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18.409: Topics in TCS: Embeddings of Finite Metric Spaces
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Lecture 16

In this lecture we will discuss several results on Lipschitz extensions of mappings. In particular we'll prove the Kirszbraun extension theorem for mappings of subsets of ℓ_2^n to ℓ_2^m. The book [4] gives a complete account of the proof and many other related results. We'll also present a result by Johnson and Lindenstrauss [2] on extensions of mappings from finite subsets of arbitrary metric spaces into ℓ_2^m. Finally, we will present a nice application of the Kirszbraun extension theorem in showing that embedding spheres into the plane requires $\Omega(\sqrt{n})$ distortion. We also show that this bound is tight.

1 Lipschitz Extensions of Mappings

Definition 1. *Let X and Y be metric spaces. Let $f : X \to Y$. If*

$$L = \sup_{x_i, x_j \in X} \frac{d_Y(f(x_i), f(x_j))}{d_X(x_i, x_j)} < \infty,$$

we say that f is a Lipschitz *map. We define the Lipschitz constant of f, $|f|_{lip}$ to be L.*

Recall: Distortion$(f) = |f|_{lip}|f^{-1}|_{lip}$. We will prove the results for mappings into ℓ_2^n. All the results hold for ℓ_2 as well.

The basic questions that we want to ask about Lipschitz extensions are the following. Let X, Y be given metric spaces, $M \subseteq X$, and $f : M \to Y$ be a Lipschitz map.

1. Can we extend f to a Lipschitz map on all of X? I.e. when does there exist $\tilde{f} : X \to Y$ such that $\tilde{f}|_M = f$ and $\tilde{f}$ is also Lipschitz?
2. If such a function $\tilde{f}$ exists, can we find any bound on $\tilde{f}$?

In the case when $X = \ell_2^n$ and $Y = \ell_2^m$, we have the following result:

Theorem 1. Kirszbraun Extension Theorem *Let $X = \ell_2^n$ and $Y = \ell_2^m$ for some $m, n \in \mathbb{N}$. Let $M \subseteq X$ and let $f : M \to Y$ be a Lipschitz map. Then, there exists $\tilde{f} : X \to Y$ such that $\tilde{f}|_M = f$ and $|\tilde{f}|_{lip} = |f|_{lip}$.*

Proof. Without loss of generality, we can assume that $|f|_{lip} = 1$ since we can scale f appropriately, and this wouldn't change the result. Hence, we have that f is non-expanding.

Claim: If $M \neq X$, it suffices to be able to extend f to at least one more point. (Informally, consider the largest subset of X that we can extend f to, and if it isn't all of X, then we can extend it be one more point and contradict maximality.) Zorn's Lemma takes care of formalizing this notion.

Let $M = \{x_i | i \in I\}$. Let $f(x_i) = y_i$. Let $x \in X - M$ and let $\|x - x_i\| = r_i$. Then

$$x \in \bigcap_{i \in I} B(x_i, r_i).$$

To extend f be one more point, we need to find $y \in Y$ such that $\|y - y_i\| \leq r_i; \forall i \in I$. That is, we want to show that $\bigcap_{i \in I} B(y_i, r_i) \neq \phi$.

Definition 2. *We say that the pair of metric spaces* (X, Y) *has* Property [K] *if whenever* $\{B(x_i, r_i)|i \in I\}$ *and* $\{B(y_i, r_i)|i \in I\}$ *are two families of closed balls in* X *and* Y *indexed over* I*, and* $d_Y(y_i, y_j) \leq d_X(x_i, x_j); \forall i, j \in I$*, then*

$$\bigcap_{i \in I} B(x_i, r_i) \neq \phi \Rightarrow \bigcap_{i \in I} B(y_i, r_i) \neq \phi.$$

What remains to show is that (ℓ_2^n, ℓ_2^m) has Property [K]. It is enough to prove it for the case when I is a finite set, since by compactness arguments we can extend it to the case of arbitrary collections of sets. Here we use the fact that if we have a collection of compact sets such that every finite sub-collection has non-empty intersection, then the intersection of all the sets is nonempty.

Let $\{B(x_i, r_i)\}$ be a finite set of balls indexed by $i = 1, 2, \ldots, k$. $\bigcap_{i \in I} B(x_i, r_i) \neq \phi$. Now consider $\bigcap_{i \in I} B(y_i, r_i)$ To show this is nonempty, it suffices to find $y \in Y$ such that $\frac{\|y - y_i\|}{r_i} \leq 1$ for $1 \leq i \leq k$.

