Lecture 9

1 Planar Graph Decompositions

In this lecture, we prove the main technical theorem, which (as we saw last lecture) implies that in any graph and for any Δ, we can either exhibit $K_{3,3}$ as a minor or find a distribution over subsets of vertices whose removal separates the graph into connected components with *weak* diameter[1] $O(\Delta)$ and with any vertex v being at distance $\Omega(\Delta)$ from the removed vertices with constant probability.

As we discussed last time, we are given a graph $G = (V, E)$ with unit edge lengths and a given $\Delta \in \mathbb{N}$. The construction goes as follows. We choose an arbitrary root a_1 in the original vertex set $V_1 = V$. From a_1 we do a breadth-first search and get a breadth-first search tree T_1 starting from a_1. Look at the vertices at distance 1, distance 2 and so on. The jth level of T_1 is the set of nodes whose distance in $G_1 = G$ is j. We are going to remove a subset of vertices in particular levels. In the stochastic construction, we choose k_1 uniformly at random in $\{0, 1, \cdots, \delta - 1\}$; as we have done the probabilistic analysis last time, we can assume here that k_1 is arbitrary with $0 \le k_1 \le \Delta - 1$. Then we remove all the vertices in the set $F_1 = \{v \in V_1 : d_{G_1}(a_1, v) \equiv k_1 \mod \Delta\}$ from G_1 and get many connected components between adjacent levels $k_1 + j\Delta$ and $k_1 + (j+1)\Delta$. We choose a connected component from $G \backslash F_1$; let V_2 be the new vertex set of the connected component we choose, and let $G_2 = G[V_2]$ be the graph induced by V_2. We can continue this process twice. Namely, we are going to choose a_i for $i = 2, 3$, and then build a breadth-first search tree T_i from a_i. Define $F_i = \{v \in V_i : d_{G_i}(a_i, v) \equiv k_i \mod \Delta\}$ where $d_{G_i}(u, v)$ is the distance between u and v in graph G_i. Again, k_i is chosen uniformly at random, but here we just assume k_i is completely arbitrary. Then we remove F_i and get connected components. Let V_{i+1} be one of the connected components in $G_i \backslash F_i$ and G_{i+1} is the graph induced by V_{i+1}. In the planar graph case, we do that three times. After we have done the third time, we focus on one of the connected components, call it V_4.

We claim that any two vertices u an v in V_4 are not far in the original graph if G does not have $K_{3,3}$ as a minor. Here is the precise statement; we should emphasize that $d(u, v)$ denotes the distance in the original graph.

Theorem 1. *If G has no $K_{3,3}$ minor, then $\forall u, v \in V_4$, $d(u, v) < 34\Delta$.*

Here we give a bit of history. The construction given here is from Klein, Plotkin and Rao [1]. They consider the case of graphs with no $K_{r,r}$ minors and show there that r levels are sufficient to get $O(\Delta)$ weak diameters. This leads to approximation algorithms for the uniform sparsest cut problem in graphs with no $K_{r,r}$ minors with a guarantee of $O(r^3 \log n)$. Their proof is along the same lines as what we are going to do today, except that we focus on planar graphs. In 1999, Rao [2] showed that the distortion can be reduced to $O(r^3 \sqrt{\log n})$. Based on those results above, in 2003, Fakcharoenphol and Talwar [3] gave an improved decomposition theorem and showed that we can get the distortion down to $O(r^2 \sqrt{\log n})$ if we choose the roots carefully. In the lecture notes of courses on metric embeddings at CMU and Chicago, they prove that the r level decomposition

[1] the maximum distance in the *original* graph between any two vertices of the component.

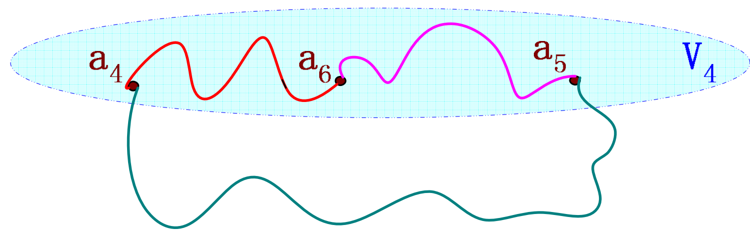

Figure 1: Path P from a_4 to a_5.

