Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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Handout 16
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# Problem set 5

This problem set is due in class on November 29th, 2005.

1. At some point during baseball season, each of $n$ teams of the American League has already played several games. Suppose team $i$ has won $w_i$ games so far, and $g_{ij} = g_{ji}$ is the number of games that teams $i$ and $j$ have yet to play. No game ends in a tie, so each game gives one point to either team and 0 to the other. You would like to decide if your favorite team (Red Sox?), say team $n$, can still win. In other words, you would like to determine whether there exists an outcome to the games to be played (remember, with no ties) such that team $n$ has at least as many victories as all the other teams (we allow team $n$ to be tied for first place with other teams).

   Show that this problem can be solved as a maximum flow problem. Explain.

2. Suppose you are given an $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$ with row sums $r_1, \cdots, r_m \in \mathbb{Z}$ and column sums $c_1, \cdots, c_n \in Z$. Some of the entries might not be integral but the row sums and column sums are. Show that there exists a rounded matrix $A'$ with the following properties:

   - row sums and column sums of $A$ and $A'$ are identical,
   - $a'_{ij} = \lceil a_{ij} \rceil$ or $a'_{ij} = \lfloor a_{ij} \rfloor$ (i.e. $a'_{ij}$ is $a_{ij}$ either rounded up or down.).

   (Hint. Think of flows.)

   By the way, this rounding is useful to the census bureau as they do not want to publish statistics that would give too much information on specific individuals. They want to be able to modify the entries without modifying row and column sums.

3. Consider the $s - t$ flow problem on a directed graph in which every directed edge has a lower bound $l(e) = 0$ and an upper bound $u(e) = 1$.

   (a) Write the dual linear program of this maximum flow problem.

   (To make the dual a bit nicer to interpret, it is useful to add a variable $f \in \mathbb{R}$ in the primal, impose that the net flow of out of $s$ equals $f$ and that the net flow out of $t$ equals $-f$, and maximize $f$. In the dual linear program, we will have one variable, say $y_v$, for every vertex $v \in V$ (which corresponds to either the flow balance constraint if $v \notin \{s, t\}$ or the newly introduced constraints if $v \in \{s, t\}$), and also one dual variable, say $z_e$ for every directed edge (arc) $e \in E$ (corresponding to the inequalities $x_e \leq u(e)$).)

   (b) Does this dual always have an optimum solution that is integral?

   (c) Show that for any *integer* solution to the dual (i.e. $y_v, z_e \in \mathbb{Z}$) you can obtain an $s - t$ cut of the same value.

(d) Now suppose you are given a non necessarily integral feasible solution for this dual of value $V$ (in the dual linear program). Show how we can obtain a cut of value at most $V$.

4. Consider an undirected graph $G = (V, E)$ with capacities $u(e)$ for all edges $e \in E$. For any two vertices $s \neq t \in V$, let $f(s,t) = f(t,s)$ to be the capacity of the minimum $s-t$ cut. In this exercise, you'll show that we can represent in a compact way all values $f(s,t)$ (there are $\binom{|V|}{2}$ such values, but only $|V|-1$ of them will be different).

   (a) Let $u_0, u_1, \cdots, u_k \in V$. Show that any $u_0u_k$-cut has to be a $u_iu_{i+1}$ cut for some $i \in \{0, \cdots, k-1\}$. Deduce that:
   $$f(u_0, u_k) \geq \min_{i=0,\cdots,k-1} f(u_i, u_{i+1}).$$

   (b) Consider the complete graph $H$ on the vertex set $V$ in which edge $(u,v)$ has a weight $f(u,v)$ (given by the capacity of the minimum $u-v$ cut in $G$). Let $T$ be a maximum weight (with respect to $f(\cdot,\cdot)$) spanning tree in $H$.

   Prove that for any two vertices $u, v \in V$, we can obtain $f(u,v)$ by looking at the values of $f(\cdot,\cdot)$ along the (unique) path $P_{uv}$ in $T$ between $u$ and $v$ and using the formula:
   $$f(u,v) = \min_{(s,t)\in P_{uv}} f(s,t).$$

5. This is a continuation of the previous exercise. There, it was shown that there exists a spanning tree $T$ which compactly encodes the *capacity* of the minimum $uv$-cut for any $u, v \in V$. Here you will show that you can also encode the cuts themselves. What you will get is a spanning tree $T$ on (the complete graph on) $V$ with labels $f(s,t)$ on its edges such that for any $u$ and $v$, if $(s,t)$ is the edge of $T$ along the path from $u$ to $v$ with smallest label then $f(u,v) = f(s,t)$ and the two connected components of $T \setminus \{(s,t)\}$ define a minimum cut for $u - v$.

   (a) For any $S \subseteq V$, let $C(S)$ be the capacity of the cut $S$, i.e.
   $$C(S) = \sum_{e\in\delta(S)} u(e).$$
   (By definition, $C(\emptyset) = C(V) = 0$.) Prove that $C(S)$ is *submodular.* (Hint. Look at the contribution of each edge to both sides of the inequality defining submodularity.)

   (b) Let $u, v \in V$ be vertices for which $f(u,v)$ is minimum over all $u \neq v$. Let $S_{uv}$ be the corresponding mincut. Our tree $T$ will contain the edge $(u,v)$ and in addition, we are going to make sure that the 2 connected components of $T \setminus \{(u,v)\}$ are $S_{uv}$ and $V \setminus S_{uv}$. Notice that if we take $s \in S_{uv}$ and $t \notin S_{uv}$ then indeed $S_{uv}$ is a minimum $st$-cut.

Now we would like to construct the tree $T$ inside $S_{uv}$ (and then we can do the same for $V \setminus S_{uv}$). Prove that, among all minimum $st$-cuts $S$ (we assume $s \in S$ and $t \notin S$), there exists one for which either $S_{st} \subset S_{uv}$ or $V \setminus S_{uv} \subset S_{st}$.

Show then how we can use this recursively to construct $T$.