18.415/6.854 Advanced Algorithms October 3, 2001

Starting solution for interior-point algorithm

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1 Transformation for the Interior Point Algorithm

In this handout, we show how a pair of dual linear programs

$$
(P)\quad \begin{array}{ll} Min & c^T x \\ s.t. & Ax = b \\ & x \geq 0 \end{array}
\qquad\qquad
(D)\quad \begin{array}{lllll} Max & b^T y & & & \\ s.t. & A^T y & + & s & = c \\ & & & s & \geq 0 \end{array}
$$

can be transformed so that we know a strictly feasible primal solution x_0 and a strictly feasible vector of dual slacks s_0 such that $x_j s_j = \mu$ for every j, where $\mu = 2^{6L}$.

Consider the pair of dual linear programs:

$$
(P')\quad \begin{array}{lrcrcrcl}
Min & c^T x & + & k_c x_{n+1} & & & & \\
s.t. & Ax & + & (b - 2^{2L} Ae) x_{n+1} & & & = & b \\
& (2^{4L} e - c)^T x & & & + & 2^{4L} x_{n+2} & = & k_b \\
& x \geq 0 & & x_{n+1} \geq 0 & & x_{n+2} \geq 0 & &
\end{array}
$$

and

$$
(D')\quad \begin{array}{lrcrcccccl}
Min & b^T y & + & k_b y_{m+1} & & & & & & \\
s.t. & A^T y & + & (2^{4L} e - c) y_{m+1} & + & s & & & = & c \\
& (b - 2^{2L} Ae)^T y & & & & & + & s_{n+1} & = & k_c \\
& & & 2^{4L} y_{m+1} & & & & + \; s_{n+2} & = & 0 \\
& & & & & s, & & s_{n+1}, \quad s_{n+2} & \geq & 0
\end{array}
$$

where $k_b = 2^{6L}(n+1) - 2^{2L} c^T e$ is chosen in such a way that $x' = (x, x_{n+1}, x_{n+2}) = (2^{2L} e, 1, 2^{2L})$ is a (strict) feasible solution to (P') and $k_c = 2^{6L}$. Notice that $(y', s') = (y, y_{m+1}, s, s_{n+1}, s_{n+2}) = (0, -1, 2^{4L} e, k_c, 2^{4L})$ is a feasible solution to (D') with $s' > 0$. x' and (y', s') serve as our initial feasible solutions.

We have to show:

1. for every j, $x_j s_j = \mu = 2^{6L}$,
2. the pair $(P') - (D')$ is equivalent to $(P) - (D)$,
3. the input size L' for (P') as defined in the lecture notes does not increase too much.

The proof of 1. is by construction. The proofs of the other statements are simple but heavily use the definition of L and the fact that vertices have all components bounded by 2^L.

In order to show that $(P') - (D')$ are equivalent to $(P) - (D)$, we consider an optimal solution x^* to (P) and an optimal solution (y^*, s^*) to (D) (the case where (P) or (D) is infeasible is an exercise). Without loss of generality, we can assume that x^* and (y^*, s^*) are vertices of the corresponding polyhedra. In particular, this means that $x^*_j, |y^*_j|, s^*_j < 2^L$.

Proposition 1 *Let* $x' = (x^*, 0, (k_b - (2^{4L}e - c)^T x^*)/2^{4L})$ *and let* $(y', s') = (y^*, 0, s^*, k_c - (b - 2^{2L}Ae)^T y^*, 0)$. *Then*

1. x' *is a feasible solution to* (P') *with* $x'_{n+2} > 0$,
2. (y', s') *is a feasible solution to* (D') *with* $s'_{n+1} > 0$,
3. x' *and* (y', s') *satisfy complementary slackness, i.e. they constitute a pair of optimal solutions for* $(P') - (D')$.

Proof:

To show that x' is a feasible solution to (P') with $x'_{n+2} > 0$, we only need to show that $k_b - (2^{4L}e - c)^T x^* > 0$ (the reader can easily verify that x' satisfy all the equalities defining the feasible region of (P')). This follows from the fact that

$$(2^{4L}e - c)^T x^* \leq n(2^{4L} + 2^L)2^L = n(2^{5L} + 2^{2L}) < n2^{6L}$$

and

$$k_b = 2^{6L}(n+1) - 2^{2L}c^T e \geq 2^{6L}(n+1) - 2^{2L} n \max_j |c_j| \geq 2^{6L} n + 2^{6L} - 2^{3L} > n2^{6L}$$

where we have used the definition of L and the fact that vertices have all their entries bounded by 2^L.

To show that (y', s') is a feasible solution to (D') with $s'_{n+1} > 0$, we only need to show that $k_c - (b - 2^{2L}Ae)^T y^* > 0$. This is true since

$$\begin{aligned}(b - 2^{2L}Ae)^T y^* &\leq b^T y^* - 2^{2L} e^T A^T y^* \\ &\leq m \max_i |b_i| 2^L + 2^{2L} nm \max_{i,j} |a_{ij}| 2^L \\ &= 2^{2L} + 2^{4L} < 2^{6L} = k_c.\end{aligned}$$

x' and (y', s') satisfy complementary slackness since

- $x^{*T} s^* = 0$ by optimality of x^* and (y^*, s^*) for (P) and (D)
- $x'_{n+1} s'_{n+1} = 0$ and

- $x'_{n+2}s'_{n+2} = 0$.

$\square$

This proposition shows that, from an optimal solution to $(P)-(D)$, we can easily construct an optimal solution to $(P')-(D')$ of the same cost. Since this solution has $s'_{n+1} > 0$, any optimal solution $\hat{x}$ to (P') must have $\hat{x}_{n+1} = 0$. Moreover, since $x'_{n+2} > 0$, any optimal solution $(\hat{y}, \hat{s})$ to (D') must satisfy $\hat{s}_{n+2} = 0$ and, as a result, $\hat{y}_{m+1} = 0$. Hence, from any optimal solution to $(P')-(D')$, we can easily deduce an optimal solution to $(P)-(D)$. This shows the equivalence between $(P)-(D)$ and $(P')-(D')$.

By some tedious but straightforward calculations, it is possible to show that L' (corresponding to $(P')-(D')$) is at most $24L$. In other words, $(P)-(D)$ and $(P')-(D')$ have equivalent sizes.