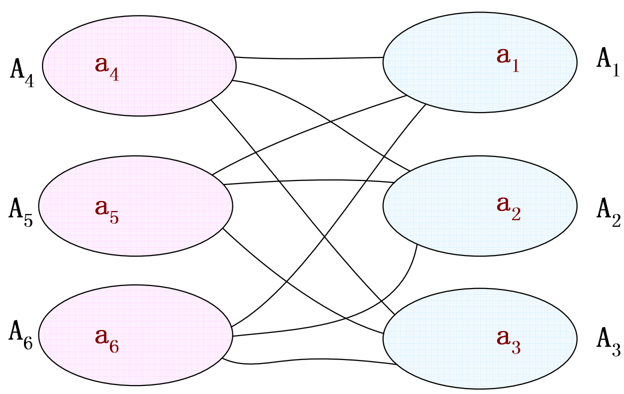

Figure 2: $K_{3,3}$ minor.

method could still get distortion $O(r^2\sqrt{\log n})$ without choosing roots carefully. However, their analysis has a flaw.

2 Preliminaries

Suppose there are two vertices a_4 and a_5 in V_4 with original distance $d(a_4, a_5) \geq 34\Delta$. a_4 and a_5 are both in V_4, but the shortest path in the original graph may go out and back in V_4. As V_4 is a connected component, there must exist a path from a_4 to a_5 within this connected component. Let's look at a path P within G_4 from a_4 to a_5. Along this path, we are going to select a vertex a_6 on P such that $d(a_4, a_6) \geq 17\Delta$ and $d(a_6, a_5) \geq 17\Delta$, see Figure 1. Notice that we use the distance in the original graph. This can be done by walking along the path and selecting the first vertex that satisfies the above two inequalities.

To obtain a contradiction, we will exhibit a $K_{3,3}$ minor (obtained by performing edge contractions and deletions). We will construct 6 disjoint sets A_i for $i = 1, \cdots, 6$; each A_i induces a connected subgraph and contains a_i. To get the minor, we will contract each A_i to a_i and this will result in a $K_{3,3}$ with a_1, a_2, a_3 on one side of the bipartition and a_4, a_5 and a_6 on the other, see Figure 2. We will also need to show that A_i and A_j for $\{1, 2, 3\}$ and $j \in \{4, 5, 6\}$ are connected by an edge. To isolate the minor, we also delete all the other edges. Next we are going to show how to find these A_i's and to argue that they are disjoint. We are also going to show that they are connected by edges.

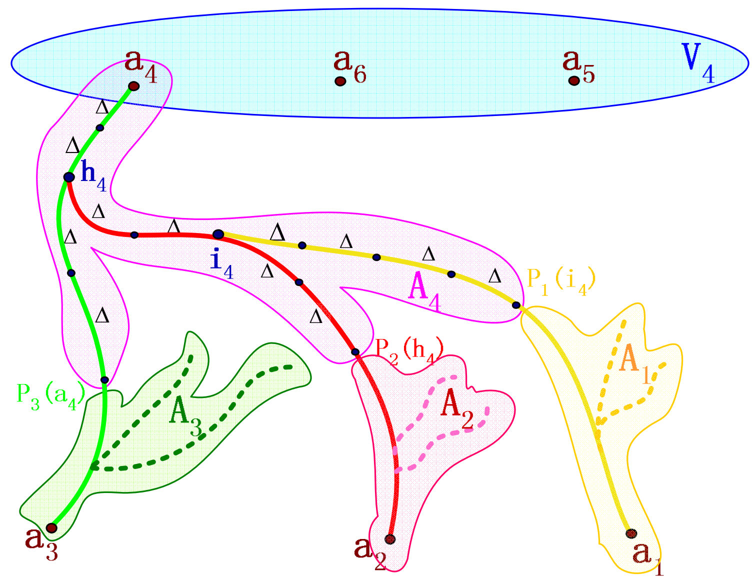

Figure 3: Vertex set and paths.

