Double affine Hecke algebras and applications

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The Calogero-Moser system

The classical Calogero-Moser system is a system of n particles of unit mass on the line with interaction potential K/r^2, where r is the distance between two particles. Thus the Hamiltonian is

$$H_{\mathrm{cl}} = \frac{1}{2}\left(\sum_{i=1}^{n} p_i^2 + \sum_{1\le i\ne j\le n} \frac{K}{(x_i - x_j)^2}\right),$$

To quantize this system, we are supposed to replace p_j with $-i\hbar\partial_j$, so setting $\hbar = 1$, we get the quantum Calogero-Moser Hamiltonian

$$H = \Delta - \sum_{1\le i\ne j\le n} \frac{K}{(x_i - x_j)^2},$$

where $\Delta := \sum_{j=1}^{n} \partial_j^2$ is the Laplace operator (and we drop the physical factor $-1/2$ for brevity).

The integrability theorem

Theorem (Calogero-Sutherland-Moser)

H defines a quantum integrable system. That is, there are unique differential operators $L_1, ..., L_n$ such that

- $L_2 = H$;
- $[L_i, L_j] = 0$ *for all* i, j;
- L_i *is homogeneous and the symbol of* L_i *is* $\sum_{j=1}^{n} \partial_j^i$.

Remark

Quantum integrability is good because it reduces the stationary Schrödinger equation

$$H\psi = \Lambda\psi$$

to the joint eigenvalue problem

$$L_i\psi = \Lambda_i\psi,\ \ i = 1, ..., n,$$

which is holonomic and therefore reduces to ODE.

Proof

Let us prove the existence of L_i. Let $K = c(c+1)$. Define the Dunkl operators on $\mathbb{C}(x_1, \ldots, x_n)$

$$D_i = \partial_i + c \sum_{j \neq i} \frac{1}{x_i - x_j} s_{ij},$$

where $s_{ij} \in S_n$ is the permutation of i and j.

Lemma (Dunkl)

$[D_i, D_j] = 0.$

The proof is by an easy computation.

Now define L_i to be the restriction of $\sum_{j=1}^n D_j^i$ to symmetric functions. A short computation shows that $L_2 = H$, and it is easy to see that the operators L_i have the required properties.

Generalization to any finite Coxeter group

Let W be a finite Coxeter group with reflection representation $\mathfrak{h}$. Then by the Chevalley-Sheppard-Todd theorem, we have $(S\mathfrak{h})^W = \mathbb{C}[P_1, ..., P_r]$ for some homogeneous polynomials $P_1, ..., P_r$. The above constructions can be generalized to this case, and are recovered in the special case $W = S_n$. Namely, assume for simplicity that W is irreducible and let $s \mapsto c_s$ be a W-invariant function on the set of reflections of W. Then the Hamiltonian of the quantum CM system is the differential operator on $\mathfrak{h}$ given by

$$H = \Delta - \sum_s \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2(x)},$$

where $\alpha_s \in \mathfrak{h}^*$ defines the reflection hyperplane of s.

Theorem (Olshanetsky-Perelomov, Heckman)

The operator H defines a unique quantum integrable system $L_1, ..., L_r$ such that $L_1 = H$, $[L_i, L_j] = 0$, and L_i is homogeneous with symbol $P_i(\partial)$.

The proof is again by using Dunkl operators. Namely, to each $y \in \mathfrak{h}$ we attach the Dunkl operator

$$D_y := \partial_y + \sum_s c_s \frac{\alpha_s(y)}{\alpha_s} s,$$

Then by Dunkl's lemma $[D_y, D_{y'}] = 0$, and L_i may be defined as the restrictions of $P_i(D)$ to W-invariant functions on $\mathfrak{h}$.

Rational Cherednik algebra (rational DAHA)

Since Dunkl operators turned out to be so useful, let us consider the algebra they define.

Definition (Drinfeld, Cherednik, E.-Ginzburg)

The rational Cherednik algebra (rational DAHA) $H_c(W, \mathfrak{h})$ is the algebra generated inside $\mathrm{End}_{\mathbb{C}}\mathbb{C}(\mathfrak{h})$ by W, $\mathfrak{h}^*$, and the Dunkl operators $D_y, y \in \mathfrak{h}$.

It turns out that this algebra can also be defined by generators and relations.

Proposition (Drinfeld, E.-Ginzburg)

The algebra $H_c(W, \mathfrak{h})$ is the quotient of $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0,\ [y, y'] = 0,\ [y, x] = (y, x) + \sum_s c_s(y, (s-1)x)s,$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$.

Adding the Planck constant

In fact, it is useful to introduce one more parameter $\hbar$ (the Planck constant), replacing the last relation by

$$[y, x] = \hbar(y, x) - \sum_s c_s(y, (s-1)x)s.$$

This defines the most general rational Cherednik algebra, $H_{\hbar,c}(W, \mathfrak{h})$, and the above theory extends to this setting (namely, in the Dunkl operators one needs to replace ∂_y by $\hbar\partial_y$). In particular, it is interesting to set $\hbar = 0$, replacing $\hbar\partial_y$ by classical momenta p_y; then the above construction gives the integrals of the classical Calogero-Moser system.

