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18.409: Topics in TCS: Embeddings of Finite Metric Spaces
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Lecture 8

Today we talk about embedding planar graph metrics into ℓ_2. Earlier we had reviewed a result of Newman and Rabinovich [3] where we exhibited a family of planar graphs for which which needed $\Omega(\sqrt{\log n})$ distortion for embedding into ℓ_2. Rao [4] showed that any planar graph metric embeds into ℓ_2 with distortion $O(1)$. The proof uses a decomposition technique of Klein, Plotkin and Rao [2]. The proof presented here partly uses a description in Chapter 15 from Matousek, and it slightly deviates from what was done originally.

Let $G = (V, E)$ be a planar graph with non-negative weights w_e on its edges. Its shortest-path metric defines a *planar graph metric* d.

Theorem 1. *Any planar graph metric d embeds into ℓ_2 with distortion $O(\sqrt{\log n})$.*

The above result shows that d embeds into ℓ_1 with distortion at most $\sqrt{O(\log n)}$, since ℓ_2 isometrically embeds into ℓ_1. Improving this bound however is still an open question, and it is conceivable that we can embed any planar graph metric with $O(1)$ distortion.
To prove the above result we use the characterization of planar graphs due to Kuratowski:

Definition 1. Graph minor: *A minor of a graph G is a graph obtained from G by a series of edge contractions and edge deletions.*

The graph $G \setminus e$ obtained from G by a *deletion* of e is $(V, E - \{e\})$. The graph G/e obtained from G by an edge *contraction* is obtained by deleting e and identifying the two end vertices of e.

Theorem 2. Kuratowski: *A graph G is planar iff it does not contain a K_5 or a $K_{3,3}$ as a minor.*

Background: There is an important result of Robertson and Seymour is the Graph Minor Project which can be formulated as follows:

Theorem 3. *If $\mathcal{G}$ is a nonempty minor-closed family of graphs, then $\mathcal{G}$ can be characterized by a finite set of excluded minors.*

A minor-closed property of a graph is one which is closed under edge deletions and contractions. Planarity is one such example of a minor-closed property. In general, embeddings on surfaces of higher genus is also minor-closed.

The theorem of Rao [4] is not just true for planar metrics, but for metrics of any family of graphs having a minor-closed property, i.e. a family characterized by a finite list of excluded minors. It was shown by Gupta, Newman, Rabinovich and Sinclair [1] that the metric of a series-parallel graph embeds into ℓ_1 with constant distortion. Such graphs are characterized by the excluded minor K_4.

We'll obtain an upper bound on the distortion for metrics of graphs that don't contain $K_{3,3}$ as a minor. Since planar graphs fall under this category, this will automatically imply the result for planar graphs.

We first prove the result assuming that all edges have unit weights, i.e. $w_e = 1\ \forall e \in E$. We use an embedding similar to a *Frechet* embedding.

Properties of random decomposition: Fix $\Delta = 2^k$, for $k \in \mathbb{N}$, $2 \leq k \leq \mathit{diameter}(G)$. We're going to construct a random decomposition of the graph as follows. We obtain a probability distribution μ on 2^V, the set of subsets of the vertex set V of G having the following properties. Let $A \subseteq V$ with $\mu(A) > 0$:

- Let $s_1, s_2, \dots s_l$ be the connected components of $G[V - A]$. We define the *weak diameter* of s_i to be $\max_{u,v \in s_i}(d_G(u,v))$, where d_G denotes the distance metric in the original graph G. Then
$$\mu(A) > 0 \Rightarrow \text{weak diameter}(s_i) < 34\Delta.$$
- $\forall v \in V, \Pr[d(v, A) > \frac{1}{4}\Delta] \geq \frac{1}{8}$.

We'll show how such a decomposition can be constructed later. We now show how to construct the required embedding given such a decomposition.

Construction of embedding: For a given A, for each connected component s_i of $G[V - A]$, we choose random signs $\sigma(s_i) \in \{-1, 1\}$ uniformly and independently.

