Lecture 3

Recall that last lecture we discussed a theorem that provides a template which essentially all distortion lower bounds of embeddings of a finite metric space (X, δ) into ℓ_2 must follow. The proof we discussed actually is easily extended to give a more general theorem concerning distortion of embeddings into ℓ_p. Furthermore, a point we did not discuss before is if one is to invoke the theorem and show that the inequalities hold, showing that they hold for dimension $d = 1$ suffices to show that they hold for all $d > 1$ as well. This is because by taking the pth power of the norm, it suffices for the inequality to hold dimension by dimension for the original inequality to hold overall. The general theorem is stated below.

Theorem 1. *A finite metric space (X, δ) has no embedding of distortion at most D into ℓ_p ($p \geq 1$) iff $\exists \eta, \varphi : \binom{X}{2} \to [0, \infty)$ such that for all sets $V = \{v_i | v_i \in \mathbb{R}^d\}_{i \in X}$ the following two inequalities hold:*

$$\sum_{(i,j)\in\binom{V}{2}} \eta(i,j)\|v_i - v_j\|_p^p \geq \sum_{(i,j)\in\binom{V}{2}} \varphi(i,j)\|v_i - v_j\|_p^p \tag{1}$$

$$\sum_{(i,j)\in\binom{V}{2}} \eta(i,j)\delta^p(i,j)D^p < \sum_{(i,j)\in\binom{V}{2}} \varphi(i,j)\delta^p(i,j) \tag{2}$$

Furthermore, showing that the inequalities hold for embeddings into ℓ_p^1 implies that they hold for embeddings into ℓ_p.

In this lecture we give two examples of how to use Theorem 1 to provide distortion lower bounds. First, we cover the existence a family of planar graph metrics which require $\Omega(\sqrt{\log n})$ distortion to be embedded into ℓ_2. We then cover an $\Omega(\log n)$ distortion bound for the embeddability of expander graphs into ℓ_2.

Planar Graph Metrics. For $G = (V, E)$ a planar graph with $|V| = n$, we define the metric space (V, δ) to be such that $\delta(u, v)$ is the shortest path distance from u to v. The family of *planar graph metrics* is the set of all such metric spaces. Recall that $c_p(\delta)$ denotes the minimum distortion required to embed the metric space δ into ℓ_p. Rao showed that for any planar graph metric, $c_2(\delta) = O(\sqrt{\log n})$ [1]. It immediately follows that $c_1(\delta) = O(\sqrt{\log n})$ since ℓ_2 embeds isometrically into ℓ_1, but improving this bound is an open problem. With the current state of knowledge, it is conceivable that $c_1(\delta) = O(1)$ for all planar graph metrics δ.

We now review a result of Newman and Rabinovich [2] showing that Rao's upper bound is tight by exhibiting a family of planar graphs with $c_2(\delta) = \Omega(\sqrt{\log n})$.

Definition 1. *The family $\mathcal{D} = \{D_m\}_{m=1}^{\infty}$ of* ***diamond graphs*** *is the set of graphs where D_1 is the cycle on 4 vertices, and D_m is obtained from D_{m-1} by replacing every edge (u, v) of D_{m-1} with four new edges (u, x), (x, v), (u, y), and (y, v), where x and y are new vertices.*

We use the following notation for discussing the diamond graphs. The set V_m (resp. E_m) is the set of vertices (resp. edges) of D_m. The set $F_m \subset \binom{V}{2}$ denotes the anti-edges of D_m, which

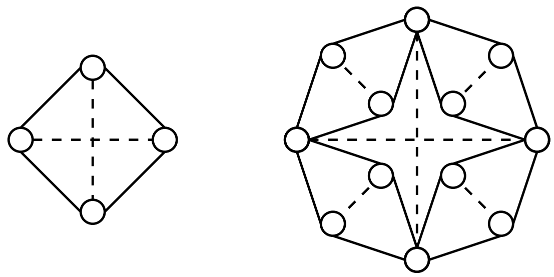

Figure 1: On the left hand side is D_1, the 4-cycle. The graph on the right hand side is D_2, the second diamond graph. Solid lines represent edges, and dashed lines represent anti-edges.

we define inductively. The anti-edges of D_1 are the diagonals of the 4-cycle. The anti-edges of D_m include all the anti-edges of D_{m-1}, in addition to the diagonals between newly introduced vertices in the same 4-cycle when replacing edges of D_{m-1}. The first two diamond graphs, along with their anti-edges, are illustrated in Figure 1.