Let g defined on Y be the following function:

$$g(t) = \max_i \left\{ \frac{\|t - y_i\|}{r_i} \right\}.$$

g is a continuous function, and g assumes lower values in some closed, bounded region of Y. Since such a region is compact, g attains its minimum at some point say y. Let $g(y) = \lambda$. Therefore,

$$\max_i \left\{ \frac{\|y - y_i\|}{r_i} \right\} = \lambda.$$

Let $\|y - y_i\| = \lambda \|x - x_i\|$ for $1 \leq i \leq l$, and let $\|y - y_i\| < \lambda \|x - x_i\|$ for $i > l$.

Claim: y must lie in the convex hull C of $y_1, y_2, \ldots, y_l$. If it did not, then there would exist a hyperplane π separating y from C. Then, by displacing y by a small amount in the direction of π, we would reduce the distance of y to y_i, for $1 \leq i \leq l$, and preserve the other strict inequalities, thus contradict the minimality of λ. Hence y is some convex combination of $y_1, y_2, \ldots, y_l$. Let $a_1, a_2, \ldots, a_l$ be such that $a_i \geq 0$, $\sum a_i = 1$ and $\sum_{i=1}^{l} a_i y_i = y$.

Consider $1 \leq i, j \leq l$. Since $\|y_i - y_j\|^2 \leq \|x_i - x_j\|^2$, we get

$$\|y_i - y_j\|^2 = \|(y_i - y) + (y - y_j)\|^2 \leq \|x_i - x_j\|^2.$$

This gives us $\|y_i - y\|^2 + \|y - y_j\|^2 + 2 \langle y_i - y, y - y_j \rangle \leq \|x_i - x_j\|^2$, thus implying $\lambda \|x_i - x\|^2 + \lambda \|x_j - x\|^2 \leq \|x_i - x_j\|^2 + 2 \langle y_i - y, y_j - y \rangle$.

Assume that $\lambda > 1$. Then we have

$$\|x_i - x\|^2 + \|x_j - x\|^2 \leq \|x_i - x_j\|^2 + 2 \langle y_i - y, y_j - y \rangle.$$

Multiplying throughout by $a_i a_j$ and summing over all $1 \leq i, j \leq l$, we get

$$\sum_{i,j} a_i a_j (\|x_i - x\|^2 + \|x_j - x\|^2 - \|x_i - x_j\|^2) < 2 \sum_{i,j} a_i a_j \langle y_i - y, y_j - y \rangle.$$

Since $\sum_{i=1}^{l} a_i y_i = y$, the R.H.S. sums to 0. Also, assuming without loss of generality that $x \equiv 0$ and simplifying the L.H.S., we deduce that $\| \sum a_i x_i \|^2 < 0$, which is a contradiction. Hence $\lambda \leq 1$. This implies $y \in \bigcap_{i \in I} B(y_i, r_i)$, and this completes the proof of the Kirszbraun Extension Theorem. $\square$

Theorem 2. Johnson and Lindenstrauss: *Let X be an arbitrary metric space, and let $M \subseteq X$ be a finite subset of X such that $|M| = n$. Let $f : M \to \ell_2^m$ be given. Then, there exists $\tilde{f} : X \to \ell_2^m$ satisfying $\tilde{f}|_M = f$ and $|\tilde{f}|_{lip} \leq O(\sqrt{\log n})|f|_{lip}$.*

We will need the following lemma:

Lemma 3. *For any metric space X, the pair (X, ℓ_∞^d) has Property [K].*

Proof. (sketch) Let $\{B(x_i, r_i)|i \in I\}$ and $\{B(y_i, r_i)|i \in I\}$ be given. Let $\|y_i - y_j\|_\infty \leq d_X(x_i, x_j); \forall i, j \in I$ and let $\bigcap_{i \in I} B(x_i, r_i) \neq \phi$.

We first observe that $B(x_i, r_i) \bigcap B(x_j, r_j) \neq \phi \Rightarrow r_i + r_j \geq d_X(x_i, x_j)$. Since $\|y_i - y_j\|_\infty \leq d_X(x_i, x_j)$, therefore $\|y_i - y_j\|_\infty \leq r_i + r_j$ which implies $B(y_i, r_i) \bigcap B(y_j, r_j) \neq \phi$. Thus, we have a collection of balls in ℓ_∞^d that intersect pairwise. We observe that ℓ_∞^d has the property that if a collection of balls in ℓ_∞^d intersect pairwise, then they all have a common point of intersection. We can see that this is true by projecting to each coordinate of ℓ_∞^d, taking the point of intersection there (of the corresponding intervals), and then concatenating them all back together. Hence, $\bigcap_{i \in I} B(y_i, r_i) \neq \phi$. □

Proof. (Johnson-Lindenstrauss Theorem)