Example

If $W = \mathbb{Z}/2$ and $\mathfrak{h}$ is the sign representation of W then $H_{\hbar,c}(W, \mathfrak{h})$ is generated by x, y, s with relations

$$sx = -xs, \; sy = -ys, \; s^2 = 1, \; [y, x] = \hbar - 2cs.$$

Examples

Example

The algebra $H_{\hbar,c}(S_n,\mathbb{C}^n)$ is generated by S_n, $x_1,\dots,x_n$, $y_1,\dots,y_n$, with relations

$$sx_i = x_{s(i)}s,\ \ sy_i = y_{s(i)}s,\ \ [x_i,x_j]=0,\ \ [y_i,y_j]=0,$$

$$[y_i,x_j]=cs_{ij},\ \ [y_i,x_i]=\hbar - c\sum_{j\neq i} s_{ij},$$

where $s\in S_n$ and $i\neq j$.

Example

$H_{0,0}(W,\mathfrak{h})=\mathbb{C}W\ltimes S(\mathfrak{h}\oplus\mathfrak{h}^*)$ and $H_{1,0}(W,\mathfrak{h})=W\ltimes D(\mathfrak{h})$. The algebra $H_{\hbar,c}$ has a filtration with $\deg\mathfrak{h}^*=\deg W=0$, $\deg\mathfrak{h}=1$, and $\mathrm{gr}H_{\hbar,c}=H_{0,0}$.

Properties of RCA

Corollary (Drinfeld, Cherednik, E. Ginzburg)

(the PBW theorem) The multiplication map $S\mathfrak{h}^* \otimes \mathbb{C}W \otimes S\mathfrak{h} \to H_c(W,\mathfrak{h})$ *is a linear isomorphism.*

Theorem (Losev)

If W *is irreducible then* $H_{\hbar,c}(W,\mathfrak{h})$ *is the universal filtered deformation of* $\mathbb{C}W \ltimes S(\mathfrak{h}\oplus\mathfrak{h}^*)$*.*

Theorem (E.-Ginzburg)

If W *is irreducible then* $H_{1,c}(W,\mathfrak{h})$ *for formal* c *is the universal formal deformation of* $H_{1,0}(W,\mathfrak{h}) = \mathbb{C}W \ltimes D(\mathfrak{h})$*.*

Remark

These results extend verbatim to the case when W is a finite complex reflection group, except the corresponding integrable systems will not have a quadratic Hamiltionian, since $\deg P_1 > 2$.

If W is crystallographic (i.e., a Weyl group) and corresponds to a reduced root system R, then the CM system attached to W admits a integrable trigonometric deformation

$$H = \Delta - \sum_s \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{4\sinh^2 \frac{1}{2}\alpha_s(x)},$$

where α_s is the positive root corresponding to s. This deformation is integrated using the trigonometric Dunkl-Cherednik operators

$$D_y^{\mathrm{trig}} := \partial_y + \sum_s c_s \frac{\alpha_s(y)}{e^{\alpha_s} - 1} s,$$

which give rise to the trigonometric Cherednik algebra $H_{\hbar,c}^{\mathrm{trig}}(W, \mathfrak{h}, R)$, a deformation of $H_{\hbar,c}(W, \mathfrak{h})$.

Trigonometric deformation

The trigonometric Cherednik algebra is generated by two subalgebras A_x, A_y which deform $\mathbb{C}W \ltimes S\mathfrak{h}^*$ and $\mathbb{C}W \ltimes S\mathfrak{h}$, respectively. The first subalgebra A_x is generated by W and commuting elements $X_i = e^{\omega_i}$ corresponding to fundamental weights, and is simply the group algebra $\mathbb{C}[\widehat{W}]$ of the extended affine Weyl group $\widehat{W}$. The second subalgebra A_y is the degenerate (or graded) affine Hecke algebra studied by Drinfeld and Lusztig. It is generated by W and $y \in \mathfrak{h}$ with relations

$$s_i(y)s_i - s_i y = c_{s_i}\alpha_i(y), \; [y, y'] = 0,$$

where s_i are the simple reflections, α_i the corresponding positive simple roots, and $y, y' \in \mathfrak{h}$.

There is also an elliptic Cherednik algebra, leading to the integrable elliptic deformation of the CM system, with Hamiltonian

$$H = \Delta - \sum_s c_s(c_s+1)(\alpha_s, \alpha_s)\wp(\alpha_s(x), \tau).$$

The trigonometric Cherednik algebra $H^{\mathrm{trig}}_{\hbar,c}$ has a q-deformation with parameters $q = e^{\varepsilon\hbar}$ and $t_s = e^{\varepsilon c_s}$, which degenerates to the trigonometric algebra as $\varepsilon \to 0$. This deformation is Cherednik's double affine Hecke algebra (DAHA) $H\!H_{q,t}(W, \mathfrak{h}, R)$.To define it, it is best to use the topological language. Namely, recall that the usual Hecke algebra $\mathcal{H}_t(W)$ is the quotient of the group algebra of the braid group B_W by the Hecke relations

$$(T_i - t_i)(T_i + t_i^{-1}) = 0.$$

This is a special case of the following general definition of the orbifold Hecke algebra, which in particular also includes affine and double affine Hecke algebras.