Theorem 2. *Any embedding of the diamond graph D_m into ℓ_2 requires distortion at least $\sqrt{m+1} = \Omega(\sqrt{\log |V_m|})$.*

Proof. Define

$$\eta(i,j) = \begin{cases} 1 & \text{if } (i,j) \in E_m, \\ 0 & \text{otherwise.} \end{cases}$$

We first prove by induction that inequality (1) of Theorem 1 holds (we can assume that the embedding is into ℓ_p^1). The case $m = 1$ amounts to proving the inequality $(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 \geq (a-c)^2 + (b-d)^2$ for all $a, b, c, d \in \mathbb{R}$. This was shown in a previous lecture when discussing the distortion of embedding the 4-cycle and hamming cube graphs into ℓ_2.

Now we proceed with the inductive step. If vertex i is mapped to $v_i \in \mathbb{R}$, then we have that

$$\sum_{(i,j)\in E_{m-1}} (v_i - v_j)^2 \geq \sum_{(i,j)\in F_{m-1}} (v_i - v_j)^2$$

Adding $\sum_{(i,j)\in F_m - F_{m-1}} (v_i - v_j)^2$ to both sides, we have:

$$\sum_{(i,j)\in F_m - F_{m-1}} (v_i - v_j)^2 + \sum_{(i,j)\in E_{m-1}} (v_i - v_j)^2 \geq \sum_{(i,j)\in F_m} (v_i - v_j)^2$$

Notice that if we associate sets of terms in the sum on the left hand side with each new 4-cycle formed by the addition of vertices going from D_{m-1} to D_m, we can apply the argument that the sum of squares of edge lengths is at least the sum of squares of the diagonals for each set of terms. Thus, the left hand side is at most $\sum_{(i,j)\in E_m} (v_i - v_j)^2$, as desired.

We now show inequality (2). The left hand side equals

$$\sum_{(i,j)\in E_m} \delta^2(i,j) D^2 = |E_m| D^2 = 4^m D^2$$

The right hand side equals $\sum_{(i,j)\in F_m} \delta^2(i,j)$. Notice that there is an anti-edge corresponding to each edge in D_k for $k < m$, plus an additional two anti-edges that remain from D_1. An anti-edge formed in D_k connects two vertices that are at distance $2^{(m-k)}$ in D_m. The right hand side thus equals

$$2 \cdot (2^m)^2 + \sum_{k=1}^{m-1} 4^k \cdot 2^{(m-k)^2} = (m+1)4^m.$$

This shows that $D \geq \sqrt{m+1} = \Omega(\sqrt{\log |V_m|})$. $\square$

Expander Graph Metrics. Bourgain showed that any finite metric space can be embedded into ℓ_2 with only $O(\log n)$ distortion [4]; this will be shown in a forthcoming lecture. Linial, London, and Rabinovich showed that this bound is actually tight [3] by demonstrating that embedding the shortest path metric of an expander graph into ℓ_2 requires $\Omega(\log n)$ distortion.

Definition 2. *For a graph $G = (V, E)$, the* ***conductance*** *(also called* ***edge expansion****) $\Phi(G)$ is defined as*

$$\Phi(G) = \min_{1 \leq |S| \leq \frac{n}{2}} \frac{|\delta(S, V \setminus S)|}{|S|}$$

where $\delta(U, V)$ is the set of edges $\{(u,v) \in E | u \in U, v \in V\}$.

Definition 3. *A family $\mathcal{G} = \{G_i\}_{i=1}^{\infty}$ of graphs is said to be a family of r-regular expander graphs when*

1. $\lim_{i\to\infty} |V(G_i)| = \infty$
2. $\exists c > 0$ *such that* $\forall i \; \Phi(G_i) \geq c$

Expanders can be most easily shown to exist by using the probabilistic method, though explicit constructions do exist. For the purposes of this lecture, we take their existence for granted.

Definition 4. *The* ***adjacency matrix*** *of G, A_G is an $n \times n$ matrix where entry i,j is 1 if $(i,j) \in V(G)$, and is 0 otherwise. The* ***Laplacian*** *L_G is an $n \times n$ matrix defined as:*

$$(L_G)_{i,j} = \begin{cases} -1 & \text{if } i \neq j \text{ and } (i,j) \in E(G), \\ d_G(v_i) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Here $d_G(v)$ is the degree of v.