Given X, and $M \subseteq X$ with $|M| = n$, and $f : M \to \ell_2^m$. Let $A = f(M)$. By the Johnson-Lindenstrauss dimension reduction lemma in ℓ_2, there exists a $1-1$ function g such that $g(A) \subseteq \ell_2^k$, where $k = O(\log n)$; and $\text{Dist}(g) \leq 2$ (say). (Note: the J-L lemma says we can get distortion $1 + \epsilon$ for any ϵ.) Since $\text{Dist}(g) = |g|_{lip}|g^{-1}|_{lip}$, we get $|g|_{lip}|g^{-1}|_{lip} \leq 2$. By scaling the function suitably we can assume without loss of generality that $|g|_{lip} \leq 1$ and $|g^{-1}|_{lip} \leq 2$.

Let I be the formal identity map from ℓ_2^k to ℓ_∞^k. Clearly $|I|_{lip} = 1$ and $|I^{-1}|_{lip} = \sqrt{k}$.

$$M \xrightarrow{f} A_{\subseteq \ell_2^m} \xrightarrow{g} g(A)_{\subseteq \ell_2^k} \xrightarrow{I} \ell_\infty^k$$

The map $I \circ g \circ f = h : M \to \ell_\infty^k$ has $|h|_{lip} \leq |f|_{lip}$. By the previous lemma, since $(X, \ell_\infty k)$ has Property [K], we can extend h to $\tilde{h} : X \to \ell_\infty^k)$ such that $|\tilde{h}|_{lip} \leq |f|_{lip}$.

Now, we have a map $g^{-1} : g(A) \to A$. By the Kirszbraun Theorem, this map can be extended to $\tilde{g^{-1}} : \ell_2^k \to \ell_2^m$ such that $|\tilde{g^{-1}}|_{lip} = |g^{-1}|_{lip}$.

Then, $\tilde{f} = \tilde{g^{-1}} \circ I^{-1} \circ \tilde{h}$ is a map such that $\tilde{f} : X \to \ell_2^m$; $\tilde{f}|_M = f$ and $|\tilde{f}|_{lip} \leq 2\sqrt{k}|f|_{lip} = O(\sqrt{\log n})|f|_{lip}$. □

2 Embeddings Spheres into the Plane

Theorem 4. *There exists a metric space in ℓ_2^3 induced by an n-point set X on the unit sphere S^2 such that any mapping $f : X \to \mathbb{R}^2$ has distortion $\Omega(\sqrt{n})$.*

To prove theorem 4 we will have to use the Borsuk-Ulam theorem. We will state it here without proof (see [3] for proof and further discussion).

Theorem 5 (Borsuk-Ulam)**.** *Every continuous map $f : S^n \to \mathbb{R}^n$ must identify a pair of antipodal points.*

Definition 3. *Suppose (S, δ) is a metric space and let $\epsilon > 0$. A subset $N \subset S$ is an ϵ-*net *for S if, for all $x \in S$, there exists $y \in N$ such that $d(x, y) < \epsilon$.*

Proof. *(Theorem 4)* Let X be a set of n points in S^2 that form a $O(1/\sqrt{n})$-net of S^2 (it is easy to see that there exists such a net of size $O(n)$). Let $f : X \to \mathbb{R}^2$ be a non-contracting embedding, so that $\forall x, y \in X$ we have $||f(x) - f(y)||_2 \leq ||x - y||_2$. Since $X \subset S^2 \subset \mathbb{R}^3$, by the Kirszbraun extension theorem we know that there exists a continuous mapping $f' : S^2 \to \mathbb{R}^2$ such that $\forall x \in X$ we have $f'(x) = f(x)$ and $\forall x, y \in S^2$ we have $||f(x) - f(y)||_2 \leq ||x - y||_2$. By the Borsuk-Ulam theorem, we know that there exist antipodal points p and q such that $f'(p) = f'(q)$. Since we chose X to be an $O(1/\sqrt{n})$-net, we know there exist points $p', q' \in X$ such that $||p - p'||_2 = O(1/\sqrt{n})$ and $||q - q'||_2 = O(1/\sqrt{n})$. Thus,

$$\begin{aligned} \left|\left|f(p') - f(q')\right|\right|_2 &= \left|\left|f'(p') - f'(q')\right|\right|_2 \\ &\leq \left|\left|f'(p') - f'(p)\right|\right|_2 + \left|\left|f'(p) - f'(q)\right|\right|_2 + \left|\left|f'(q) - f'(q')\right|\right|_2 \\ &\leq \left|\left|p - p'\right|\right|_2 + \left|\left|q - q'\right|\right|_2 \\ &= O(1/\sqrt{n}) \end{aligned} \tag{1}$$

