Euclidean TSP (part I)

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These notes are based on scribe notes by Marios Papaefthymiou and Mike Klugerman.

1 Euclidean TSP

Consider the travelling salesman problem in the plane. Given n points in the plane, we would like to find a tour that visits all of them and that minimizes the distance travelled, where the distance between two points is given by the Euclidean distance. This problem will be denoted by ETSP (Euclidean TSP).

In a companion set of notes, we will present very recent algorithms of Arora and Mitchell that produce in polynomial time a tour which is within $1+\epsilon$ of the optimum, for any fixed $\epsilon > 0$. In this set of notes, we first present some preliminaries and older related results. Before we continue, we should point out that it is not known whether the Euclidean TSP is in $\mathcal{NP}$. Even if we are presented with a candidate tour for the "yes-no" version of the problem, we do not know how to avoid computing a possibly exponential number of decimal digits, in order to calculate the square root required for the Euclidean distance.

First, we present an algorithm which generates a path of length no more than $\frac{\sqrt{N}}{k}$ (k a constant to be determined) given that the points all lie within a unit square. Here we use the "Strips Method". First, break the square down into $\frac{\sqrt{N}}{c}$ horizontal strips of equal height. The "Salesman's" strategy will then be the following: He will begin at the left side of the topmost strip and travel to the right along the horizontal line splitting the strip in half. If at any point the salesman reaches a spot where a point, p, in the problem is located in the strip directly above or below him, he travels directly to p, and back to the center line. When he reaches the end of the line, he travels along the edge of the square to the middle of the next strip and then travels left across the middle line. The salesman goes back and forth in this way until he has passed through all strips. At the end, the salesman travels from the lower right corner to the upper left to finish the loop (see Figure 1, part a).

Analysis: Let

1. A = length travelled horizontally across each strip = 1.
2. B = distance travelled vertically along edge of square.
3. C = worst case distance travelled getting to each point and back.

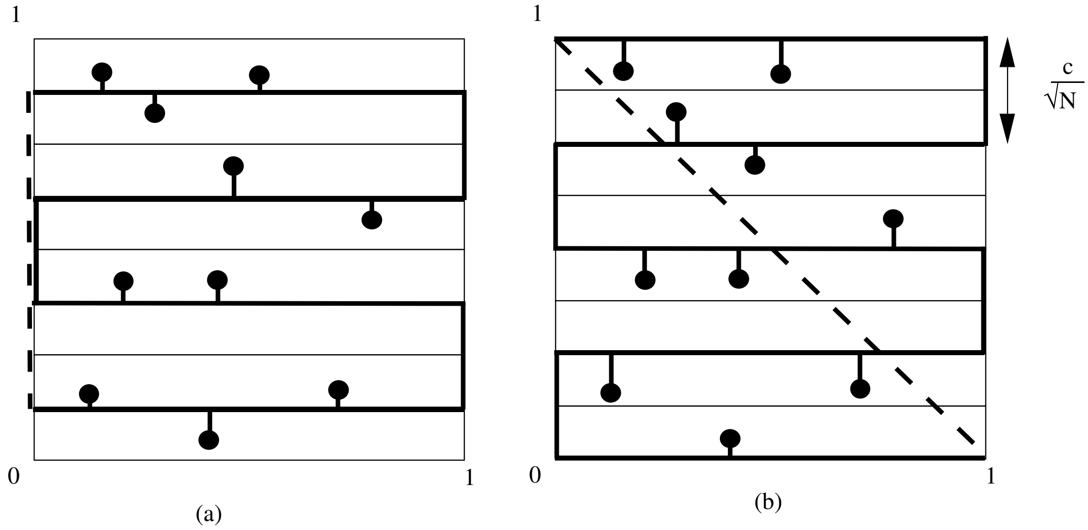

Figure 1: (a) First path followed by salesman in the strips method. (b) Alternative path.

4. D = distance travelled closing the loop.

The total distance,

$$\begin{aligned} Z_{SM} &\leq \frac{\sqrt{N}}{c}A + B + NC + D \\ &\leq \frac{\sqrt{N}}{c}1 + 1 + N2\frac{c}{2\sqrt{N}} + \sqrt{2} \\ &= \sqrt{N}(\frac{1}{c} + c) + 1 + \sqrt{2} \\ &= 2\sqrt{N} + \sqrt{2} + 1 \qquad\qquad \text{for } c = 1 \end{aligned}$$

Now, we can improve this method in the following way. We consider another path, as well. The saleman travels along the boundary lines of the strips rather than along the middle of the strip, and treats what were the middle lines as the new boundary lines (see Figure 1, part b).