Before we begin our proof, we introduce some notations first. When we do the breadth-first search from a_i, $i \in \{1, 2, 3\}$, we get the breadth-first search tree T_i. Let's look at the path $P_i(u)$ in T_i from a_i to vertex u. We are going to partition the vertices of $P_i(u)$ into two pieces, $P_i^+(u)$ and $P_i^-(u)$:

$$P_i^+(u) = \{v \in P_i(u) : d_{P_i(u)}(u, v) \le 4\Delta\}$$

$$P_i^-(u) = \{v \in P_i(u) : d_{P_i(u)}(u, v) > 4\Delta\}$$

Now let's show how to construct the six vertex sets $A_1, \ldots, A_6$. Look at the first path from a_3 to a_4, which is $P_3(a_4)$. Define h_4 to be the vertex at distance 2Δ from a_4 along this path $P_3(a_4)$. Next we do the same thing from a_2 to h_4. That is, we look at the 4Δ last edges along this path $P_2(h_4)$ and define i_4 to be the vertex at distance 2Δ from h_4. Then we look at the 4Δ last edges along the path $P_1(i_4)$ from a_1 to i_4. More generally, for $j \in \{4, 5, 6\}$, we define h_j to be the vertex on path $P_3(a_j)$ such that $d_{P_3(a_j)}(a_j, h_j) = 2\Delta$ and define i_j to be the vertex on path $P_2(h_j)$ such that $d_{P_2(h_j)}(i_j, h_j) = 2\Delta$.

Now let's define what are those six vertex sets (see Figure 3):

- $A_1 = P_1^-(i_4) \cup P_1^-(i_5) \cup P_1^-(i_6)$
- $A_2 = P_2^-(h_4) \cup P_2^-(h_5) \cup P_2^-(h_6)$
- $A_3 = P_3^-(a_4) \cup P_3^-(a_5) \cup P_3^-(a_6)$
- $A_4 = P_3^+(a_4) \cup P_2^+(h_4) \cup P_1^+(i_4)$
- $A_5 = P_3^+(a_5) \cup P_2^+(h_5) \cup P_1^+(i_5)$

- $A_6 = P_3^+(a_6) \cup P_2^+(h_6) \cup P_1^+(i_6)$

In the next section, we are going to show two things, namely that these six sets are disjoint and that they are connected in a proper way.

3 Proof of the Main Theorem

To show that they are connected in a proper way is trivial. Why? Let's take A_3 and A_4 as an example. Obviously, A_3 and A_4 are connected by that edge between $P_3^+(a_4)$ and $P_3^-(a_4)$. Similarly, we can find other connecting edges. Hence, we know that for $\forall i \in \{1, 2, 3\}$ and $\forall j \in \{4, 5, 6\}$, there exists an edge between A_i and A_j. So the only thing that remains to show is that those A_i's are disjoint.

Claim 2. *$\forall v \in A_i$, $i \in \{4, 5, 6\}$, $d(v, a_i) \leq 8\Delta$.*

Proof. $\forall v\ A_i$, we can exhibit a path between v and a_i and the distance is not large. Let's take $i = 4$ as an example. For $v \in P_3^+(a_4)$, $d(v, a_4) \leq 4\Delta$; for $v \in P_2^+(h_4)$, $d(v, a_4) \leq 6\Delta$ and for $v \in P_1^+(i_4)$, $d(v, a_4) \leq 8\Delta$. Similarly, we can prove that the inequalities hold for A_5 and A_6. □

Since $d(a_i, a_j) \geq 17\Delta$ holds for $4 \leq i < j \leq 6$, we can get the following corollary.

Corollary 3. *A_4, A_5 and A_6 are disjoint.*

Now we are going to show that A_1, A_2 and A_3 are disjoint in the following way.

