18.415/6.854 Advanced Algorithms September 19, 2001

Problem Set Solution 1

1. **Consider $P = \{x : Ax \leq b, x \geq 0\}$, where A is $m \times n$. Show that if x is a vertex of P then we can find sets I and J with the following properties.**

   (a) $I \subseteq \{1, \ldots, m\}$, $J \subseteq \{1, \ldots, n\}$ and $|I| = |J|$.

   (b) A_J^I is invertible where A_J^I is the submatrix of A corresponding to the rows in I and the columns in J.

   (c) $x_j = 0$ for $j \notin J$ and $x_J = (A_J^I)^{-1} b^I$ where b^I denotes the restriction of b to the indices in I.

   (Hint: Consider $Q = \{(x, s) : Ax + Is = b, x \geq 0, s \geq 0\}$.)

   Using the hint we turn our attention to $Q = \{(x, s) : Ax + Is = b, x \geq 0, s \geq 0\}$. If we let (x, s) be a pair such that $x \in P$ and s is the unique vector of slack variables associated with x ($s = b - Ax$), it is not hard to show that if x is a vertex of P then (x, s) is a vertex of Q. Assume that x is a vertex of P but there is a (y, t) such that $(x, s) \pm (y, t) \in Q$ and $(y, t) \neq 0$. Then we have $A(x \pm y) + (s \pm t) = b, x \pm y \geq 0, s \pm t \geq 0$. This implies that $A(x + y) \leq b$ and $A(x - y) \leq b$. Since x is a vertex, this implies $y = 0$. Then solving for t in $Ax + (s + t) = b$ we find that it must be zero as well which contradicts $(y, t) \neq 0$.

   We can now take advantage of the fact that Q is in the special form ($Ax = b, x \geq 0$). If (x, s) is a vertex of Q then there is a subset $B \subseteq \{1, \ldots, n + m\}$ such that $|B| = m$ and

   (a) $(x, s)_N = 0$ for $N = \{1, \ldots, n + m\} \setminus B$

   (b) $(A|I)_B$ is non singular

   (c) $(x, s)_B = (A|I)_B^{-1} b \geq 0$.

   Let $J \subseteq B$ be the set of columns involving A of $(A|I)$. Notice that if $|J| = k$, k of the x variables are basic and $m - k$ of the s variables are basic. So $x_j = 0$ for $j \notin J$ and and k of the s variables are zero. We take the rows corresponding to the zero components of s as the set I. Then

   $$A^I x = A_J^I x_J = b^I$$

   and A_J^I is invertible, so

   $$x_J = (A_J^I)^{-1} b^I.$$

2. **In his paper in FOCS 92, Tomasz Radzik needs a result of the following form (Page 662 of the Proceedings):**

   Lemma 1 ***Let*** $c \in \mathbb{R}^n$ ***and*** $y_k \in \{0,1\}^n$ ***for*** $k = 1, \ldots, q$ ***such that*** $2|y_{k+1}c| \leq |y_k c|$ ***for*** $k = 1, \ldots, q-1$***. Assume that*** $|y_q c| = 1$***. Then*** $q \leq f(n)$***.***

   In other words, given any set of n (possibly negative) numbers, one cannot find more than $f(n)$ subsums of these numbers which decrease in absolute value by a factor of at least 2.

   Radzik proves the result for $f(n) = O(n^2 \log n)$ and conjectures that $f(n) = O^*(n)$ where O^* denotes the omission of logarithmic terms. Using linear programming, you are asked to improve his result to $f(n) = O(n \log n)$.

   (a) **Given a vector c and a set of q subsums satisfying the hypothesis of the Lemma, write a set of inequalities in the variables $x_i \geq 0, i = 1 \ldots n$, such that $x_i = |c_i|$ is a feasible vector, and for any feasible vector x' there is a corresponding vector c' satisfying the hypothesis of the Lemma for the same set of subsums.**

   We have a set of inequalities of the form $2|y_{i+1}c| \leq |y_i c|$ for $1 \leq i \leq q-1$ and $|y_1 c| = 1$, and we have a vector c which satisfies them. To obtain a linear system, we need to remove the absolute value signs. Let $y'_i = y_i sgn(y_i c)$. Then $|y_i c| = y'_i c$. The system becomes $2y'_{i+1}c \leq y'_i c$ for $1 \leq i \leq q-1$ and $y'_1 c = 1$ and the original c is still feasible. To limit the solution space to vectors of the form $x \geq 0$ a similar trick is used. We replace elements c_j that are negative by $-x_j$ and non-negative elements by x_j. The linear system that remains has a solution $x \geq 0$, namely $x_j = |c_j|$. A solution c' to the original inequalities can be obtained from any feasible x in the newly constructed set of inequalities by negating the value of the i^{th} element of x if the i^{th} element of c was negative. This results in the same number of inequalities as we had originally, namely q.

