Lecture 20

The aim of this lecture is to outline the gluing of embeddings at different scales described in James Lee's paper *Distance scales, embeddings, and metrics of negative type* from SODA 2005 [1].

We begin by recalling some definitions. A map $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) is said to be C-Lipschitz if

$$d_Y(f(x), f(y)) \leq C d_X(x, y)$$

for all $x, y \in X$. The infimum of all C such that f is C-Lipschitz is denoted by $\|f\|_{\text{Lip}}$. If f is bijective and $\|f^{-1}\|_{\text{Lip}}$ is finite, we say that f is bi-Lipschitz. The distortion of a bi-Lipschitz map is defined to be $\|f\|_{\text{Lip}}\|f^{-1}\|_{\text{Lip}}$. If $f : X \to Y$ is a 1-Lipschitz map such that for all $x, y \in X$ satisfying $\tau \leq d_X(x, y) \leq 2\tau$ we have

$$d_Y(f(x), f(y)) \geq \frac{\tau}{K},$$

then we say that f is a scale-τ embedding with deficiency K. Finally, recall that $c_2(X)$ is the minimum distortion required to embed a metric space X in L_2.

The goal of this lecture is to prove the following "gluing lemma":

Lemma 1. *Suppose that for each $m \in \mathbb{Z}$ there exists a scale-2^m embedding $\phi_m : X \to L_2$ with deficiency K. Then $c_2(X) = O(\sqrt{K \log n})$.*

For planar graphs, we obtain embeddings with deficiency $K = O(1)$, so the gluing lemma tells us that we can embed a planar graph into L_2 with distortion $O(\sqrt{\log n})$, a result we obtained before by other means. We will prove this using a series of technical lemmas, some of whose proofs will be deferred.

Lemma 2. *There exists a map $M : X \to L_2$ such that*

1. $\|M\|_{\text{Lip}} = O(\sqrt{\log n})$
2. *For all $m \in \mathbb{Z}$ and for all $x, y \in X$ satisfying $2^m \leq d(x, y) \leq 2^{m+1}$ and $\log \frac{|B(x,2^{m-1})|}{|B(x,2^{m-2})|} < 1$, we have*

$$\|M(x) - M(y)\| \geq \Omega(1) d(x, y).$$

Lemma 3. *For every $m \in \mathbb{Z}$, there exists a map $f_m : X \to L_2$ with $\|f_m\|_{\text{Lip}} \leq 1$ and such that for all $x, y \in X$ satisfying $2^m \leq d(x, y) \leq 2^{m+1}$,*

$$\|f_m(x) - f_m(y)\| \geq \frac{d(x, y)}{1 + O\left(\log \frac{|B(x,2^{m+1})|}{|B(x,2^{m-3})|}\right)}$$

Lemmas 2 and 3 will be proved in the next lecture; Lemma 2 follows from a variant of the analysis of Bourgain's embedding into Hilbert space with logarithmic distortion while Lemma 3 follows from an embedding somewhat similar to Rao's embedding for planar graph metrics.

In the following, $L_2^{\leq D}$ will denote the space with the same underlying point set as L_2 but with norm $\|x\| = \min\{\|x\|_2, D\}$.

Lemma 4. *There exists a map $G : L_2^{\leq D} \to L_2$ with distortion 2 such that for every $x \in L_2$, $\|G(x)\|_2 \leq 2D$. In particular,*

$$\frac{1}{2}\min\{D, \|x-y\|_2\} \leq \|G(x)-G(y)\|_2 \leq \min\{D, \|x-y\|_2\}.$$

We will use the following notation in the sequel:

$$\rho_m(x,y) = \begin{cases} x & \text{if } |B(x,2^m)| > |B(y,2^m)| \\ y & \text{otherwise.} \end{cases}$$

The final ingredient used in the proof of lemma 1 is the following result.

Lemma 5. *Given for every $m \in \mathbb{Z}$ a 1-Lipschitz map $h_m : X \to L_2$, there exists a map $H : X \to L_2$ which satisfies*

1. $\|H\|_{\mathrm{Lip}} = O(\sqrt{\log n})$.
2. *For every $m \in \mathbb{Z}$ and every $x, y \in X$ such that $2^m \leq d(x,y) < 2^{m+1}$, we have*

$$\|H(x)-H(y)\|_2 \geq C\sqrt{\left\lfloor \log \frac{|B(\rho_{m-3}(x,y), 2^{m+1})|}{|B(\rho_{m-3}(x,y), 2^{m-3})|} \right\rfloor} \|h_m(x)-h_m(y)\|_2$$

for some constant C.

Proof. By using lemma 4, for each map $h_m : X \to L_2$, we obtain a "truncated" map $\hat{h}_m : X \to L_2$ satisfying

$$\frac{1}{2}\min\{2^m, \|h_m(x)-h_m(y)\|\} \leq \|\hat{h}_m(x)-\hat{h}_m(y)\| \leq \|h_m(x)-h_m(y)\|$$

and $\hat{h}_m(x) \leq 2^{m+1}$. Let $R(x,t) = \sup\{R : |B(x,R)| \leq 2^t\}$. Let $\rho : \mathbb{R} \to \mathbb{R}^+$ be any $O(1)$-Lipschitz map that (1) has support contained in $[2^{-4}, 2^4]$, (2) is identically 1 on $[2^{-3}, 2^3]$, and (3) is at most 1 everywhere. Define

$$\rho_{m,t}(x) = \rho\left(\frac{R(x,t)}{2^m}\right)$$

and let

$$\psi_t(x) = \bigoplus_{m \in \mathbb{Z}} \rho_{m,t}(x)\hat{h}_m(x).$$

The map H we seek will then be given by

$$H = \psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_{\lceil \log n \rceil}.$$

