1 Minimum Cost Circulation Problem

Theorem 1 *Let f be a circulation. The following are equivalent:*

(i) f is of minimum cost.

(ii) G_f has no negative cost directed cycles.

(iii) $\exists p : c_p(v,w) \geq 0 \;\; \forall (v,w) \in E_f$, where $c_p(v,w) = c(v,w) + p(v) - p(w)$.

Proof: $i \Rightarrow ii$ and $iii \Rightarrow i$ were proven last lecture. All that remains is the proof of $ii \Rightarrow iii$:
Let G' be obtained from the residual graph G_f by adding a vertex s linked to all other vertices by edges of cost 0 (the costs of these edges do not matter). Let $p(v)$ be the length of the shortest path from s to v in G' with respect to the costs.
These quantities are well-defined since G_f does not contain any negative cost directed cycles, and every vertex is reachable from s. By definition of the shortest paths, $p(w) \leq p(v) + c(v,w) \;\; \forall (v,w) \in E_f$. This implies that $c_p(v,w) \geq 0 \;\; \forall (v,w) \in E_f$. □

2 Klein's Algorithm for MCCP

Klein's Cycle canceling algorithm:

1. Let f be any circulation.
2. While G_f contains a negative cycle Γ do
   push $\delta = \min_{(v,w)\in\Gamma} u_f(v,w)$ along Γ.

Argument for Correctness:

If the algorithm terminates, then the circulation found must be optimum. Furthermore, if all capacities and costs are integers, then the algorithm will terminate.

Why?

- $f(v,w)$ is always an integer, thus $\delta = \min_{(v,w)\in\Gamma} u_f(v,w) \geq 1$
- If $|c(v,w)| \leq C$ and $|f(v,w)| \leq U$, then the absolute value of the cost of the optimal circulation is at most mCU

Therefore, the algorithm terminates after $O(mCU)$ iterations.

Remark 1 *If the edge capacities in the graph are irrational, then the algorithm is not correct.*

The cycle canceling algorithm can be applied to the Max-Flow Problem by making appropriate modifications to the graph G. Let G' be obtained by setting the cost of all edges within G to 0. Furthermore, select two vertices s and t from within the graph, and add the directed edges (s,t) and (t,s), where $c(s,t) = 1, c(t,s) = -1$ and both edges have infinite capacity. Now, solving for the maximum flow between s to t is equivalent to solving for the minimum cost circulation, which contains s and t. In this circumstance, Klein's Algorithm reduces to the Ford-Fulkerson Algorithm.

Ford-Fulkerson Augmenting Path Algorithm:

1. Begin with zero flow: $f = 0$.
2. While G_f contains a directed path P from s to t do
   push $\delta = \min_{(v,w)\in P} u_f(v,w)$ along P.

The running time given above for Klein's Cycle-Canceling Algorithm is not polynomial. The negative cost cycle in Klein's Algorithm (or the directed path in the Ford-Fulkerson Algorithm) must be chosen appropriately to insure a polynomial running time.

Candidates for Cycles in Klein's Algorithm:

1. The most negative cost cycle in G_f?
   Finding this cycle is an NP-Hard problem, so it would not be a viable choice.
2. The negative cycle in G_f which would yield the maximum cost improvement?
   Finding this cycle is again an NP-Hard problem.
   However for the Max-Flow Problem, this choice reduces to finding the st-path with maximum residual capacity. Such a path can be found in $O(m)$ time, $m = |E|$. The resulting Max-Flow algorithm is known as the "fattest" path algorithm (Edmonds-Karp '72). The number of iterations necessary is $O(m \log U)$, thus the running of the algorithm is $O(m^2 \log U)$.
3. Minimum Mean-Cost Cycle?
   Define the mean cost of a cycle Γ to be:

$$\frac{c(\Gamma)}{|\Gamma|} = \frac{\Sigma_{(v,w)\in\Gamma} c(v,w)}{|\Gamma|} \tag{1}$$

   where $|\Gamma|$ denotes the number of edges in Γ. The minimum mean cost of all cycles of the residual graph G_f would thus be:

$$\mu(f) = \min_{cycles\ \Gamma\ in\ G_f} \frac{c(\Gamma)}{|\Gamma|} \tag{2}$$

   The minimum mean-cost cycle can be determined in strongly polynomial time by using a modified version of the Bellman-Ford Algorithm. More precisely, the minimum mean cost cycle can be found in $O(mn)$ time. Using this method to solve the Min-Cost Circulation Problem yields the Goldberg-Tarjan Algorithm, which runs in polynomial time. Using this method to solve the Max-Flow Problem yields what is known as the "shortest" augmenting path algorithm (Edmonds-Karp). This Max-Flow Algorithm is able to find the augmenting path in $O(m)$ time, and requires $O(mn)$ iterations to arrive at the solution. Thus, the total running time is $O(m^2 n)$.