The salesman then takes the shorter of this path and the path described above. We can see why this improves our bound by looking at the sum of the lengths of the two paths. Nearly each argument in the sum above will double in size. The difference is that the sum of the distance travelled to each point and back will remain the same $(\frac{c}{2\sqrt{N}})$ (see Figure 2).

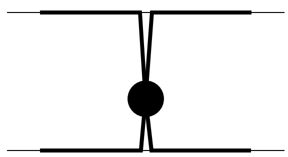

Figure 2: Distance travelled to each point and back.

So the total will be:

$$\begin{aligned}
Z_{2SM} &\leq 2\frac{\sqrt{N}}{c}A + 2B + NC + 2D \\
&\leq 2\frac{\sqrt{N}}{c}1 + 2 + N2\frac{c}{2\sqrt{N}} + 2\sqrt{2} \\
&= \sqrt{N}(\frac{2}{c} + c) + 2 + 2\sqrt{2} \\
&= 2\sqrt{2}\sqrt{N} + 2\sqrt{2} + 2 \qquad\qquad \text{for } c = \sqrt{2}
\end{aligned}$$

Thus, one of the two paths has length $\leq \sqrt{2}\sqrt{N} + \sqrt{2} + 1$. The first proof of this $\sqrt{2N}$ bound is due to Few [2]. This has been improved to $.984\sqrt{2}\sqrt{N}$ by Karloff [3].

2 Karp's partitioning scheme

We now present Karp's partitioning algorithm [4] for the Euclidean TSP problem on n points. A deterministic analysis of the algorithm shows that the length of the tour given by this algorithm does not exceed the length of the optimum tour by more than a factor of $o(\sqrt{n})$. A probabilistic analysis of the algorithm shows that if the points are uniformly distributed in the unit square, then the length of the tour yielded by the algorithm approaches the optimum length with probability 1, as the number of points in the plane becomes arbitrarily large.

Given n points in $[0,1]^2$, the algorithm proceeds as follows.

Algorithm Partition

1. Choose s such that $s! \leq n$ ($s = \lg n / \lg\lg n$ is a good choice, for example). In fact, any s such that $s! = O(n^k)$ for some k will do.

2. Divide the unit square into $\sqrt{n/s}$ vertical strips (see Figure 3). Each strip must contain exactly $\sqrt{ns}$ points.

3. Divide each vertical strip into $\sqrt{n/s}$ horizontal strips, so that every one of the n/s rectangles contains s points.

4. For each rectangle Q_i solve the TSP optimally for the s points inside Q_i. Even brute-force solutions will cost $s! = O(n^k)$ steps.

5. For each Q_i select a vertex v_i. Then construct a tour on $\{v_i \mid i = 1, \ldots, n/s\}$ of length L no greater than $\sqrt{2}\sqrt{n} + 1 + \sqrt{2}$. This can be done using double STRIPS, for instance. In fact, any tour of length less than $c\sqrt{n}$, for a constant c, will do.

6. Shortcut the resulting graph to get a tour of length Z_{KARP}.

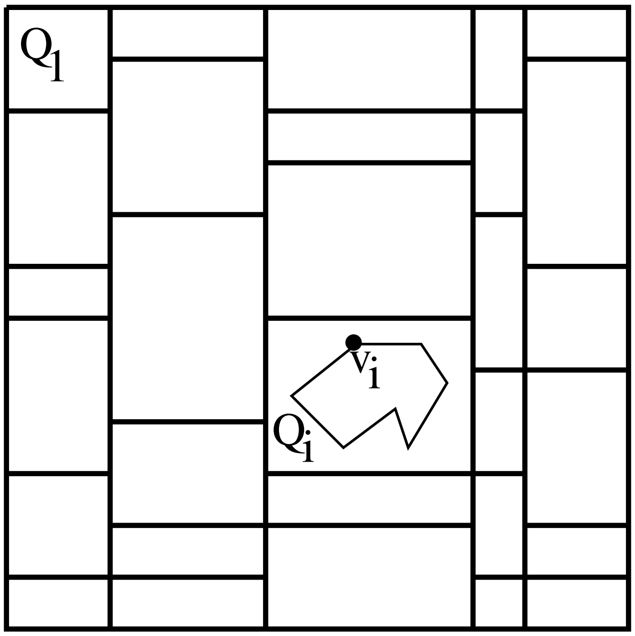

Figure 3: Karp's partitioning algorithm.

3 Determinisitc Analysis of Karp's Algorithm

In this section we give a deterministic analysis of Karp's partitioning algorithm. We show that the length Z_{KARP} of the tour yielded by the partitioning algorithm does not exceed the length Z_{TSP} of the optimal tour by more than an additive factor of $O(\sqrt{n/s}) = o(\sqrt{n})$.