Claim 4. *$\forall i \in \{1, 2, 3\}$, $A_i \subseteq V_i \backslash V_{i+1}$.*

Proof. Let's take $i = 3$ as an example. We claim that A_3 is in V_3 but not in V_4. Recall that the paths $P_3(a_4)$, $P_3(a_5)$ and $P_3(a_6)$ are obtained when we do breadth-first search in V_3. T_3 is the corresponding breadth-first search tree and we can define the levels for vertices in T_3. Recall that V_4 is one of the connected components in $V_3 \backslash F_3$, and the levels of vertices in V_4 are between $k_3 + j\Delta$ and $k_3 + (j + 1)\Delta$ for some integer j, as shown in Figure 4. Hence, the level difference between every two vertices within V_4 is at most $\Delta - 1$. If we trace back at least Δ from any vertex in V_4, we are out of V_4. Because the vertices in A_3 are all ancestors of a_4, a_5 or a_6 and they are at distance at least 4Δ from a_4 or a_5 or a_6 in T_3, therefore cannot be within V_4. In this way, we prove that $A_3 \subseteq V_3 \backslash V_4$.

The case of A_2 is similar. We do breadth-first search in V_2 and get the breadth-first search tree T_2. We know that h_4, h_5 and h_6 are all in one connected component V_3, so their level differences in T_2 are at most $\Delta - 1$. As we trace back Δ, we get out of V_3. Hence, $A_3 \subseteq V_2 \backslash V_3$. It is similar for A_1. □

Therefore, $A_i \subseteq V_i \backslash V_{i+1}$ holds for $1 \leq i \leq 3$, which implies the following corollary.

Corollary 5. *A_1, A_2 and A_3 are disjoint.*

In the following part, we are going to prove that some paths are disjoint which will help us show that A_i is disjoint from A_j for $i \leq 3$ and $j \geq 4$. First, we consider the paths $P_3^+(a_i)$ and $P_3^-(a_j)$ with $i \neq j$ and show that they are disjoint.

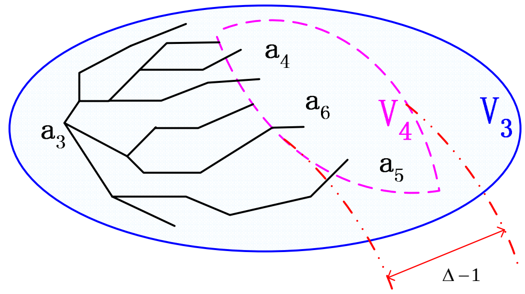

Figure 4: Levels in V_3.

Claim 6. *$P_3^+(a_i) \cap P_3^-(a_j) = \emptyset$, $i \neq j$ and $i, j \in \{4,5,6\}$.*

Proof. Assume that $i = 5$ and $j = 6$; other choices are similar. Assume that there exists a vertex $v \in P_3^-(a_5) \cap P_3^+(a_6)$.

$v \in P_3^+(a_6)$ means that $d_{G_3}(v, a_6) \leq 4\Delta$ according to the definition of $P_3^+(a_6)$. Because a_4 a_5 and a_6 are all in V_4, their level differences are at most $\Delta - 1$ in T_3. If $v \in P_3^-(a_5)$, we can derive that $d_{G_3}(v, a_5) \leq 4\Delta + (\Delta - 1) = 5\Delta - 1$. By the triangle inequality, we get $d_{G_3}(a_5, a_6) \leq d_{G_3}(v, a_5) + d_{G_3}(v, a_6) \leq 9\Delta - 1$, which contradicts our assumption $d(a_5, a_6) \geq 17\Delta$. □

We prove an analogous statement for T_2 first, and then T_1 .

Claim 7. *$P_2^+(h_i) \cap P_2^-(h_j) = \emptyset$, for $i \neq j$ and $i, j \in \{4,5,6\}$.*

Proof. If there exists a vertex $v \in P_2^+(h_i) \cap P_2^-(h_j)$ for $i \neq j$ and $i, j \in \{4,5,6\}$, we know that $d_{G_2}(v, h_i) \leq 4\Delta$ which implies that $d_{G_2}(v, a_i) \leq 4\Delta + 2\Delta = 6\Delta$. Because h_i and h_j are both within V_3, their level difference is at most $\Delta - 1$ in T_2. So we know that $d_{G_2}(v, h_j) \leq 4\Delta + (\Delta - 1)$ which implies that $d_{G_2}(v, a_j) \leq 4\Delta + (\Delta - 1) + 2\Delta = 7\Delta - 1$. By the triangle inequality, we get $d_{G_2}(a_i, a_j) \leq 13\Delta - 1$ which contradicts our assumption $d(a_i, a_j) \geq 17\Delta$ for $i \neq j$ and $i, j \in \{4,5,6\}$. □