3 The Goldberg-Tarjan Algorithm

Goldberg-Tarjan Algorithm:

1. Begin with zero flow: $f = 0$.
2. While $\mu(f) < 0$ do
   push $\delta = \min_{(v,w)\in\Gamma} u_f(v,w)$ along a minimum mean cost cycle Γ of G_f.

Analysis of Goldberg-Tarjan Algorithm:

In order to analyze this algorithm, it is necessary to define the concept of proximity measure for a circulation f.

Definition 1 *A circulation f is ϵ-optimal if there exists p such that $c_p(v,w) \geq -\epsilon \ \forall (v,w) \in E_f$.*

Definition 2 *$\epsilon(f) =$ minimum ϵ such that f is ϵ-optimal.*

The following theorem states that the minimum mean cost $\mu(f)$ of all cycles in G_f is equal to $-\epsilon(f)$, as defined above.

Theorem 2 *For any circulation f, $\mu(f) = -\epsilon(f)$.*

Proof:

- $\mu(f) \geq -\epsilon(f)$
  By definition, there exists p such that $c_p(v,w) \geq -\epsilon(f) \ \forall (v,w) \in E_f$. Thus, it is implied that $c_p(\Gamma) \geq -\epsilon(f)|\Gamma|$ for any directed cycle $\Gamma \in G_f$. But for any $\Gamma \in G_f$, $c(\Gamma) = c_p(\Gamma)$. Thus, dividing both sides by $|\Gamma|$ yields that the mean cost of any directed cycle $\Gamma \in G_f$ is at least $-\epsilon(f)$. Therefore, $\mu(f) \geq -\epsilon(f)$.
- $\epsilon(f) \leq -\mu(f)$
  Consider $\mu(f)$. For every cycle $\Gamma \in G_f$, it is the case that $\frac{c(\Gamma)}{|\Gamma|} \geq \mu(f)$. Let $c'(v,w) = c(v,w) - \mu(f) \ \forall (v,w) \in E_f$. With respect to this new cost function c' every cycle $\Gamma \in G_f$ will have nonnegative cost. Now, let G' be obtained by adding a new node s to G_f and adding directed edges from s to $v \ \forall v \in V$, all with zero cost. Let $p(v)$ be the cost with respect to c' of the shortest path from s to v in the new graph G'. For all edges (v,w), $p(w) \leq p(v) + c'(v,w) = p(v) + c(v,w) - \mu(f)$. This implies that $c_p(v,w) \geq \mu(f) \ \forall (v,w) \in E_f$. Therefore, $\epsilon(f) \leq -\mu(f)$.
- $\mu(f) \geq -\epsilon(f)$ and $\epsilon(f) \leq -\mu(f) \Rightarrow \epsilon(f) = -\mu(f)$.

□

Remark 2 *Along the minimum mean cost cycle Γ, $c_p(v,w) = -\epsilon(f) \ \forall (v,w) \in \Gamma$.*

Having completed the necessary definitions and proofs, we may now proceed with the analysis of the Goldberg-Tarjan Algorithm. The following theorem considers only one iteration of the algorithm.

Theorem 3 *Let f be a circulation and let f' be the circulation obtained by canceling the minimum mean cost cycle Γ of G_f. Then, $\epsilon(f') \leq \epsilon(f)$.*

Proof: By definition, there exists p such that $c_p(v,w) \geq -\epsilon(f) \ \forall (v,w) \in E_f$. In the case of the minimum mean cost cycle Γ of G_f, $c_p(v,w) = -\epsilon(f) \ \forall (v,w) \in \Gamma$. After performing the one cycle-canceling step, we obtain the new residual graph $G_{f'}$. We claim that $c_p(v,w) \geq -\epsilon(f) \ \forall (v,w) \in E_{f'}$. In the case of all edges $(v,w) \in E_{f'} \bigcap E_f$, the claim is certainly true. In the case of all edges

$(v, w) \in E_{f'} \backslash E_f$, it must be true that $(w, v) \in \Gamma$. For all $(w, v) \in \Gamma$, $c_p(w, v) = -\epsilon(f)$, and thus $c_p(v, w) = \epsilon(f) \geq -\epsilon(f)$. Therefore, $c_p(v, w) \geq -\epsilon(f)$ holds true for all $(v, w) \in E_{f'}$. $\square$

The above theorem shows that by completing a single iteration of the Goldberg-Tarjan algorithm, it is impossible to generate a new flow which is farther from optimality than the original.