Consider the map

$$\phi_{A,\sigma} : v \rightarrow \begin{cases} \sigma(s_i) d(v, A) & \text{if } v \in s_i \\ 0 & v \in A. \end{cases}$$

The above map has the following non-expanding property:

Proposition 4. $\forall A \in 2^V, \forall \sigma, \forall u, v \in V, u \in s_i, v \in s_j$

$$|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)| \leq d(u, v).$$

Proof. We fix some A, σ, u, v. Let $x \in A$ be such that $d(u, A) = d(u, x)$, and similarly let $y \in A$ be such that $d(v, A) = d(v, y)$.

If $i = j$, then $|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)| = |d(u, A) - d(v, A)|$. Since

$$\begin{aligned} d(u, A) - d(v, A) &= d(u, x) - d(v, y) \\ &\leq d(u, x) - d(v, x) \\ &\leq d(u, v). \end{aligned}$$

Similarly we have $d(u, A) - d(v, A) \leq d(u, v)$, and hence $|d(u, A) - d(v, A)| \leq d(u, v)$. Thus in this case we get

$$|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)| \leq d(u, v).$$

If $i \neq j$, since there is no path from u to v in $G[V - A]$, hence $d(u, v) = d(u, x_1) + d(x_1, y_1) + d(y_1, v)$ for some $x_1, y_1 \in A$.

$$\begin{aligned} d(u, v) &= d(u, x_1) + d(x_1, y_1) + d(y_1, v) \\ &\geq d(u, x_1) + d(v, y_1) \\ &\geq d(u, A) + d(v, A). \end{aligned}$$

Moreover, $|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)| \leq d(u, A) + d(v, A)$. Hence, $|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)| \leq d(u, v)$. $\square$

Let $l(A)$ be the number of connected components of $G[V - A]$. Then, the probability of choosing a given (A, σ) is given by $\gamma(A, \sigma) = \mu(A)2^{-l(A)}$.

For $u, v \in V$, let $E_\gamma[|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)|]$ be the expected value of $[|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)|]$ taken over all choices of A, σ. By the lemma above, $E_\gamma[|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)|] \leq d(u, v)$.

Proposition 5. *Let $u, v \in V$ such that $34\Delta \leq d(u, v) \leq 68\Delta$. Then $E_\gamma[|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)|] \geq \frac{d(u,v)}{68 \cdot 32}$.*

Proof. Let $u \in s_i$ and $v \in s_j$. Since the weak diameter of each component of $G[V \setminus A] < 34\Delta$, $\therefore i \neq j$. Since $\Pr[\sigma(s_i) \neq \sigma(s_j)] = \frac{1}{2}$,

$$\begin{aligned} E_\gamma[|\phi_{A,\sigma}(u) - \phi_{A,\sigma}(v)|] &\geq \frac{1}{2} E_\gamma[d(u, A) + d(v, A)] \\ &\geq \frac{1}{2}(\frac{\Delta}{4} \cdot \frac{1}{8} + \frac{\Delta}{4} \cdot \frac{1}{8}) \\ &\geq \frac{d(u, v)}{68 \cdot 32}. \end{aligned}$$

□

Theorem 6. *Every planar graph metric d such that every edge has unit weight, can be embedded into ℓ_2 with distortion $O(\sqrt{\log n})$.*

Proof. First of all, subdivide every edge into a path of length 136, to make sure that all distances between the original vertices are at least 34Δ when $\Delta \geq 4$. Consider the map $\phi_\Delta : V \to \ell_2^N$,for some $N \in \mathbb{N}$ with $\phi_\Delta(v) = (\sqrt{\gamma(A, \sigma)}\sigma(s_i)d(v, A))_{A,\sigma}$ (for $v \in s_i$), i.e. the coordinate value of $\phi_\Delta(v)$ corresponding to (A, σ) is $\sqrt{\gamma(A, \sigma)}\sigma(s_i)d(v, A)$. By proposition 4,

$$\forall u, v \in V, \|\phi_\Delta(u) - \phi_\Delta(v)\|_2 \leq d(u, v).$$

Also, by proposition 5, for all $u, v \in V$ with $34\Delta \leq d(u, v) \leq 68\Delta$,

$$\|\phi_\Delta(u) - \phi_\Delta(v)\|_2 \geq \frac{d(u, v)}{2176}.$$

To construct the final embedding $\eta : V \to \ell_2$, we let $\eta(v)$ be the concatenation of the vectors ϕ_Δ for $\Delta \in \{2^j | 4 \leq 2^j \leq 2 \text{ diameter}(G)\}$, where diameter$(G)$ denotes the diameter in the original graph. This choice of values for Δ is to make sure that for every pair of vertices u, v of the original graph, there exists a Δ with $34\Delta \leq d(u, v) \leq 68\Delta$.

