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Lecture 2

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# 1 Vertices of polyhedral sets

Last time, we defined the *vertex* of a linear program, and we proved one direction of the theorem below (see notes on Linear Programming). Now we prove the other direction.

**Theorem 1** *Let $x \in P$, where $P = \{x : Ax = b, x \geq 0\}$. Define $A_x$ as the submatrix of $A$ of columns $j$ for which $x_j > 0$. Then $x$ is a vertex of $P$ if and only if $A_x$ has linearly independent columns.*

**Proof of Theorem 1:** Last time we proved that if the columns of $A_x$ are linearly dependent, then $x$ is not a vertex. Now, we show that if $x$ is not a vertex, then the columns of $A_x$ are linearly dependent. Assume that $x$ is not a vertex. Then by the definition of a vertex, $\exists y \neq 0$ such that $x + y, x - y \in P$. This means that $Ax + Ay = b$ and $Ax - Ay = b$, so $Ay = 0$, and also $x - y \geq 0$ and $x + y \geq 0$, therefore $y_j = 0$, for every $j$ such that $x_j = 0$. Thus, since $y \neq 0$, $A_y$, the submatrix containing columns $j$ of $A$ such that $y_j > 0$, has dependent columns. Since every column in $A_y$ is a column in $A_x$, therefore $A_x$ has linearly dependent columns. □

# 2 Bases and basic feasible solutions

Let $P = \{x : Ax = b, x \geq 0\}$, where $A$ is an $m \times n$ matrix. We define a basis, a basic solution, and a basic feasible solution for $P$ as follows:

**Definition 1** *A subset $B$ of $\{1, 2, \ldots, n\}$ of size $m$ is called a* basis *if $A_B$, the matrix consisting of columns of $A$ that correspond to the indices in $B$, is invertible.*

**Definition 2** *A vector $x$ is called a* basic solution *to $Ax = b$ if and only if there is a basis $B$ such that*

- *$x_j = 0$ if $j \notin B$,*
- *$A_B x_B = b$, or equivalently, $x_B = A_B^{-1} b$.*

**Definition 3** *$x$ is a* basic feasible solution *if in addition to the conditions above, we have $A_B^{-1} b \geq 0$.*

Without loss of generality, we can assume $Rank(A) = m$ ($A$ has full row rank). The following lemma shows a correspondence between vertices and basic feasible solutions of $P$.

**Theorem 2** *Let $A$ be an $m \times n$ matrix with full row rank. Then for every $x$, $x$ is a basic feasible solution if and only if $x$ is a vertex.*

**Proof:** By the definition above, we can see that if $x$ is a basic feasible solution, then $x$ is a vertex. Also, if $x$ is a vertex, we show that it is a basic feasible solution. To show this, let $S = \{j : x_j > 0\}$ and consider three cases:

- $|S| > m$. This case cannot happen since by Theorem 1 columns of $A_S$ must be linearly independent, but there cannot be more than $m$ linearly independent columns.

- $|S| = m$. In this case, we are done since $x$ is a vertex and $|S| = m$, so $x$ is a basic feasible solution corresponding to the basis $S$.
- $|S| < m$. By Theorem 1, the columns of $A_S$ is a linearly independent subset of the set of columns of $A$. By basic linear algebra, since the rank of $A$ is $m$, we can augment $S$ to find a set $A_B$ of $m$ linearly independent columns of $A$. Now, $B$ is a basis and $x$ is a basic feasible solution corresponding to $B$.

$\square$

**Remark 1** *Note that the above correspondence between vertices of $P$ and basic feasible solutions is not one-one, i.e., there can be many bases corresponding to the same vertex. When several bases correspond to the same vertex, we say we have* degeneracy.

