Lecture 4

Definition 1. Given $G = (V, E)$, the Uniform Sparsest Cut Problem is the problem of computing

$$\beta(G) = \min_{\emptyset \neq S \subseteq V} \frac{|\delta(S, V \setminus S)|}{|S||V \setminus S|}$$

where $\delta(A, B) = \{(u, v) \in E | u \in A, v \in B\}$.

Remark 1. We are often interested not only in the value $\beta$, but also in the argument, $S$ of the equation.

Finding the Sparsest Cut is an NP-hard problem, but is also related to expansion. Recall:

$$\phi(G) = \min_{S \subseteq V, |S| \leq |V|/2} \frac{|\delta(S, V \setminus S)|}{|S|}.$$

Thus, $\frac{\phi(G)}{n} \leq \beta(G) \leq 2\frac{\phi(G)}{n}$.

So, if we can find an approximate solution for the sparsest cut problem, we can also find an approximation for expansion. For example, if we have a family of $r$-regular expanders, then we know that $\beta(G) = \Theta(1)$.

Before we introduce the non-uniform version of the sparsest cut problem, we define the concurrent multicommodity flow problem.

Definition 2. The Concurrent Multi-Commodity Flow Problem: We are given $G = (V, E, c)$ where $c : E \to \mathbb{R}^+$ and $k$ “commodities” such that $\forall i \in [k]$, we have source/destination pairs $(s_i, t_i) \in V \times V$ and demand $D_i$. The goal is to maximize $\alpha$ such that $\forall i$ we can simultaneously send $\alpha D_i$ units between $s_i$ and $t_i$ with respect to the capacities $c$. Let $\alpha(G)$ denote the maximum such value.

Example: Using the complete bipartite graph $K_{2,3}$, we set $k = 4$ and set $\forall e : c(e) = 1$ and $\forall i \in [4] : D_i = 1$. We set our sources and sinks as shown in Figure 1. For this example, we see a valid concurrent flow in Figure 2 with $\alpha = 3/4$.

In our example, each path between source and destination for any commodity has length 2, so $\alpha D_i$ uses $2\alpha$ units of capacity. As we have 4 units of demand and a total capacity of 6 in the graph, we see that $8\alpha \leq 6 \Rightarrow \alpha \leq \frac{3}{4}$. Thus, for our instance, $\alpha(G) = \frac{3}{4}$.

To get an upper bound on $\alpha(G)$, we can look at every cut and the requirement through this cut. This leads to the following (non-uniform) Sparsest Cut problem:

Definition 3. Let

$$\beta(G) = \min_S \frac{\sum_{e \in \delta(S, V \setminus S)} c(e)}{\sum_{s_i, t_i : s_i \in S \text{ XOR } t_i \in S} D_i}.$$

Observation 1. It follows that $\alpha(G) \leq \beta(G)$ since in the cut $\delta(S, V \setminus S)$ we would need $\sum_{s_i, t_i : s_i \in S \text{ XOR } t_i \in S} D_i$ if we were to fully satisfy the demands. In our example, $\alpha(G) = \frac{3}{4} < \beta(G) = 1$. 

1
Figure 1: The complete bipartite graph with sizes 2 and 3.

**Observation 2.** For another example, we look again at constant-degree expanders. From before we saw that the expansion $\phi(G)$ is constant, and thus $\beta(G) \geq \phi(G)/n = \Omega(1/n)$.

In order to calculate $\alpha$, we note that in an expander graph, $\sum_{e \in E} c(e) = \frac{m}{2} = \Theta(n)$ and for most $s_i, t_i$ pairs, the path length between them will be $\Omega(\log n)$. So, $\alpha(G) = \frac{\Theta(n)}{\Omega(n^2 \log \log n)} = O(\frac{1}{n \log n})$. Therefore, $\frac{\beta(G)}{\alpha(G)} = \Theta(\log n)$.

**Remark 2.** In some cases, $\alpha(G) = \beta(G)$. The first is simple: $k = 1$. Equality follows directly from the equivalence of Max-Flow and Min-Cut. T.C. Hu proved that the two are also equal when $k = 2$ in 1963.