Note that for an r-regular graph G, $L_g = rI - A$. Thus, eigenvalues λ of A_G are in correspondence with eigenvalues μ of L_G by the mapping $\lambda_i \mapsto r - \mu_i$. We label the n eigenvalues of L_G so that $\mu_i \leq \mu_{i+1}$ for $i = 1, 2, \ldots, n-1$. Notice that the vector $\sum_{i=1}^n e_i$ is an eigenvector of L_G with eigenvalue 0, so one of the μ_i is 0. Also, for any x we have that $x^T L_G x = \sum_{(u,v)\in E(G)} (x_u - x_v)^2$. Therefore L_G is positive semidefinite, and so $\mu_1 = 0$.

We will use the following Lemma without proof:

Lemma 3. *For any graph G:*

$$2\Phi(G) \geq \mu_2(G) \geq \frac{\Phi^2(G)}{4r}$$

The relevance of Lemma 3 to us is that $\mu_2(G) = \Theta(1)$ for G an expander.

Theorem 4. *For a graph G an expander, the shortest-path metric δ for G has $c_2(\delta) = \Omega(\log n)$.*

Proof. We will again use an η, φ argument. Observe that for any embedding $x_1, x_2 \ldots, x_n$ into ℓ_2^1, we can assume $\sum_i x_i = 0$ without affecting the distortion. This can be achieved by a simple translation. Now, let $v_1, v_2, \ldots, v_n$ be the unit eigenvectors for L_G. The v_i are orthonormal and $v_1 = (1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n})$. Thus, any x satisfying $\sum_i x_i = 0$ is orthogonal to v_1 and we can write $x = \sum_{i=2}^n \alpha_i v_i$. Now we prove a useful inequality:

$$x^T L_G x = \sum_{i=2}^n x^T(\alpha_i L_G v_i) = \sum_{i=2}^n \alpha_i \mu_i x^T v_i = \sum_{i=2}^n \alpha_i^2 \mu_i \geq \mu_2 \sum_{i=1}^n \alpha_i^2 = \mu_2 \|x\|^2. \tag{3}$$

Note in the inequality that $\alpha_1 = 0$. We will define:

$$\eta(i,j) = \begin{cases} 1 & \text{if } (i,j) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Now calculating the left-hand side of (1) from Theorem 1:

$$\sum_{(i,j)\in\binom{V}{2}} \eta(i,j)(x_i - x_j)^2 = \sum_{(i,j)\in E(G)} (x_i - x_j)^2 = \sum_{i \in V(G)} r x_i^2 - 2 \sum_{(i,j)\in E(G)} x_i x_j = x^T L_G x.$$

Now we define $\varphi(i,j) = \mu_2(G)/n$ for all $(i,j) \in \binom{V}{2}$. Then we have:

$$\begin{aligned} \sum_{(i,j)\in\binom{V}{2}} \varphi(i,j)(x_i - x_j)^2 &= \frac{\mu_2(G)}{n}\left[(n-1)\sum_{i=1}^n x_i^2 - 2\sum_{i<j} x_i x_j\right] \\ &= \frac{\mu_2(G)}{n}\left[n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2\right] \\ &= \mu_2(G)\|x\|. \end{aligned}$$

Applying inequality 3 shows that inequality (1) holds. For the second part, $\sum_{(i,j)\in E(G)} \delta^2(i,j) = |E| = nr/2$. Now, notice that for any r-regular graph the number of vertices at distance $o(\log_r n)$ from any vertex u is $o(n)$. Thus, $E[\delta^2(i,j)] \geq \alpha \log_r^2 n$ for some positive constant α. Thus for inequality (2) we have:

$$\sum_{(i,j)\in\binom{V}{2}} \frac{\mu_2(G)}{n}\delta^2(i,j) \geq \alpha \frac{\mu_2(G)}{n} \log_r^2 n \binom{n}{2}.$$

We thus need

$$\frac{nr}{2}D^2 < \frac{\mu_2(G)}{n} \log_r^2 n \binom{n}{2}$$

to make inequality (2) hold. This implies that $D = \Omega(\sqrt{\mu_2(G)/r}\log n) = \Omega(\log n)$, as $r = O(1)$, and $\mu_2(G) = O(1)$ by Lemma 3. $\square$

References

[1] Satish Rao. Small distortion and volume preserving embeddings for planar and Euclidean metrics. *Symposium on Computational Geometry*, 300–306, 1999.

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