Claim 8. *$P_1^+(i_j) \cap P_1^-(i_k) = \emptyset$, for $j \neq k$ and $j, k \in \{4,5,6\}$.*

Proof. If there exists a vertex $v \in P_1^+(i_j) \cap P_1^-(i_k)$, we know $d_{G_1}(v, i_j) \leq 4\Delta$ which implies that $d_{G_1}(v, a_j) \leq 8\Delta$. As before, we also get $d_{G_1}(v, i_k) \leq 4\Delta + (\Delta - 1)$ which implies that $d_{G_1}(v, a_k) \leq 9\Delta - 1$. Hence we get $d_{G_1}(a_j, a_k) \leq 17\Delta - 1$ which contradicts our assumption. □

This last claim justifies our choice of 34Δ for the weak diameter. Now we are going to show that A_i is disjoint from A_4, A_5, A_6, for each $i = 1, 2, 3$. We start with $i = 1$ as this is the easiest case.

Claim 9. *A_1 is disjoint from A_j, $j \in \{4,5,6\}$.*

Proof. First we show that A_1 is disjoint from A_4. Both of them consist of three paths:

$$A_1 = P_1^-(i_4) \cup P_1^-(i_5) \cup P_1^-(i_6)$$

$$A_4 = P_3^+(a_4) \cup P_2^+(h_4) \cup P_1^+(i_4)$$

By definition, $P_1^+(i_4) \cap P_1^-(i_4) = \emptyset$. By Claim 8, $P_1^+(i_4)$ is disjoint from both $P_1^-(i_5)$ and $P_1^-(i_6)$.

Notice that $P_3^+(a_4) \cup P_2^+(h_4) \subseteq V_2$, and $A_1 \subseteq V_1 \backslash V_2$ (shown in Claim 4), so we derive that $P_3^+(a_4) \cup P_2^+(h_4)$ is disjoint from A_1. Hence we conclude that A_1 is disjoint from A_4. Similarly we can prove that A_1 is disjoint from A_5 and A_6. □

Claim 10. *A_2 is disjoint from A_j, $j \in \{4,5,6\}$.*

Proof. $A_2 = P_2^-(h_4) \cup P_2^-(h_5) \cup P_2^-(h_6)$. We will show that A_2 is disjoint from A_4, without loss of generality. First $P_2^+(h_4) \cap A_2 = \emptyset$ because of Claim 7. Secondly $P_3^+(a_4) \cap A_2 = \emptyset$ because $A_2 \subseteq V_2 \backslash V_3$ and $P_3^+(a_4) \subseteq V_3$. Similarly, we can see that $(P_1^+(i_4) \backslash V_2) \cap A_2 = \emptyset$. Finally, let's consider $v \in P_1^+(i_4) \cap V_2$ and $w \in A_2$, and we are going to show that $v \neq w$. Assume that $w \in P_2^-(h_k)$. We derive that

$$\begin{aligned} d_{G_2}(a_2, v) &\geq d_{G_2}(a_2, i_4) - (\Delta - 1) \\ &= d_{G_2}(a_2, h_4) - 2\Delta - (\Delta - 1) \\ &= d_{G_2}(a_2, h_4) - 3\Delta + 1 \\ &\geq d_{G_2}(a_2, h_k) - (\Delta - 1) - 3\Delta + 1 \\ &\geq d_{G_2}(a_2, w) + 2 \end{aligned}$$

the first inequality following from the fact that $v \in P_1^+(i_4) \cap V_2$ implies that $P_1^+(i_4) \backslash P_1^+(v) \subseteq V_2$, the second inequality from the definition of V_3, and the last inequality from the fact that $w \in A_2$. The fact that $d_{G_2}(a_2, v) \geq d_{G_2}(a_2, w) + 2$ means that v and w can not be equal, i.e, $(P_1^+(i_4) \cap V_2) \cap A_2 = \emptyset$. Similarly, we can show that A_2 is disjoint from A_5 and A_6. □

Let's do the same thing for A_3.

