Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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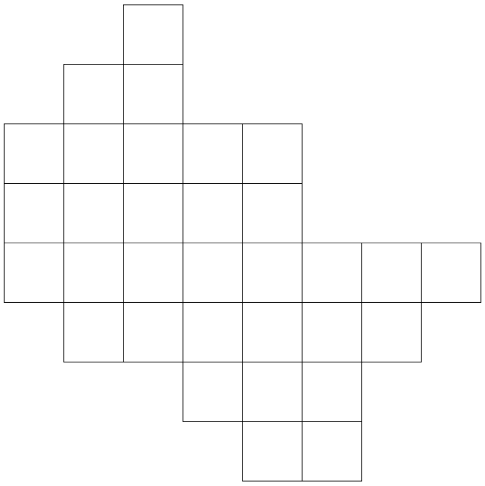
Handout 6
September 22th, 2005

Problem set 2

This problem set is due in class on September 29th, 2005.

1. Given a bipartite graph $G = (V, E)$ with bipartition A, B ($V = A \cup B$), Hall's theorem gives a condition for the existence of a matching of size $|A|$ (i.e. every vertex in A is matched): For every $S \subseteq A$ we have $|N(S)| \geq |S|$, where $N(S) = \{b \in B : \exists a \in S$ with $(a, b) \in E\}$. Deduce Hall's theorem from König's theorem.

2. Can the following figure be tiled by dominoes (a domino being 2 adjacent squares)? Give a tiling or prove that no tiling exists.



3. Consider a bipartite graph $G = (V, E)$ in which every vertex has degree k (a so-called k-regular bipartite graph). Prove that such a graph always has a perfect matching in two different ways:

   (a) by using König's theorem,

   (b) by using the linear programming formulation we saw in lecture for the assignment problem.

4. Using the previous problem, show that the edges of a k-regular bipartite graph G can be partitioned into k matchings (i.e. the number of colors needed to color the edges of a k-regular bipartite graph such that no two edges with an endpoint in common have the same color is precisely k).

5. In this exercise, you will do a little experiment with the (minimum cost) assignment problem. Take a complete bipartite graph with n vertices on each side of the bipartition, and let us assume that all c_{ij} (for $i, j \in \{1, \cdots, n\}$) are all independent uniform random variables between 0 and 1. Take 5 different values for n (the largest being a few hundreds) and for each compute the *minimum* cost assignment value for 5 instances. Any guess on how this value increases as n goes to infinity. Does it seem to converge? To what value? Surprised? (Yes, you should be, and it is normal that you do not understand why.)

   To solve the assignment problem, you can use MATLAB (it is available on athena). To access MATLAB on athena, you need to type:

   ```
   >> add matlab
   >> matlab
   ```

   If you have never used MATLAB before, you can find some tutorial on the 18.06 webpage `http://web.mit.edu/18.06/www/`. There is a MATLAB m-file at `http://www-math.mit.edu/~goemans/bghungar.m` which implements the Hungarian method (just put it in the directory from which you run MATLAB). If C is an $n \times n$ matrix, then

   ```
   >> a=bghungar(C)
   ```

   gives a vector a such that $(1, a(1)), (2, a(2)), \cdots$ is the *maximizing* matching. So to get the value of the *minimum* assignment, you can just do:

   ```
   >> a=bghungar(-C);
   >> v=sum(diag(C(1:n,a(1:n))))
   ```

6. An edge cover of a graph is a subset F of edges such that for every vertex $v \in V$ there exists an edge of F adjacent to it. Consider a (not necessarily bipartite) connected graph $G = (V, E)$ with $n = |V|$ vertices, and assume that the maximum matching has size $\nu(G)$. Prove that the minimum cardinality of an edge cover is precisely $n - \nu(G)$.

7. (Tricky.) Show that any 3-regular 2-connected graph $G = (V, E)$ (not necessarily bipartite) has a perfect matching. (A 2-connected graph has at least 2 edges in every cutset; a cutset being the edges between S and $V \setminus S$ for some vertex set S.)