By the triangle inequality we get

$$\left|\left|p' - q'\right|\right|_2 + \left|\left|p' - p\right|\right|_2 + \left|\left|q' - q\right|\right|_2 \geq ||p - q||_2 = 2$$

$$\left|\left|p' - q'\right|\right|_2 \geq 2 - O(1/\sqrt{n}) \geq 2 - \frac{c}{\sqrt{n}}$$

and therefore

$$\left|\left|p' - q'\right|\right|_2 = \Omega(1). \tag{2}$$

Let D be the distortion of f. Since f is non-contracting, we know that

$$D \geq \frac{||p - q||_2}{||f(p) - f(q)||_2}.$$

From equations (1) and (2), we get

$$\frac{||p - q||_2}{||f(p) - f(q)||_2} \geq \frac{\Omega(1)}{O(1/\sqrt{n})} = \Omega(\sqrt{n})$$

which means that $D = \Omega(\sqrt{n})$. □

We now prove that the bound given in theorem 4 is tight. Recall that for a metric space $M = (X, \delta)$, $c_p^d(M)$ denotes the minimum distortion of any embedding of M into l_p^d.

Theorem 6. *If $M = (X, \delta)$ is a metric in ℓ_2^3 induced by an n-point subset X of the unit sphere S^2, then $c_2^2 = O(\sqrt{n})$.*

Proof. Let K be a cap of S^2 such that the size of K is $\Omega(1/n)$ and $K \cap X = \emptyset$ (such a cap must exist since caps of size less than $4\pi/n$ centered at all points in X do not cover fully S^2). Let p_0 be the center of K and let p_0' be its antipode in S^2. Without loss of generality, assume $p_0 = (0, 0, 1)$ and $p_0' = (0, 0, -1)$. Define $f : X \to \mathbb{R}^2$ as follows: for $p = (x, y, z) \in X$,

$$f(p) = \begin{cases} \left(\rho_{S^2}(p, p_0') \frac{x}{\sqrt{x^2+y^2}}, \rho_{S^2}(p, p_0') \frac{y}{\sqrt{x^2+y^2}}\right) & \text{if } p \neq p_0 \text{ and } p \neq p_0' \\ (0, 0) & \text{if } p = p_0 \text{ or } p = p_0' \end{cases}$$

where $\rho_{S^2}(u,v)$ denotes the geodesic distance between u and v in S^2, that is, $\rho_{S^2}(u,v) = \arccos(\langle u,v\rangle)$.

Informally, think of $f(S^2)$ in the following way: cut K off S^2 and unfold what is left into the xy-plane. If we see it this way, it is clear that f is non-contracting. It is also easy to see that the expansion of f is maximized for points p, q on the perimeter of K that are antipodals with respect to K. (For a formal proof of this, see [1]).

So pick points p and q on the perimeter of K such that they are antipodals with respect to K, and let ϕ_K be the angle of K (if O denotes the origin, $\phi_K = \angle pOq$). Let $r_K = \frac{\phi_K}{2}$. Then

$$||f(p) - f(q)||_2 = ||f(p)||_2 + ||f(q)||_2 = (\pi - r_K) + (\pi - r_K) = 2\pi - 2r_K$$

Moreover, $||p-q||_2 = 2\sin r_K$, so the expansion of f for p and q is

$$\frac{||f(p) - f(q)||_2}{||p-q||_2} = \frac{2\pi - 2r_K}{2\sin r_K} = \frac{\pi - r_K}{\sin r_K}$$

and this is the maximum expansion for all points in X.

Since f is non-contracting, we have that for all $x, y \in X$

$$||x-y||_2 \leq ||f(x) - f(y)||_2 \leq \left(\frac{\pi - r_K}{\sin r_K}\right) ||x-y||_2 \,.$$

Now, since K is of size $\Omega(1/n)$, we must have that $r_K = \Omega(1/\sqrt{n})$. We can also assume that $r_K \leq \frac{\pi}{2}$ because otherwise, we can let K be a smaller cap. We know the function $g(\theta) = \frac{\sin\theta}{\theta}$ is decreasing in the interval $(0, \frac{\pi}{2}]$. Thus, for $0 < \theta \leq \frac{\pi}{2}$ we have $\frac{sin\theta}{\theta} \geq \frac{2}{\pi}$. In particular, we have $\sin r_K \geq \frac{2r_K}{\pi}$. Therefore,

$$\frac{\pi - r_K}{\sin r_K} \leq \frac{\pi(\pi - r_K)}{2r_K} < \frac{\pi^2}{2r_K} = O(\sqrt{n})$$

We can thus conclude that f has distortion $O(\sqrt{n})$. □

References

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