Now, we relate these values to embeddings and their distortions. We will first derive two very similar looking expressions for $\beta(G)$ and $\alpha(G)$.

**Lemma 1.** $\beta(G) = \min_{(v, d) \text{ i.e. embeddable}} \frac{\sum_{e \in E} c(e) \cdot d(e)}{\sum_{i \in [k]} D_i d(s_i, t_i)}$.

**Proof.** Use the cut metric induced by $S$: $d(u, v) = 1 \iff u \in S \ XOR \ v \in S$. Then

$$\beta(G) = \min_{\text{cut metrics } d} \frac{\sum_{e \in E} c(e) \cdot d(e)}{\sum_i D_i d(s_i, t_i)} = \min_{d \in \text{CUT}_n} \frac{\sum_{e \in E} c(e) \cdot d(e)}{\sum_i D_i d(s_i, t_i)}.$$

The last equality follows from the fact that $\text{CUT}_n$ is precisely the cone of all the cut metrics and that $\beta(G)$ is the ratio of two linear functions. More precisely, if we let $a_i$ and $b_i$ be the numerator and denominator for cut $i$, we obtain $\min_{i} \left\{ \frac{a_i}{b_i} \right\}$ when we optimize over the cuts while we get

$$\min_{\lambda \geq 0} \frac{\sum_i \lambda_i a_i}{\sum_i \lambda_i b_i}$$

when we optimize over $\text{CUT}_n$. The two quantities are equal for all choices of $a_i, b_i$.

Recall from Lecture 1 (see Lemma 4 there) that $\text{CUT}_n$ is exactly the set of all $l_1$-embeddable (semi)-metrics, and this completes the proof of the lemma. 

\[\square\]
Figure 2: A solution providing as much concurrent flow as possible.

Lemma 2. \[\alpha(G) = \min_{(V,d) \text{ t}_\infty \text{-embeddable}} \frac{\sum_{e \in E} c(e) \cdot d(e)}{\sum_{i} D_i d(s_i, t_i)}\]

Proof. Formulate \(\alpha(G)\) as a linear program. We will do this in a seemingly straightforward and stupid way. For all \(i\), we enumerate all paths between \(s_i\) and \(t_i\). So, \(P_{ij}\) is the \(j^{th}\) such path between \(s_i\) and \(t_i\) and \(x_{ij}\) is the amount of flow on \(P_{ij}\). Then,

\[
\begin{align*}
\alpha(G) &= \max \alpha \\
\text{s.t.} & \quad \alpha D_i - \sum_j x_{ij} \leq 0 \quad i = 1, \cdots, k \quad (1) \\
& \quad \sum_{i=1}^{k} \sum_{j : e \in P_{ij}} x_{ij} \leq c(e) \quad e \in E \quad (2) \\
& \quad \alpha \geq 0 \\
& \quad x_{ij} \geq 0 \quad \forall i, j
\end{align*}
\]

By strong duality, we know that \(\alpha(G)\) also equals to (using dual variables \(h_i\) for constraints (1) and \(l_e\) for constraints (2)): 
\[ \alpha(G) = \min \sum_{e \in E} c(e) \]
\[ \text{s.t.} \]
\[ \sum_i D_i h_i \geq 1 \]
\[ \sum_{e \in P_{ij}} l_e - h_i \geq 0 \quad \forall i, j \] (3)
\[ h_i \geq 0 \quad \forall i \]
\[ l_e \geq 0 \quad \forall e \] (4)

Observe that we can assume that (3) is an equality (by decreasing some of the \(h_i\)'s without violating any other constraint), and similarly that for every \(i\), \(h_i\) equals the shortest path length with respect to \(l_e\) between \(s_i\) and \(t_i\) (i.e. for every \(i\), there exists \(j\) with (4) an equality).

