Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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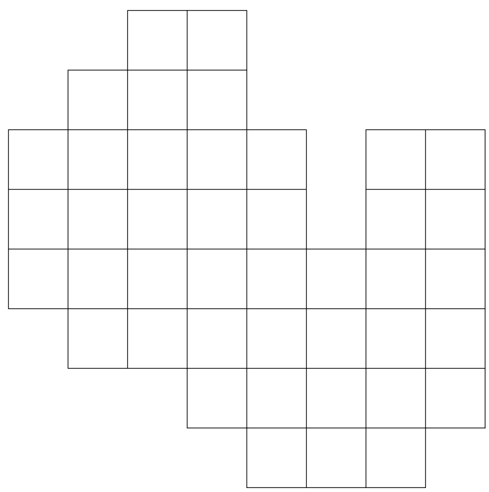
Handout 4
February 11th, 2009

Problem set 1

This problem set is due in class on February 18th, 2009. A large random subset of the problems will be graded.

1. Consider the problem of perfectly tiling a subset of a checkerboard (i.e. a collection of unit squares, see example below) with dominoes (a domino being 2 adjacent squares).
   (a) Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.
   (b) Can the following figure be tiled by dominoes? Give a tiling or a short proof that no tiling exists.



2. Exercise 1-2 on Page 5 of the notes on bipartite matchings.

3. Exercise 1-11 on Pages 10 and 11 of the notes on bipartite matchings.

4. Given a graph $G = (V, E)$, its edge coloring number is the smallest number of colors needed to color the edges in E so that any two edges having a common endpoint have a different color.
   (a) Show that the edge coloring number of a *bipartite* graph G is always equal to its maximum degree Δ (i.e. the maximum over all vertices v of the number of edges incident to v). (Use the previous problem.)
   (b) Give an example of a non-bipartite graph for which the edge coloring number is (strictly) greater than Δ.

5. Consider a bipartite graph $G = (V, E)$ with bipartition (A, B) ($V = A \cup B$). Let $\mathcal{I} = \{X \subseteq A : \text{there exists a matching } M \text{ of } G \text{ such that all vertices of } X \text{ are matched}\}$.

   Show that

   (a) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

   (b) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$ then there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

   (Later in the class, we will discuss matroids, and properties (i) and (ii) form the definition of independent sets of a matroid.)

6. Consider a bipartite graph $G = (V, E)$ with bipartition (A, B). For $X \subseteq A$, define $\text{def}(X) = |X| - |N(X)|$ where $N(X) = \{b \in B : \exists a \in X \text{ with } (a, b) \in E\}$. Let

$$\text{def}_{max} = \max_{X \subseteq A} \text{def}(X).$$

   Since $\text{def}(\emptyset) = 0$, we have $\text{def}_{max} \geq 0$.

   (a) Generalize Hall's theorem by showing that the maximum size of a matching in a bipartite graph G equals $|A| - \text{def}_{max}$.

   (b) For any 2 subsets $X, Y \subseteq A$, show that

$$\text{def}(X \cup Y) + \text{def}(X \cap Y) \geq \text{def}(X) + \text{def}(Y).$$

7. (Optional for any undergrad. Required for any graduate student.) For the assignment problem (minimum cost perfect matching problem in a complete bipartite graph), the greedy algorithm (which repeatedly finds the minimum cost edge disjoint from all the previously selected edges) can lead to a solution whose cost divided by the optimum cost can be arbitrarily large (even for graphs with 2 vertices on each side of the bipartition).

   Suppose now that the cost comes from a metric, even just a line metric. More precisely, suppose that the bipartition is $A \cup B$ with $|A| = |B| = n$ and the ith vertex of A (resp. the jth vertex of B) is associated with $a_i \in \mathbb{R}$ (resp. $b_j \in B$). Suppose that the cost between these vertices is given by $c_{ij} = |a_i - b_j|$.

   Consider the greedy algorithm: select the closest pair of vertices, one from A and from B, match them together, delete them, and repeat until all vertices are matched. For these line metric instances, is the cost of the greedy solution always upper bounded by a constant (independent of n) times the optimum cost of the assignment? If so, prove it; if not, give a family of examples (parametrized by n) such that the corresponding ratio becomes arbitrarily large.