system of linear inequalities. More precisely, you'll show that $conv(S_k)$ is given by:

$$P_k = \{x : \begin{array}{ll} \sum_j x_{ij} \leq 1 & \forall i \in A \\ \sum_i x_{ij} \leq 1 & \forall j \in B \\ \sum_i \sum_j x_{ij} \leq k & \\ x_{ij} \geq 0 & i \in A, j \in B\}. \end{array}$$

   Without the last constraint, we have shown in lecture that the resulting matrix is totally unimodular.

   (a) With the additional constraint, is the resulting matrix totally unimodular? Either prove it or disprove it.

   (b) Show that P_k indeed equals $conv(S_k)$.

   (Hint: If you haven't shown that the matrix is TUM, you could focus on a vertex of P_k and use the fact that it must belong to a face of dimension 1 (an edge) of the bipartite matching polytope (without the additional cardinality constraint), and then use the previous optional exercise (without proving it).)

   (c) Suppose now that instead of a cardinality constraint on all the edges, our edges are partitioned into E_1 and E_2 and we only impose that our matching has at most k edges from E_1 (and as many as we'd like from E_2). Is it still true that the convex hull of all such matchings is given by simply replacing $\sum_i \sum_j x_{ij} \leq k$ by

$$\sum_i \sum_{j:(i,j)\in E_1} x_{ij} \leq k?$$