Thus, from the optimum solution \(l, h\) of the above dual program, we can construct the shortest path metric \(d\) corresponding to the lengths \(l_e\), and we have that

\[ \alpha(G) = \frac{\sum_{e \in E} c(e) \cdot d(e)}{\sum_i D_i d(s_i, t_i)} . \]

Vice versa, if we have an \(l_\infty\)-embeddable metric \(d\) which minimizes \(\sum_{e \in E} c(e) d(e) / \sum_i D_i d(s_i, t_i)\), without loss of generality (without increasing the ratio), we can assume that this metric is the shortest path metric corresponding to \(d_e\) for \(e \in E\) (i.e. the distance separating the endpoints of a non-edge is indeed given by the shortest path). Thus, it leads to a feasible solution of the above dual program and this proves equality and completes the proof.

Linial, London and Rabinovich, '95 and Aumann and Rabani '95 proposed to solve the linear program to get the value \(\alpha\) and the \(l_1\)-embeddable metric \(d\) that optimizes it. Now, embed \(d\) into the \(l_1\)-metric \(l\) with distortion \(\gamma\) (with \(\gamma\) chosen as small as possible); thus we have that \(l(u, v) \leq d(u, v) \leq \gamma l(u, v)\) for all \(u, v\). Then

\[ \beta(G) \leq \frac{\sum c(e) l(e)}{\sum_i D_i l(s_i, t_i)} \leq \gamma \frac{\sum c(e) d(e)}{\sum_i D_i d(s_i, t_i)} = \gamma \alpha(G) , \]

showing that \(\frac{\beta(G)}{\alpha(G)} \leq \gamma\). In addition, the \(l_1\)-embeddable metric \(l\) can be decomposed into at most \(\binom{n}{2}\) cut metrics (see Lemma 4 from Lecture 1), and the best of these gives a cut \(S\) with

\[ \frac{\sum_{e \in \delta(S, V \setminus S)} c(e)}{\sum_{s_i, t_i \in S \text{ XOR } s_{i'} \in S} D_i} \leq \gamma \alpha(G) \leq \gamma / \beta(G) , \]

and therefore approximates the sparsest cut within a ratio of \(\gamma\).

What remains is to establish how small \(\gamma\) can be. In general, Bourgain proved that \(\gamma\) can be \(O(\log n)\) for any metric. (We will see a proof of this in the next lecture.) This implies that for all \(G, k, c, D : \frac{\beta(G)}{\alpha(G)} \leq O(\log n)\). This logarithmic gap between sparsest cut and concurrent flow was
already known for the uniform case; this was established (without embeddings of metric spaces) by Leighton and Rao ’88.

If the graph $G$ is planar, Rao has shown that for any planar graph metric $d$, we have $c_1(d) \leq c_2(d) = O(\sqrt{\log n})$. Thus for every planar graph independently of the number of commodities), we have that $\frac{\beta(G)}{\alpha(G)} = O(\sqrt{\log n})$.

Last time we noted that a regular expander graph metric $d$ satisfies $c_2(d) = \Omega(\log n)$. Today we saw that $\frac{\beta(G)}{\alpha(G)} = \Omega(\log n)$ for expander graphs, implying that $c_1(d) = \Theta(\log n)$. Thus, Bourgain’s result cannot be improved for general graphs.

We have two closing remarks.

**Remark 3.** The embedding of any metric into $l_1$ with distortion $O(\log n)$ can done efficiently. This implies that the Linial-London-Rabinovich result is algorithmic.

**Remark 4.** Suppose we don’t care about concurrent flow, and just finding an approximation to the sparsest cut. Instead of optimizing over $l_\infty$ metrics (which was convenient since we could use linear programming), we can try instead to optimize over a more restrictive class of metrics which nevertheless includes all cut metrics (so that the resulting bound is a lower bound on $\beta(G)$).

One class for which this has been successful is the class of negative type metrics. A metric $(X,d)$ is a negative-type metric if $(X,\sqrt{d})$ is isometrically embeddable into $l_2$.

This only begs more questions. One question is: can we optimize in polynomial time over negative-type metrics? It turns out that we can—at least, approximately—using Semi-Definite Programming. The clear follow up to this is: what distortion do we need to embed negative type metrics into $l_1$? According to Arora, Lee and Naor (2005), that distortion is $O(\sqrt{\log n \log \log n})$. 