   (b) **Prove that there must exist a vector c' satisfying the hypothesis of the Lemma, with c' of the form $(d_1/d, d_2/d, \ldots, d_n/d)$ for some integers $|d|, |d_1|, \ldots, |d_n| = 2^{O(n \log n)}$.**

   (Hint: see Problem 1.)

   The polytope defined by the inequalities above is nonempty. This means that the polytope has a vertex. Our system looks like $Ax \geq b$ and $x \geq 0$, where A is a q by n matrix containing entries between -3 and 3 (since every entry is the difference of two integers, one being ± 2 or 0, the other being ± 1 or 0) and b has one nonzero element which is ± 1. From the first problem, we know that the nonzero components, say y, of a vertex satisfy

$A'y = b'$ where A' is an invertible submatrix of A and b' is a subvector of b. Notice that $|det(A')| \leq n!3^n = 2^{O(n \log n)}$. As in class (by Cramer's rule), we know that we can set d to be $|det(A')|$ and the (nonzero) d_i's to be determinants of submatrices of A'. By the same argument, these determinants are also upper bounded by $2^{O(n \log n)}$, proving the result.

(c) **Deduce from the above that $f(n) = O(n \log n)$.**
Multiplying the vector c' by d yields an integer solution to $2y'_{i+1}x \leq y'_i x$ for $1 \leq i \leq q-1$ with elements of value $2^{O(n \log n)}$. Thus the largest sum that can be obtained by a subset is $2^{O(n \log n)}$. As the first subset sums to at least one (since the d_i's are integers), the number of times the sum can double is at most $O(n \log n)$.

3. **The maximum flow problem on the directed graph $G = (V, E)$ with capacity function u (and lower bounds 0) can be formulated by the following linear program:**

$$\max w$$

subject to

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} w & i = s \\ 0 & i \neq s, t \\ -w & i = t \end{cases}$$

$$x_{ij} \leq u_{ij}$$

$$0 \leq x_{ij}.$$

(x_{ij} represents the flow on edge (i, j); the flow has to be less or equal to the capacity on any edge and flow conservation must be satisfied at every vertex except the source s, where we try to maximize the flow, and the sink t.)

(a) **Show that its dual is equivalent to:**

$$\min \sum_{(i,j) \in E} u_{ij} y_{ij}$$

subject to

$$z_i - z_j + y_{ij} \geq 0 \qquad (i, j) \in E$$

$$z_s = 0, z_t = 1$$

$y_{ij} \geq 0.$

This is an immediate consequence of the definition of the dual. If one takes the dual of the system of equations and inequalities above, then one gets

$$\min \sum_{(i,j)\in E} u_{ij} y_{ij}$$

$$z_i - z_j + y_{ij} \geq 0 \qquad \forall (i,j) \in E$$

$$z_t - z_s = 1$$

$$y_{ij} \geq 0.$$

Since adding a constant to all z_i's doesn't change anything, we can require that $z_s = 0$ and $z_t = 1$.

(b) **A cut is a set of edges of the form $\{(i,j) \in E : i \in S, j \notin S\}$ for some $S \subset V$ and its value is**

$$W = \sum_{(i,j)\in E: i\in S, j\notin S} u_{ij}.$$

It separates s from t if $s \in S$ and $t \notin S$.

Show that a cut of value W separating s from t corresponds to a feasible solution (y, z) of the dual program such that

$$W = \sum_{(i,j)\in E} u_{ij} y_{ij}.$$

For a cut defined by $S \subset V$, we define $z_i = 0$ for $i \in S$, $z_i = 1$ for $i \notin S$, $y_{ij} = 1$ for $i \in S, j \notin S$, $(i,j) \in E$ and $y_{ij} = 0$ otherwise. Obviously, (y, z) is a feasible solution and its value is

$$\sum_{(i,j)\in E} u_{ij} y_{ij} = \sum_{i\in S, j\notin S} u_{ij} = W$$

(c) **Given any (not necessarily integral) optimal solution y^*, z^* of the dual linear program and an optimal solution x^* of the primal linear program, show how to construct from z^* a cut separating s from t of value equal to the maximum flow.**

(Hint: Consider the cut defined by $S = \{i : z_i \leq 0\}$ and use complementary slackness conditions.)