Claim 11. *A_3 is disjoint from A_j, $j \in \{4,5,6\}$.*

Proof. Consider $A_3 = P_3^-(a_4) \cup P_3^-(a_5) \cup P_3^-(a_6)$ and $A_4 = P_3^+(a_4) \cup P_2^+(h_4) \cup P_1^+(i_4)$ as an example.

According to Claim 6, we can get $P_3^+(a_4) \cap A_3 = \emptyset$. For those vertices in $P_2^+(h_4) \backslash V_3$, we can see that they are disjoint from A_3 because of $A_3 \subseteq V_3$. Now we consider $P_2^+(h_4) \cap V_3$. Suppose v is an arbitrary vertex in $P_2^+(h_4) \cap V_3$, and u is an arbitrary vertex in $P_3^-(a_k) \subseteq A_3$, see Figure 5.

As in the previous claim, we have:

$$\begin{aligned} d_{G_3}(a_3, v) &\geq d_{G_3}(a_3, h_4) - (\Delta - 1) \\ &= d_{G_3}(a_3, a_4) - 2\Delta - (\Delta - 1) \\ &= d_{G_3}(a_3, a_4) - 3\Delta + 1 \\ &\geq d_{G_3}(a_3, a_k) - (\Delta - 1) - 3\Delta + 1 \\ &\geq d_{G_3}(a_3, u) + 2 \end{aligned}$$

showing that $u \neq v$, i.e. $(P_2(h_4) \cap V_3) \cap A_3 = \emptyset$.

The analysis for $P_1(i_4) \cap A_3 = \emptyset$ is similar. Hence, A_3 is disjoint from A_4. Similarly we can prove that A_3 is also disjoint from A_5 and A_6. □

We have shown that these six sets are disjoint and they are connected by edges, which means that we have found a $K_{3,3}$ minor.

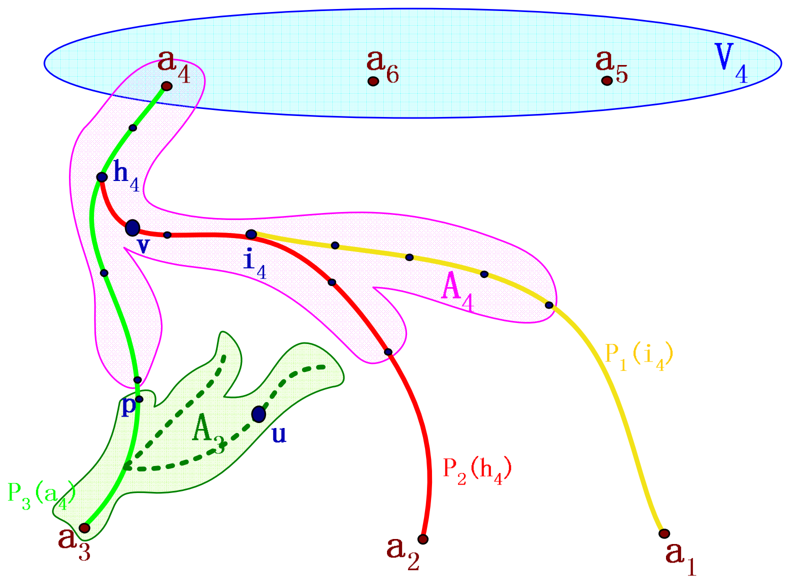

Figure 5: A_3 and A_4.

References

[1] Philip N. Klein, Serge A. Plotkin, Satish Rao, *Excluded minors, network decomposition, and multicommodity flow*, STOC 1993: 682-690.

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[3] Jittat Fakcharoenphol, Kunal Talwar, *An Improved Decomposition Theorem for Graphs Excluding a Fixed Minor*, RANDOM-APPROX 2003: 36-46.