Since the map $x \mapsto R(x,t)$ is 1-Lipschitz (if it were not, then we could assume $R(x,t) > d(x,y) + R(y,t)$, which easily leads to a contradiction), we have

$$|\rho_{m,t}(x)-\rho_{m,t}(y)| \leq \frac{O(1)}{2^m}|R(x,t)-R(y,t)| \leq O(1)\frac{d(x,y)}{2^m}.$$

Now

$$\|\psi_t(x)-\psi_t(y)\|^2 = \sum_{m \in \mathbb{Z}} \|\rho_{m,t}(x)\hat{h}_m(x)-\rho_{m,t}(y)\hat{h}_m(y)\|^2,$$

and we wish to bound this from above. From the definition of $\rho_{m,t}$, for fixed t there are only $O(1)$ values $m \in \mathbb{Z}$ for which $\rho_{m,t}(x)$ or $\rho_{m,t}(y)$ is nonzero. For such m, we can write

$$\begin{aligned}\|\rho_{m,t}(x)\hat{h}_m(x) - \rho_{m,t}(y)\hat{h}_m(y)\| &\leq \|\hat{h}_m(x)\| \cdot |\rho_{m,t}(x) - \rho_{m,t}(y)| + |\rho_{m,t}(y)| \cdot \|\hat{h}_m(x) - \hat{h}_m(y)\| \\ &\leq 2^{m+1} \cdot O(1)\frac{d(x,y)}{2^m} + d(x,y) \\ &\leq O(1) \cdot d(x,y).\end{aligned}$$

After summing over t, it follows that $\|H\|_{\mathrm{Lip}} = O(\sqrt{\log n})$.

Now for the opposite inequality, fix $x, y \in X$ such that $2^m \leq d(x,y) < 2^{m+1}$. Whenever $\rho_{m,t}(x) = \rho_{m,t}(y) = 1$, we have

$$\|\psi_t(x) - \psi_t(y)\| \geq \|\hat{h}_m(x) - \hat{h}_m(y)\| \geq \frac{1}{4}\|h_m(x) - h_m(y)\|.$$

We will count the number of values t among $1, 2, \ldots, \lceil \log n \rceil$ for which $\rho_{m,t}(x) = \rho_{m,t}(y) = 1$.

Now $\rho_{m,t}(x) = 1$ if $R(x,t) \in [2^{m-3}, 2^{m+3}]$, which occurs if $t \in [\log |B(x, 2^{m-3})|, \log |B(x, 2^{m+3})|]$, and similarly for $\rho_{m,t}(y)$. We can assume that $\rho_{m-3}(x,y) = x$, so $|B(x, 2^{m-3})| > |B(y, 2^{m-3})|$. In this case, if

$$t \in [\log |B(x, 2^{m-3})|, \log |B(x, 2^{m+1})|],$$

then $\rho_{m,t}(x) = \rho_{m,t}(y) = 1$. Hence the desired equality holds for at least

$$\left\lfloor \log \frac{|B(x, 2^{m+1})|}{|B(x, 2^{m-3})|} \right\rfloor.$$

values of t. It follows that

$$\|H(x) - H(y)\|^2 \geq \frac{1}{4} \left\lfloor \log \frac{|B(\rho_{m-3}(x,y), 2^{m+1})|}{|B(\rho_{m-3}(x,y), 2^{m-3})|} \right\rfloor \cdot \|h_m(x) - h_m(y)\|^2.$$

□

Proof of Lemma 1. Let $F : X \to L_2$ and $\Phi : X \to L_2$ be the maps obtained by applying lemma 5 to the given collection $\{\phi_m\}$ and the collection $\{f_m\}$ provided by lemma 3. Letting M denote the map of lemma 1, the final embedding will be $\Psi = F \oplus \Phi \oplus M$. We clearly have

$$\|\Psi\|_{\mathrm{Lip}} \leq \|F\|_{\mathrm{Lip}} + \|\Phi\|_{\mathrm{Lip}} + \|M\|_{\mathrm{Lip}} = O(\sqrt{\log n}).$$

Now let $x, y \in X$ be fixed, so $d(x,y) \in [2^m, 2^{m+1})$ for some m; without loss of generality, we can assume $x = \rho_{m-3}(x,y)$. Let $A = \log \frac{|B(x,2^{m+1})|}{|B(x,2^{m-3})|}$. Then

$$\begin{aligned}\|\Psi(x) - \Psi(y)\|^2 &= \|F(x) - F(y)\|^2 + \|\Phi(x) - \Phi(y)\|^2 + \|M(x) - M(y)\|^2 \\ &\geq \Omega(1) d(x,y)^2 \left(\frac{\lfloor A \rfloor}{(1+A)^2} + \frac{\lfloor A \rfloor}{K^2} + 1_{\{A<1\}} \right) \\ &\geq \Omega(1)\frac{d(x,y)^2}{K},\end{aligned}$$

by looking at the contributions of the first 2 terms based on whether $A \leq K$ or not. We conclude the distortion of Ψ is $O(\sqrt{K \log n})$. □

References

[1] James R. Lee, "Distance scales, embeddings, and metrics of negative type", Proceedings of SODA 2005. Full version available at `http://www.cs.washington.edu/homes/jrl/papers/soda05-full.pdf`.