**Remark 2** *By the above theorem, the number of vertices of $P$ cannot exceed the number of bases, which is at most $\binom{n}{m}$. In fact, it is proved (the proof is not easy) that the number of vertices is at most*

$$\binom{n - \lfloor (m+1)/2 \rfloor}{n-m} + \binom{n - \lfloor (m+2)/2 \rfloor}{n-m}.$$

*This bound is tight.*

**Theorem 3** *If $min\{c^T x : Ax = b, x \geq 0\}$ is finite (not unbounded), then for every $x \in P$, there exists a vertex $x' \in P$, such that $c^T x' \leq c^T x$.*

**Proof of Theorem 2:** If $x$ is a vertex, then we are done. Suppose $x$ is not a vertex, then $\exists y \neq 0$, such that $x + y, x - y \in P$. Thus, $Ay = 0$, and $y_j = 0$ for every $j$ such that $x_j = 0$. Assume $c^T y \leq 0$ (the case $c^T y \geq 0$ is similar). Then, for every $\lambda > 0$, $A(x + \lambda y) = b$ and $c^T(x + \lambda y) \leq c^T x$. Also, since $y_j = 0$ for every $j$ such that $x_j = 0$, if we take $\lambda = min_{j: y_j < 0} \frac{x_j}{-y_j} > 0$, we will have $x + \lambda y \geq 0$, and furthermore, the number of nonzero entries of $x + \lambda y$ is at least one less than the number of nonzero entries of $x$. Thus, if we take $x' = x + \lambda y$, then $x' \in P$, $c^T x' \leq c^T x$, and the number of nonzero entries of $x'$ is at least one less than that of $x$. Now, if $x'$ is a vertex, we are done; otherwise, we repeat the same process on $x'$. Since this process decreases the number nonzero entries of $x$, we will eventually stop, i.e., we will find a $x'$ that is a vertex. $\square$

**Corollary 4** *If $P = \{x : Ax = b, x \geq 0\} \neq \emptyset$, then there exists a vertex of $P$.*

# 3 The simplex method

Theorem 3 shows that for every bounded linear program, there is a vertex of the polyhedron that minimizes the objective function. This is the main idea behind the simplex algorithm. The simplex algorithm was proposed by Dantzig in 1947. It focuses on vertices to solve linear programming problems.

**Sketch of the algorithm:**

1. We start with a basis $B$.
2. At every step, if we can't prove that $B$ is optimum, we replace a varible in $B$ with a variable outside $B$, to obtain another basis. This process is called *pivoting*, and it is done according to a *pivot rule*.

Simplex algorithm is known to work very well in practice. However, almost for every known pivot rule, we know that the worst case running time of the simplex algorithm is exponential (i.e., there is an example on which the simplex algorithm takes an exponential number of steps to find the optimum). We do not know whether there is a polynomial time simplex algorithm.

The complexity of the Simplex algorithm is "related" to the Hirsch conjecture, which says the diameter of the skeleton of a polyhedron in $\mathbb{R}^d$ with $n$ facets is at most $n - d$.

## 4 Duality in Linear Programming

We know from basic linear algebra that if a system of linear equations is not feasible, then it is possible to get a contradiction by multiplying each equation by a coefficient and adding them up. In other words,

**Theorem 5** *Either $Ax = b$ has a solution, or $A^T y = 0$, $b^T y \neq 0$ has a solution, but not both.*

The following theorem, known as Farkas' Lemma proves something similar for linear inequalities.

**Theorem 6** *(Farkas' Lemma) Either $Ax = b$, $x \geq 0$ has a solution, or $A^T y \geq 0$, $y^T b < 0$ has a solution, but not both.*

It is obvious that both cases in the Farkas Lemma cannot occur at the same time, so we just have to prove that if $Ax = b$, $x \geq 0$ does not have a solution, then there is a $y$ such that $A^T y \geq 0$ and $y^T b < 0$. This fact is usually proved using the simplex algorithm. However, here we present a different proof using the following lemma.

**Theorem 7** *(Projection Theorem) Let $K$ be a nonempty, closed convex set in $\mathbb{R}^n$, and $b \notin K$. The projection $p$ of $b$ onto $K$ is a point $p \in K$ that minimizes the distance $||b - p||$. Then for every $z \in K$, $(b-p)^T (z-p) \leq 0$.*

**Proof:** If this is not true, by moving from $p$ a little bit toward $z$, we obtain another point in $K$ that is closer to $b$. This is a contradiction with the definition of $p$. $\square$