We divide the vertices into two sets defined as follows:

$$\begin{aligned} S &= \{i \in V \mid z_i^* \leq 0\} \\ \overline{S} &= \{i \in V \mid z_i^* > 0\} \end{aligned}$$

Every edge (i,j) with $i \in S$ and $j \notin S$ satisfies $z_i^* - z_j^* + y_{ij}^* \geq 0$. Since $z_i^* \leq 0$ and $z_j^* > 0$ we have that $y_{ij}^* > 0$, which by complementary slackness implies that $x_{ij}^* = u_{ij}^*$. Every edge (j,i) with $i \in S$ and $j \notin S$ satisfies $z_j^* - z_i^* + y_{ji}^* > 0$, since $z_j^* > 0$ and $z_i^* \leq 0$. By complementary slackness we have that $x_{ji}^* = 0$. Thus, we can write

$$\sum_{i\in S, j\notin S} u_{ij} \quad = \quad \sum_{i\in S, j\notin S} x_{ij}^* - \sum_{i\notin S, j\in S} x_{ij}^* = \sum_{i\in S}\left(\sum_j x_{ij}^* - \sum_j x_{ji}^*\right) = w^*,$$

which is the value of the maximum flow.

(d) **Deduce the max-flow–min-cut theorem: the value of the maximum flow from s to t is equal to the value of the minimum cut separating s from t.**

From (b) and weak duality, the value of any cut is greater or equal to the maximum flow value. By the analysis above, we can find a cut which is equal to the maximum flow. Thus, the minimum cut value must be the same as the maximum flow value.

4. **Consider the following property of vector sums.**

Theorem 2 ***Let*** $v_1, \ldots, v_n$ ***be*** $d-$***dimensional vectors such that*** $\|v_i\| \leq 1$ ***for*** $i = 1, \ldots, n$ ***(where*** $\|.\|$ ***denotes*** **any** ***norm) and***

$$\sum_{i=1}^n v_i = 0.$$

Then there exists a permutation π ***of*** $\{1, \ldots, n\}$ ***such that***

$$\left\|\sum_{j=1}^k v_{\pi(j)}\right\| \leq d$$

for $k = 1, \ldots, n$.

In this problem, you are supposed to prove this theorem by using linear programming techniques.

(a) **Suppose we have a nested sequence of sets**

$$\{1, \ldots, n\} = V_n \supset V_{n-1} \supset \ldots \supset V_d$$

where $|V_k| = k$ **for** $k = d, d+1, \ldots, n$. **Suppose further that we have numbers** λ_{ki} **satisfying:**

$$\sum_{i\in V_k} \lambda_{ki} v_i = 0, \qquad (1)$$

$$\sum_{i \in V_k} \lambda_{ki} = k - d, \tag{2}$$

$$0 \le \lambda_{ki} \le 1 \qquad i \in V_k, \tag{3}$$

for $k = d, \ldots, n$. Define a permutation π as follows: set $\pi(1), \ldots, \pi(d)$ to be elements of V_d in any order, and set $\pi(k)$ to be the unique element in $V_k \setminus V_{k-1}$ for $k = d+1, \ldots, n$.

Show that this permutation satisfies the conditions of Theorem 2.

For $k \le d$, the theorem is trivial. By the definition of π and λ_{ki}, for $k > d$ we have

$$\left\| \sum_{j=1}^{k} v_{\pi(j)} \right\| = \left\| \sum_{i \in V_k} v_i \right\| = \left\| \sum_{i \in V_k} (1 - \lambda_{ki}) v_i \right\| \le \sum_{i \in V_k} (1 - \lambda_k) = d.$$

(b) **Show that there exist λ_{ni}, $i = 1 \ldots n$, satisfying (1), (2) and (3) for $k = n$.**

We choose simply

$$\lambda_{ni} = 1 - \frac{d}{n}.$$

Then

$$\sum_{i \in V_n} \lambda_{ni} v_i = (1 - \frac{d}{n}) \sum_{i \in V_n} v_i = 0$$

and

$$\sum_{i \in V_n} \lambda_{ni} = n - d.$$

(c) **Suppose we have constructed $V_n, \ldots, V_{k+1}$ and λ_{ji} for $j = k+1, \ldots, n$ and $i \in V_j$ satisfying (1), (2) and (3) for $k+1, \ldots, n$ (where $k \ge d$). Prove that the following system of $d+1$ equalities ((4) contains d equalities), $k+1$ inequalities and $k+1$ nonnegativity constraints has a solution with at least one $\beta_i = 0$:**

$$\sum_{i \in V_{k+1}} \beta_i v_i = 0, \tag{4}$$

$$\sum_{i \in V_{k+1}} \beta_i = k - d, \tag{5}$$

$$0 \le \beta_i \le 1 \qquad i \in V_{k+1}. \tag{6}$$

Deduce the existence of the nested sequence and the λ's as described in (a).