First, we prove the following claim, that bounds the length Z_i^* of the locally optimum tour in Q_i, constructed in Step 4 of the algorithm, in terms of the length Z_i of the edges of the optimum tour in Q_i and the perimeter of Q_i.

Claim 1 *Let C be the optimal tour on all n points. Let Z_i be the length of the edges of C in Q_i. Let Z_i^* be the length of the optimal tour in Q_i. Then*

$$Z_i^* \le Z_i + \frac{3}{2} P(Q_i),$$

where $P(Q_i)$ denotes the perimeter of Q_i.

Proof: Let's focus on some Q_i (see Figure 4). For each edge crossing the boundary of Q_i we define a new boundary point (cfr. A through F on the figure). There is an even number of boundary points. We connect all boundary points in their order of appearance on the boundary with a cycle. This cycle is of length less than or equal to $P(Q_i)$. We also add edges which constitute a short matching, i.e. one of length no greater than $(1/2)P(Q_i)$, on the boundary nodes. One of the two ways of selecting every other edge of the cycle constructed above constitutes such a short matching. The purpose of these additions is to create an eulerian graph. Now, we can short-cut to get a tour on the inside nodes and the boundary nodes. This tour can be further short-cut to obtain a tour on the inside nodes only. Clearly, the length l of this tour satisfies

$$\begin{aligned} Z_i^* &\le l \\ &\le Z_i + \frac{3}{2} P(Q_i), \end{aligned}$$

since the total length of the edges in the eulerian graph is at most $Z_i + P(Q_i) + (1/2)P(Q_i)$. $\square$

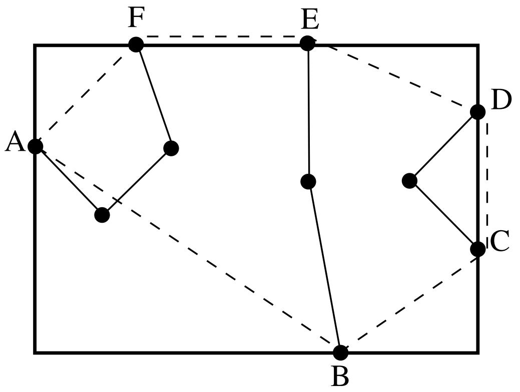

Figure 4: A subrectangle Q_i. Its boundary intersections are denoted by A through F.

Now, we can prove the following theorem, which provides a performance guarantee for the partitioning algorithm.

Theorem 2 *Let Z_{TSP} be the length of the optimal tour on the given n points and let Z_{KARP} be the length of the tour yielded by Karp's partitioning algorithm. Then*

$$Z_{TSP} \leq Z_{KARP} \leq Z_{TSP} + O(\sqrt{n/s}),$$

where $s! \leq n$.

Proof: By the definition of Z_{KARP} and the previous claim we have:

$$\begin{aligned}
Z_{KARP} &\leq \sum_{i=1}^{n/s} Z_i^* + L \\
&\leq \sum_{i=1}^{n/s} Z_i + \frac{3}{2}\sum_{i=1}^{n/s} P(Q_i) + \sqrt{2}\sqrt{n/s} + O(1) \\
&= Z_{TSP} + \frac{3}{2}\left(2\sqrt{n/s} + 2\sqrt{n/s}\right) + \sqrt{2}\sqrt{n/s} + O(1) \\
&= Z_{TSP} + O(\sqrt{n/s}).
\end{aligned}$$

□

4 Probabilistic Analysis of Karp's Algorithm

In this section we present a probabilistic analysis of Karp's partitioning algorithm. We state three theorems, one of them along with its proof, and we conclude with two important corollaries about the properties of the solutions yielded by the algorithm.

Let $X_1, \ldots, X_n$ be n independent random variables uniformly distributed in $[0,1]^2$. The following theorem gives an upper and a lower bound on the expected length of the optimum tour on these n points.

Theorem 3 $E[TSP(X_1, \ldots, X_n)] = \Theta(\sqrt{n})$.

Proof: For convenience let us denote $E[TSP(X_1, \ldots, X_n)]$ by E_n. Applying the double STRIPS method we deduce an upper bound for E_n:

$$E_n \leq \sqrt{2}\sqrt{n} + 1 + \sqrt{2}.$$

We furthermore claim that $E_n \geq \sqrt{n}/2$. First, since any edge incident to X_i has length at least $d(X_i, \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\}) = \min_{j \neq i} ||X_j - X_i||$, we observe that

$$\begin{aligned}
E_n &\geq E\left[\sum_{i=1}^{n} d(X_i, \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\})\right] \\
&= \sum_{i=1}^{n} E\left[d(X_i, \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\})\right] \\
&= nE\left[d(X_n, \{X_1, \ldots, X_{n-1}\})\right].
\end{aligned}$$

