Problem set 3

This problem set is due in class on October 13th, 2005 (or in my office on October 12th for those observing Yom Kippur).

1. A *stable set* S (sometimes, it is called also an independent set) in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in S. If we let P denote the convex hull of all (incidence vectors of) stable sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for P.

   (a) Give a graph G for which P is *not* equal to

   $$\begin{aligned}\{x \in \mathbb{R}^{|V|} : \quad & x_i + x_j \leq 1 \quad && \text{for all } (i,j) \in E \\ & x_i \geq 0 && \text{for all } i \in V\}\end{aligned}$$

   (b) Show that if the graph G is bipartite then P equals

   $$\begin{aligned}\{x \in \mathbb{R}^{|V|} : \quad & x_i + x_j \leq 1 \quad && \text{for all } (i,j) \in E \\ & x_i \geq 0 && \text{for all } i \in V\}\end{aligned}$$

2. Suppose we have n activities to choose from. Activity i starts at time t_i and ends at time u_i (or more precisely just before u_i); if chosen, activity i gives us a profit of p_i units. Our goal is to choose a subset of the activities which do not overlap (nevertheless, we can choose an activity that ends at t and one that starts at t) and such that the total profit (i.e. sum of profits) of the selected activities is maximum.

   (a) Defining x_i as a variable that represents whether activity i is selected ($x_i = 1$) or not ($x_i = 0$), write an integer program of the form $\max\{p^T x : Ax \leq b, x \in \{0,1\}^n\}$ that would solve this problem.

   (b) Show that the matrix A is totally unimodular, implying that one can solve this problem by solving the linear program $\max\{p^T x : Ax \leq b, 0 \leq x_i \leq 1 \text{ for every } i\}$.

3. Given two extreme points a and b of a polyhedron P, we say that they are adjacent if the line segment between them forms an edge (i.e. a face of dimension 1) of the polyhedron P. This can be rephrased by saying that a and b are adjacent on P if and only if there exists a cost function c such that a and b are the only two extreme points of P minimizing $c^T x$ over P.

   Consider the polyhedron (polytope) P defined as the convex hull of all perfect matchings in a (not necessarily bipartite) graph G. Give a necessary and sufficient condition for two matchings M_1 and M_2 to be adjacent on this polyhedron (hint: think about $M_1 \triangle M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$) and prove that your condition is necessary and sufficient.

4. Let $e_k \in \mathbb{R}^n$ ($k = 0, \ldots, n-1$) be a vector with the first k entries being 1, and the following $n-k$ entries being -1. Let $S = \{e_0, e_1, \ldots, e_{n-1}, -e_0, -e_1, \ldots, -e_{n-1}\}$, i.e. S consists of all vectors consisting of $+1$ followed by -1 or vice versa. In this problem set, you will study $conv(S)$.

   (a) Consider any vector $a \in \{-1, 0, 1\}^n$ such that (i) $\sum_{i=1}^n a_i = 1$ and (ii) for all $j = 1, \ldots, n-1$, we have $0 \leq \sum_{i=1}^j a_i \leq 1$. (For example, for $n = 5$, the vector $(1, 0, -1, 1, 0)$ satisfies these conditions.) Show that $\sum_{i=1}^n a_i x_i \leq 1$ and $\sum_{i=1}^n a_i x_i \geq -1$ are valid inequalities for $conv(S)$.

   (b) How many such inequalities are there?

   (c) Show that any such inequality defines a facet of $conv(S)$.

   (This can be done in several ways. Here is one approach, but you are welcome to use any other one as well. First show that either e_k or $-e_k$ satisfies this inequality at equality, for any k. Then show that the resulting set of vectors on the hyperplane are affinely independent (or uniquely identifies it).)

   (d) Show that the above inequalities define the entire convex hull of S.

   (Again this can be done in several ways. One possibility is to consider the 3rd technique from lecture, which shows that for any cost function, there exists a inequality in our description satisfied by equality by all optimum solutions.)