By the induction hypothesis, we have

$$\beta_i = \frac{k-d}{k+1-d}\,\lambda_{k+1,i}$$

satisfying our inequalities, so the polytope of feasible solutions is nonempty. We want to find a solution with at least one zero coordinate. Consider a vertex of the polytope and suppose that for each i, $\beta_i > 0$. There are $k+1$ coordinates summing up to $k-d$, so at most $k-d-1$ of them can be equal to 1. (If $k-d$ coordinates are equal to 1, the rest is zero.) Therefore we have at least $d+2$ coordinates $\beta_i, 0 < \beta_i < 1$. Let's denote this set of coordinates by J. The corresponding vectors $v_j, j \in J$ cannot be affinely independent, so there exists a linear combination with $\gamma \neq 0$ such that

$$\sum_{j \in J} \gamma_j v_j = 0,$$

$$\sum_{j \in J} \gamma_j = 0.$$

For a small enough $\epsilon > 0$, we obtain two feasible solutions by replacing the coordinates of β_j for $j \in J$ by $\beta_j \pm \epsilon\gamma_j$, which contradicts the assumption that β_i is a vertex. Therefore a vertex has always a zero coordinate and by removing this coordinate we obtain the subset V_k and the corresponding coefficients $\lambda_{ki} = \beta_i$ which completes the induction.

5. **Consider the following optimization problem with "robust conditions":**

$$\min\{c^T x : x \in \mathbb{R}^n; Ax \geq b \textbf{ for any } A \in F\},$$

where $b \in \mathbb{R}^m$ and F is a set of $m \times n$ matrices:

$$F = \{A : \forall i,j; a_{ij}^{min} \leq a_{ij} \leq a_{ij}^{max}\}.$$

(a) **Considering F as a polytope in $\mathbb{R}^{m \times n}$, what are the vertices of F?**

F is an $(m \times n)$-dimensional product of intervals. It has 2^{mn} vertices $A^{(k)}$ where each coordinate $a_{ij}^{(k)}$ is either $a_{ij}^{\min}$ or $a_{ij}^{\max}$.

(b) **Show that instead of the conditions for all $A \in F$, it is enough to consider the vertices of F. Write the resulting linear program. What is its size?**

Suppose that x satisfies

$$A^{(k)}x \geq b$$

for every vertex $A^{(k)}$. Any $A \in F$ can be written as a convex linear combination of the vertices:

$$A = \sum_k \lambda_k A^{(k)},$$

$$\sum_k \lambda_k = 1, \lambda_k \geq 0.$$

By taking the corresponding linear combination of inequalities (with non-negative coefficients), we get

$$\sum_k \lambda_k A^{(k)}x \geq \sum_k \lambda_k b$$

which is

$$Ax \geq b.$$

Therefore x is a feasible solution if and only if it satisfies the condition for every vertex of F. We can write the optimization problem in the following form:

$$\min\{c^T x : \forall k; A^{(k)}x \geq b\}.$$

This is a linear program; however, it has an exponential number of inequalities, namely $m2^{mn}$, in n variables.

(c) **Derive a more efficient description of the linear program: Write the condition on x given by one row of A, for all choices of A. Formulate this condition as a linear program. Use duality and formulate the original problem as a linear program. What is the size of this one?**

Let us consider a fixed vector x. It is feasible if the following condition is satisfied for each row a_i of the matrix A:

$$\forall a_{ij} \in [a_{ij}^{\min}, a_{ij}^{\max}]; \sum_j a_{ij}x_j \geq b_i.$$

We can regard this condition as a linear programming problem:

$$b_i \leq \min\left\{\sum_j x_j a_{ij} : a_{ij} \geq a_{ij}^{\min}, -a_{ij} \geq -a_{ij}^{\max}\right\}.$$

Note that the variables are now a_{ij}, while x is fixed! By duality, we get an equivalent condition for a linear program with variables p_{ij}, q_{ij}:

$$b_i \leq \max\left\{\sum_j p_{ij}a_{ij}^{\min} - \sum_j q_{ij}a_{ij}^{\max} : p_{ij} - q_{ij} = x_j, p_{ij} \geq 0, q_{ij} \geq 0\right\}.$$

This means that x is feasible if there exist $p_{ij} \geq 0, q_{ij} \geq 0$ such that

$$p_{ij} - q_{ij} = x_j$$

and

$$\sum_j p_{ij} a_{ij}^{\min} - \sum_j q_{ij} a_{ij}^{\max} \geq b_i.$$

All together, we can write our optimization problem as the following:

$$\begin{array}{lrl} \min\{c^T x : & p_{ij} - q_{ij} - x_j = 0 & \forall i, j \\ & \sum_j p_{ij} a_{ij}^{\min} - \sum_j q_{ij} a_{ij}^{\max} \geq b_i & \forall i \\ & p_{ij}, q_{ij} \geq 0 & \forall i, j\} \end{array}$$

which is a linear program in variables x_j, p_{ij}, q_{ij}. It has $2mn + n$ variables, mn equalities, m inequalities and $2mn$ nonnegativity constraints. The linear program from part (b) has size which is exponential in the size of this one.