Since $E[Y] = \int_0^\infty Pr[Y \geq y]dy$ for any nonnegative random variable Y, we can bound the expectation in the following way:

$$\begin{aligned} E\left[d(X_n, \{X_1, \ldots, X_{n-1}\})\right] &= \int_0^\infty Pr\left[d(X_n, \{X_1, \ldots, X_{n-1}\}) \geq r\right] dr \\ &\geq \int_0^{1/\sqrt{\pi}} (1 - \pi r^2)^{n-1} dr. \end{aligned}$$

where we had used the fact that:

$$\begin{aligned} Pr[d(X_n, \{X_1, \ldots, X_{n-1}\}) \geq r \mid X_n] &= Pr[\min_{i<n} ||X_i - X_n|| \geq r \mid X_n] \\ &= \prod_{i<n} Pr[||X_i - X_n|| \geq r \mid X_n] \\ &\geq (1 - \pi r^2)^{n-1}, \end{aligned}$$

for all X_n. The last integral can be seen to be equal to

$$\frac{1}{2} \frac{\Gamma(n)}{\Gamma(n + \frac{1}{2})},$$

which is greater than or equal to

$$\frac{1}{2} \frac{1}{\sqrt{n}}.$$

Here is a quick way to show that this integral is $\Theta(\frac{1}{\sqrt{n}})$:

$$\begin{aligned} \int_0^{1/\sqrt{\pi}} (1 - \pi r^2)^{n-1} dr &\geq \int_0^{1/\sqrt{\pi n}} (1 - \pi r^2)^{n-1} dr \\ &\geq \int_0^{1/\sqrt{\pi n}} (1 - \frac{1}{n})^{n-1} dr \\ &\sim \frac{e^{-1}}{\sqrt{\pi n}}. \end{aligned}$$

It follows that

$$E_n \geq n \frac{1}{2\sqrt{n}} = \frac{\sqrt{n}}{2},$$

thus completing the proof of the theorem. □

We can prove a stronger result than that of Theorem 3. Specifically, we can show that the ratio $E_n/\sqrt{n}$ is not only bounded but it also approaches a limit β, as the number of points n approaches infinity.

Theorem 4 *There exists a number β such that*

$$\lim_{n \to \infty} \frac{E[TSP(X_1, \ldots, X_n)]}{\sqrt{n}} = \beta.$$

The limit β is known to be in the range $0.625 \leq \beta \leq 0.9204$.

An even stronger theorem was proven by Beardwood, Halton and Hammersley [1]:

Theorem 5 *There exists a number β such that*

$$Pr\left[\lim_{n\to\infty} \frac{TSP(X_1,\ldots,X_n)}{\sqrt{n}} = \beta\right] = 1.$$

The proofs of Theorems 4 and 5 are omitted.

Based on Theorem 2 we can show that Karp's algorithm is optimal in expected value.

Corollary 6 $1 \leq \frac{Z_{KARP}}{Z_{TSP}} \leq 1 + o(1)$.

Proof: From Theorem 2 we have

$$Z_{KARP} \leq Z_{TSP} + \frac{c\sqrt{n}}{\sqrt{s}}.$$

Therefore

$$E[Z_{KARP}] \leq E[Z_{TSP}] + \frac{c\sqrt{n}}{\sqrt{s}}.$$

From Theorem 3 we have

$$1 \leq \frac{E[Z_{KARP}]}{E[Z_{TSP}]} \leq 1 + \frac{c\sqrt{n}}{\sqrt{s}\sqrt{n}/2} = 1 + \frac{2c}{\sqrt{s}} = 1 + o(1),$$

since $1/\sqrt{s} \to 0$ as $n \to \infty$. $\square$

Based on Theorem 5 we can show that the length of the tour yielded by Karp's algorithm approaches the optimal length almost surely, as the number of points approaches infinity.

Corollary 7 $Pr\left[\lim_{n\to\infty} \frac{Z_{KARP}}{Z_{TSP}} = 1\right] = 1$.

References

[1] J. Beardwood, J. Halton, J. Hammersley, "The shortest path through many points", *Proc. Cambridge Philos. Soc.*, **55**, 299–327, 1959.

[2] L. Few, "The shortest path and shortest road through n points", *Mathematika*, **2**, 141–144, 1955.

[3] H.J. Karloff, "How long can a Euclidean Traveling Salesman Tour be", *SIAM J. Disc. Math.*, **2**, 91–99, 1989.

[4] R. Karp, "Probabilistic Analysis of Partitioning Algorithms for the Traveling Salesman Problem in the plane", *Math. of Operations Research*, **2**, 1977.