18.415/6.854 Advanced Algorithms	September 10, 2001

Problem Set 1

Due September 19, 2001

1. Consider $P = \{x : Ax \leq b, x \geq 0\}$, where A is $m \times n$. Show that if x is a vertex of P then we can find sets I and J with the following properties.

   (a) $I \subseteq \{1, \ldots, m\}$, $J \subseteq \{1, \ldots, n\}$ and $|I| = |J|$.

   (b) A_J^I is invertible where A_J^I is the submatrix of A corresponding to the rows in I and the columns in J.

   (c) $x_j = 0$ for $j \notin J$ and $x_J = (A_J^I)^{-1} b^I$ where b^I denotes the restriction of b to the indices in I.

   (Hint: Consider $Q = \{(x, s) : Ax + Is = b, x \geq 0, s \geq 0\}$.)

2. In his paper in FOCS 92, Tomasz Radzik needs a result of the following form (Page 662 of the Proceedings):

   Lemma 1 *Let $c \in \mathbb{R}^n$ and $y_k \in \{0,1\}^n$ for $k = 1, \ldots, q$ such that $2|y_{k+1}c| \leq |y_k c|$ for $k = 1, \ldots, q-1$. Assume that $y_q c = 1$. Then $q \leq f(n)$.*

   In other words, given any set of n (possibly negative) numbers, one cannot find more than $f(n)$ subsums of these numbers which decrease in absolute value by a factor of at least 2.

   Radzik proves the result for $f(n) = O(n^2 \log n)$ and conjectures that $f(n) = O^*(n)$ where O^* denotes the omission of logarithmic terms. Using linear programming, you are asked to improve his result to $f(n) = O(n \log n)$.

   (a) Given a vector c and a set of q subsums satisfying the hypothesis of the Lemma, write a set of inequalities in the variables $x_i \geq 0, i = 1 \ldots n$, such that $x_i = |c_i|$ is a feasible vector, and for any feasible vector x' there is a corresponding vector c' satisfying the hypothesis of the Lemma for the same set of subsums.

   (b) Prove that there must exist a vector c' satisfying the hypothesis of the Lemma, with c' of the form $(d_1/d, d_2/d, \ldots, d_n/d)$ for some integers $|d|, |d_1|, \ldots, |d_n| = 2^{O(n \log n)}$.
   (Hint: see Problem 1.)

   (c) Deduce that $f(n) = O(n \log n)$.

(d) (Not part of the problem set; only for those who like challenges... A guaranteed A+ for anyone getting this part without outside help.) Show that $f(n) = \Omega(n \log n)$.

3. The maximum flow problem on the directed graph $G = (V, E)$ with capacity function u (and lower bounds 0) can be formulated by the following linear program:

$$\max w$$

subject to

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} w & i = s \\ 0 & i \neq s, t \\ -w & i = t \end{cases}$$

$$x_{ij} \leq u_{ij}$$

$$0 \leq x_{ij}.$$

(x_{ij} represents the flow on edge (i, j); the flow has to be less or equal to the capacity on any edge and flow conservation must be satisfied at every vertex except the source s, where we try to maximize the flow, and the sink t.)

(a) Show that its dual is equivalent to:

$$\min \sum_{(i,j)\in E} u_{ij} y_{ij}$$

subject to

$$z_i - z_j + y_{ij} \geq 0 \qquad (i, j) \in E$$

$$z_s = 0, z_t = 1$$

$$y_{ij} \geq 0.$$

(b) A cut is a set of edges of the form $\{(i, j) \in E : i \in S, j \notin S\}$ for some $S \subset V$ and its value is

$$W = \sum_{(i,j)\in E: i\in S, j\notin S} u_{ij}.$$

It separates s from t if $s \in S$ and $t \notin S$.
Show that a cut of value W separating s from t corresponds to a feasible solution y, z of the dual program such that

$$W = \sum_{(i,j)\in E} u_{ij} y_{ij}.$$

(c) Given any (not necessarily integral) optimal solution y^*, z^* of the dual linear program and an optimal solution x^* of the primal linear program, show how to construct from z^* a cut separating s from t of value equal to the maximum flow.

