Lecture notes on matroid union

From any matroid $M = (E, \mathcal{I}(M))$, one can construct a dual matroid $M^* = (E, \mathcal{I}(M^*))$.

Theorem 1 *Let $\mathcal{I}(M^*) = \{X \subseteq E(M) : E(M) \setminus X$ contains a base of $M\}$. Then $M^* = (E, \mathcal{I}(M^*))$ is a matroid with rank function*

$$r_{M^*}(X) = |X| + r_M(E \setminus X) - r_M(E).$$

There are several ways to show this. One is to first show that indeed the size of the largest subset of X in $\mathcal{I}(M^*)$ has cardinality .$|X| + r_M(E \setminus X) - r_M(E)$ and then show that r_{M^*} satisfies the three conditions that a rank function of a matroid needs to satisfy (the third one, submodularity, follows from the submodularity of the rank function for M).

One can use Theorem 1 and matroid intersection to get a good characterization of when a graph $G = (V, E)$ has two edge-disjoint spanning trees. Indeed, letting M be the graphic matroid of the graph G, we get that G has two edge-dsjoint spanning trees if and only if

$$\max_{S \in \mathcal{I}(M) \cap \mathcal{I}(M^*)} |S| = |V| - 1.$$

For the graphic matroid, we know that $r_M(F) = n - \kappa(F)$ where $n = |V|$ and $\kappa(F)$ denotes the number of connected components of (V, F). But by the matroid intersection theorem, we can write:

$$\begin{aligned}
\max_{S \in \mathcal{I}(M) \cap \mathcal{I}(M^*)} |S| &= \min_{E_1 \subseteq E} r_M(E_1) + r_{M^*}(E \setminus E_1) \\
&= (n - \kappa(E_1)) + (|E \setminus E_1| + \kappa(E) - \kappa(E_1)) \\
&= n + 1 + |E \setminus E_1| - 2\kappa(E_1),
\end{aligned}$$

where we replaced $\kappa(E)$ by 1 since otherwise G would even have one spanning tree. Rearranging terms, we get that G has two edge-dsjoint spanning trees if and only if for all $E_1 \subseteq E$, we have that $E \setminus E_1 \geq 2(\kappa(E_1) - 1)$. If this inequality is violated for some E_1, we can add to E_1 any edge that does not decrease $\kappa(E_1)$. In other words, if the connected components of E_1 are $V_1, V_2, \cdots, V_p$ then we can assume that $E_1 = E \setminus \delta(V_1, V_2, \cdots V_p)$ where $\delta(V_1, \cdots, V_p) = \{(u, v) \in E : u \in V_i, v \in V_j \text{ and } i \neq j\}$. Thus we have shown:

Theorem 2 *G has two edge-disjoint spanning trees if and only if for all partitions V_1, $V_2, \cdots V_p$ of V, we have*

$$|\delta(V_1, \cdots, V_p)| \geq 2(p - 1).$$

Going back to the spanning tree game, it is now clear that if the graph does not have two edge-disjoint spanning trees then player 1 has a winning strategy. He/she just needs to delete edges from $\delta(V_1, \cdots, V_p)$ for the partition given by theorem 2.

Theorem 2 can be generalized.

Theorem 3 *G has k edge-disjoint spanning trees if and only if for all partitions $V_1, V_2, \cdots V_p$ of V, we have*
$$|\delta(V_1, \cdots, V_p)| \geq k(p-1).$$

From two matroids $M_1 = (E, \mathcal{I}(M_1))$ and $M_2 = (E, \mathcal{I}(M_2))$, we can also define its union by $M_1 \cup M_2 = (E, \mathcal{I})$ where $\mathcal{I} = \{S_1 \cup S_2 : S_1 \in \mathcal{I}(M_1), S_2 \in \mathcal{I}(M_2)\}$.

We can show that:

Theorem 4 *$M_1 \cup M_2$ is a matroid with rank function*
$$r_{M_1 \cup M_2}(X) = min_{F \subseteq E} \{|E \setminus F| + r_{M_1}(F) + r_{M_2}(F)\}.$$

Proof: To show that it is a matroid, assume that $X, Y \in \mathcal{I}$ with $|X| < |Y|$. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ where $X_1, Y_1 \in \mathcal{I}(M_1)$ and $X_2, Y_2 \in \mathcal{I}(M_2)$. We can furthermore assume that the X_i's are disjoint and so are the Y_i's. Finally we assume that among all choices for X_1, X_2, Y_1 and Y_2, we choose the one maximizing $|X_1 \cap Y_1| + |X_2 \cap Y_2|$. Since $|Y| > |X|$, we can assume that $|Y_1| > |X_1|$. Thus, there exists $e \in (Y_1 \setminus X_1)$ such that $X_1 \cup \{e\}$ is independent for M_1. The maximality implies that $e \notin X_2$ (otherwise consider $X_1 \cup \{e\}$ and $X_2 \setminus \{e\}$). But this implies that $X \cup \{e\} \in \mathcal{I}$ as desired.

We now show the expression for the rank function. The fact that it is $\leq$ is obvious as an independent set $X \in \mathcal{I}$ has size $|X \setminus F| + |X \cap F| \leq |E \setminus F| + r_{M_1}(F) + r_{M_2}(F)$ and this is true for any F.

For the converse, notice that $X \in \mathcal{I}$ is such that $X = X_1 \cup X_2$ with $X_1 \in \mathcal{I}(M_1)$ and $X_2 \in \mathcal{I}(M_2)$. We can furthermore assume that X_1 and X_2 are disjoint and that $r_{M_2}(X_2) = r_{M_2}(E)$ (otherwise add elements to X_2 and possibly remove them from X_1). Thus we can assume that $|X| = |X_1| + r_{M_2}(E)$ and that $X_1 \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2^*)$. The proof is completed by using the matroid intersection theorem and Theorem 1:

$$\begin{aligned}
r_{M_1 \cup M_2}(E) &= \max_{X_1 \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2^*)} (|X_1| + r_{M_2}(E)) \\
&= \min_{E_1 \subseteq E} \left(r_{M_1}(E_1) + r_{M_2^*}(E \setminus E_1) + r_{M_2}(E)\right) \\
&= \min_{E_1 \subseteq E} (r_{M_1}(E_1) + |E \setminus E_1| + r_{M_2}(E_1) - r_{M_2}(E) + r_{M_2}(E)) \\
&= \min_{E_1 \subseteq E} (|E \setminus E_1| + r_{M_1}(E_1) + r_{M_2}(E_1)),
\end{aligned}$$

as desired. △