Since $\exists\Delta$ such that $|\phi_\Delta(u) - \phi_\Delta(v)| \geq \frac{d(u,v)}{2176}$ (as we have subdivided the original edges), hence $\|\eta(u) - \eta(v)\|_2 \geq \frac{d(u,v)}{2176}$. As diameter$(G) < n$, where $n = |V|$, we have at most $\log(n)$ possible values of Δ. Hence $\|\eta(u) - \eta(v)\|_2 \leq \sqrt{\log(n)}d(u, v)$. Thus we have an embedding into ℓ_2 with distortion $O(\sqrt{\log n})$.

□

Existence of decomposition. Fix $a_1 \in V$ arbitrarily. This will serve as the node of a breadth-first search tree. We pick an integer $k_1 \in \{0, 1, \ldots, \Delta - 1\}$ uniformly at random. Let $A_1 = \{v \in V : d(v, a_1) \equiv k_1(\mathrm{mod}\Delta)\}$. When we remove the vertices of A_1 from G, the remaining vertices get partitioned into connected components. This is the first level of the decomposition. We repeat the same process for every one of these components. For each component, we pick a new node a_2 for a breadth-first search tree and we pick k_2 at random from $\{0, 1, \ldots, \Delta - 1\}$. Let A_2 be the union over all components of all vertices at a distance congruent to k_2 modulo Δ from the tree node. At the second level of the decomposition, we get all the connected components of $G - (A_1 \cup A_2)$. To get A_3, we repeat the process over again for each connected component. Let $A = A_1 \cup A_2 \cup A_3$.

Proposition 7. *Let* $\Delta \equiv 0 \pmod 4$*. Then,* $\Pr[d(v, A) \geq \frac{\Delta}{4}] \geq \frac{1}{8}$.

Proof. We observe that $\Pr[d(v, A_1) \geq \frac{\Delta}{4}] \geq \frac{1}{2}$. This is so as $\Pr[d(v, A_1) \geq \frac{\Delta}{4}] = \Pr[d(v, a_1) \equiv r + k_1(\mathrm{mod}\Delta)$, where $r \in \{\lceil \frac{\Delta}{4} \rceil, \lceil \frac{\Delta}{4} \rceil + 1, \ldots, \lfloor \frac{3\Delta}{4} \rfloor\}$. Since k_1 was chosen at random and $\Delta \equiv 0 \pmod 4$, $\Pr[d(v, A_1) \geq \frac{\Delta}{4}] \geq \frac{1}{2}$.

Similarly we have $\Pr[d(v, A_2) \geq \frac{\Delta}{4} | v \notin A_1] \geq \frac{1}{2}$, and $\Pr[d(v, A_3) \geq \frac{\Delta}{4} | v \notin (A_1 \cup A_2)] \geq \frac{1}{2}$. Hence, $\Pr[d(v, A) \geq \frac{\Delta}{4}] \geq \frac{1}{8}$. $\square$

It remains to show that the weak diameter of each component in the resulting decomposition is less than 34Δ. Let us assume the contrary, i.e. let $a_4, a_5 \in s_i$, where s_i is a connected component of $G[V - A]$ and $d_G(a_4, a_5) \geq 34\Delta$. Let $P_{4,5}$ be a path from a_4 to a_5 in s_i. Then there exists $a_6 \in P_{4,5}$ such that $d_G(a_4, a_6) \geq 17\Delta$ and $d_G(a_5, a_6) \geq 17\Delta$.

The remainder of the proof will be covered in the next lecture.

References

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[3] Ilan Newman and Yuri Rabinovich. A lower bound on the distortion of embedding planar metrics into Euclidean Space. *Symposium on Computational Geometry,* 94-96, 2002.

[4] Satish Rao. Small distortion and volume preserving embeddings for planar and Euclidean metrics. *Symposium on Computational Geometry,* 300-306, 1999.