(Hint: Consider the cut defined by $S = \{i : z_i \leq 0\}$ and use complementary slackness conditions.)

(d) Deduce the max-flow–min-cut theorem: the value of the maximum flow from s to t is equal to the value of the minimum cut separating s from t.

4. Consider the following property of vector sums.

Theorem 2 *Let $v_1, \ldots, v_n$ be $d-$dimensional vectors such that $\|v_i\| \leq 1$ for $i = 1, \ldots, n$ (where $\|.\|$ denotes* any *norm) and*

$$\sum_{i=1}^{n} v_i = 0.$$

Then there exists a permutation π of $\{1, \ldots, n\}$ such that

$$\left\| \sum_{j=1}^{k} v_{\pi(j)} \right\| \leq d$$

for $k = 1, \ldots, n$.

In this problem, you are supposed to prove this theorem by using linear programming techniques.

(a) Suppose we have a nested sequence of sets

$$\{1, \ldots, n\} = V_n \supset V_{n-1} \supset \ldots \supset V_d$$

where $|V_k| = k$ for $k = d, d+1, \ldots, n$. Suppose further that we have numbers λ_{ki} satisfying:

$$\sum_{i \in V_k} \lambda_{ki} v_i = 0, \tag{1}$$

$$\sum_{i \in V_k} \lambda_{ki} = k - d, \tag{2}$$

$$0 \leq \lambda_{ki} \leq 1 \qquad\qquad i \in V_k, \tag{3}$$

for $k = d, \ldots, n$. Define a permutation π as follows: set $\pi(1), \ldots, \pi(d)$ to be elements of V_d in any order, and set $\pi(k)$ to be the unique element in $V_k \setminus V_{k-1}$ for $k = d+1, \ldots, n$.

Show that this permutation satisfies the conditions of Theorem 2.

(b) Show that there exist λ_{ni}, $i = 1 \dots n$, satisfying (1), (2) and (3) for $k = n$.

(c) Suppose we have constructed $V_n, \dots, V_{k+1}$ and λ_{ji} for $j = k+1, \dots, n$ and $i \in V_j$ satisfying (1), (2) and (3) for $k+1, \dots, n$ (where $k \geq d$). Prove that the following system of $d+1$ equalities ((4) contains d equalities), $k+1$ inequalities and $k+1$ nonnegativity constraints has a solution with at least one $\beta_i = 0$:

$$\sum_{i \in V_{k+1}} \beta_i v_i = 0, \tag{4}$$

$$\sum_{i \in V_{k+1}} \beta_i = k - d, \tag{5}$$

$$0 \leq \beta_i \leq 1 \qquad\qquad i \in V_{k+1}. \tag{6}$$

Deduce the existence of the nested sequence and the λ's as described in (a).

5. Consider the following optimization problem with "robust conditions":

$$\min\{c^T x : x \in \mathbb{R}^n; Ax \geq b \text{ for any } A \in F\},$$

where $b \in \mathbb{R}^m$ and F is a set of $m \times n$ matrices:

$$F = \{A : \forall i, j; a_{ij}^{min} \leq a_{ij} \leq a_{ij}^{max}\}.$$

(a) Considering F as a polytope in $\mathbb{R}^{m \times n}$, what are the vertices of F?

(b) Show that instead of the conditions for all $A \in F$, it is enough to consider the vertices of F. Write the resulting linear program. What is its size? Is this polynomial in the size of the input, namely m, n and the sizes of b, c, a_{ij}^{min} and a_{ij}^{max}?

(c) Derive a more efficient description of the linear program: Write the condition on x given by one row of A, for all choices of A. Formulate this condition as a linear program. Use duality and formulate the original problem as a linear program. What is the size of this one? Is this polynomial in the size of the input?