Problem set 6

This problem set is due in class on December 8th, 2005.
Let's go back to the problem from the first problem set:

> The goal of this exercise is to familiarize you with the intrinsic complexity of combinatorial optimization problems. Consider the 7×7 grid drawn in Figure 1. Given an assigment of signs (+ or -) to the vertices of this grid, a *violation* corresponds to either a thick (solid) edge with two different signs at its endpoints or a thin (dashed) edge with identical signs at its endpoints. The goal is to find an assignment of signs which minimize the total number of violations in the grid.

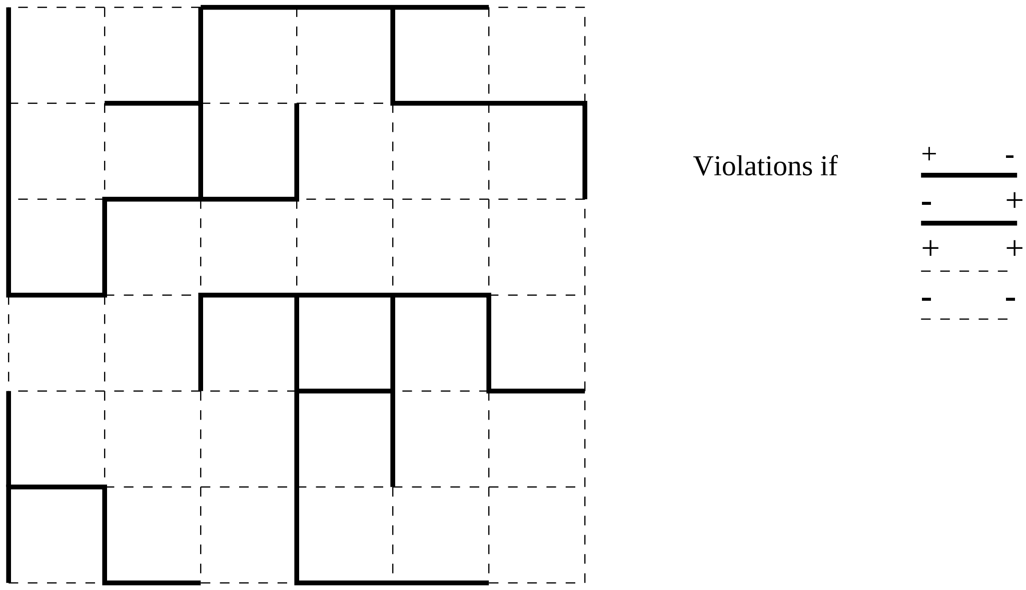

Figure 1: The instance

Consider the general problem for an arbitrary $n \times m$ grid G with all edges either dashed or solid.

1. Consider the dual graph $G^* = (V^*, E^*)$ to G. This is obtained by placing a vertex $v \in V^*$ in every face (including the outside face) of G and an edge between two vertices of G^* (i.e. between two faces of G) if the corresponding two faces (of G) are adjacent (touching). Thus, in the example above, V^* has 36+1=37 vertices (one for the outside face and 36 for the little squares); the vertex of G^* corresponding to the outside face

will have 24 edges incident to it, while the other vertices of G^* will have 4 edges incident to it. Every edge of E corresponds exactly to an edge of E^*; so we could even label them the edges of G and G^* the same way and assume that $E = E^*$. Thus when we refer to the set F of edges with violations, we can view them as edges in G or in G^*.

Argue that for every face F of G (i.e. a vertex v^* of G^*), the number of violated edges along it (i.e. adjacent to v^*) must be even if F has an even number of solid edges, and odd if F has an odd number of solid edges. Let T denote the set of vertices v^* of G^* (i.e. faces of G) with an odd number of solid edges incident to it. A *T-join* F in $G^* = (V^*, E^*)$ is a set $F \subseteq E^*$ such that F has odd degree precisely at the vertices v^* in T.

Show that, from any T-join F, one can find an assignment of signs for the problem such that the violated edges are precisely the edges of F.

2. This means that we have just translated the problem into finding a T-join F of minimum cardinality in the graph $G^* = (V^*, E^*)$. Now, we will construct an auxiliary graph H with weights on the edges. $H = (V(H), E(H))$ is the complete graph on the vertex set $V(H) = T$. Given two vertices u and v in T, the weight w_{uv} of the edge $(u, v) \in E(H)$ is equal to the number of edges in the shortest path between u and v in the graph G^*.

   Show that the problem of finding a minimum cardinality T-join in G^* is equivalent to finding a minimum weight perfect matching in H, by showing that

   - from every perfect matching M in H of weight w, one (you!) can construct a T-join F of cardinality s in G^*, and
   - from every T-join F in G^* of cardinality s, one (you!) can construct a perfect matching M in H of weight (at most) s.

3. In class, we saw how to find a *maximum* weight matching in a graph, while here we need to find a *minimum* weight *perfect* matching. Show a way to transform the latter problem into the former.