Let X be a simply connected complex manifold and G be a discrete group of automorphisms of X acting on X properly discontinuously (i.e., we have a complex orbifold X/G). Let $X^\circ \subset X$ be the set of points with trivial stabilizer. The braid group of X, G is the group $B := \pi_1(X^\circ/G)$. We have a surjective homomorphism $\phi : B \to G$ corresponding to gluing back points with nontrivial stabilizer. Let us describe the kernel of ϕ. To this end, define a reflection hypersurface to be a codimension 1 connected component of X^g for some $g \in G$. For every reflection hypersurface $Y \subset X$, we have a small counterclockwise loop $T_Y \subset X^\circ/G$ around the image of Y in X/G, which defines a conjugacy class in B. Then the Seifert-van Kampen theorem yields

Proposition

$\mathrm{Ker}\phi$ *is generated by the relations* $T_Y^{n_Y} = 1$, *where* n_Y *is the order of the stabilizer* G_Y *of a generic point of* Y.

Orbifold Hecke algebras

Now to each reflection hypersurface Y attach nonzero parameters $\mathbf{t}_j(Y)$, $j = 1, ..., n_Y$ which are the same for conjugate reflection hypersurfaces Y.

Definition (E.)

The orbifold Hecke algebra $\mathcal{H}_{\mathbf{t}}(G, X)$ is the quotient of $\mathbb{C}B$ by the relations

$$\prod_{j=1}^{n_Y} (T_Y - \mathbf{t}_j(Y)) = 0.$$

This is clearly a deformation of the group algebra $\mathbb{C}G$ of G, which, by the above proposition, is recovered when $\mathbf{t}_j(Y) = e^{2\pi ij/n_Y}$.

Theorem (E.)

(Formal PBW theorem) If $H^2(X, \mathbb{C}) = 0$ then this deformation is flat in the formal sense (i.e., when $\mathbf{t}_j(Y) = e^{\tau_{j,Y}} e^{2\pi ij/n_Y}$, where $\tau_{j,Y}$ are formal parameters).

Finite, affine and double affine Hecke algebras

Example

Let $X = \mathfrak{h}$.
1. If $G = W$ then $\mathcal{H}_{\mathbf{t}}(G, X)$ is the usual Hecke algebra $\mathcal{H}_t(W)$.
2. If $G = \widehat{W} = W \ltimes P^\vee$, where $P^\vee$ is the coweight lattice (in the crystallographic case) then $\mathcal{H}_{\mathbf{t}}(G, X)$ is the affine Hecke algebra $\mathcal{H}_t(\widehat{W})$.

Definition (Cherednik)

If $G = W \ltimes (P^\vee \oplus \tau P^\vee)$, where $\mathrm{Im}\tau > 0$, then $\mathcal{H}_{\mathbf{t}}(G, X)$ is the double affine Hecke algebra $\mathbb{H}_{q,t}(W, \mathfrak{h}, R)$.

From this description it is easy to see that as the trigonometric Cherednik algebra is q-deformed, the two subalgebras A_x, A_y deform into two copies of the affine Hecke algebra $\mathcal{H}_t(\widehat{W})$.

Explicit presentation

Let us describe by explicit generators and relations the DAHA for $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$ (i.e., $G = S_n \ltimes \mathbb{Z}^{2n}$), which we call $H\!\!\!H_{q,t}(n)$.

Proposition (Cherednik)

$H\!\!\!H_{q,t}(n)$ is generated by invertible elements X_i, Y_i, $i = 1, ..., n$, and T_i, $i = 1, ..., n-1$, with relations

$$(T_i - t^{1/2})(T_i + t^{-1/2}) = 0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \ \ T_i T_j = T_j T_i \ (|i-j| \geq 2),$$

$$T_i X_i T_i = X_{i+1}, \ \ T_i X_j = X_j T_i \ (j \neq i, i+1), \ [X_i, X_j] = 0,$$

$$T_i Y_i T_i = Y_{i+1}, \ \ T_i Y_j = Y_j T_i \ (j \neq i, i+1), \ [Y_i, Y_j] = 0,$$

$$X_1^{-1} Y_2^{-1} X_1 Y_2 = T_1^2, \ \ Y_i \tilde{X} = q \tilde{X} Y_i, \ \ X_i \tilde{Y} = q^{-1} \tilde{Y} X_i,$$

where $\tilde{X} := \prod_i X_i$ and $\tilde{Y} = \prod_i Y_i$.

Polynomial representation of DAHA

The DAHA contains two Laurent polynomial subalgebras $\mathbb{C}[X^{\pm 1}] \subset A_x$ and $\mathbb{C}[Y^{\pm 1}] \subset A_y$ isomorphic to the group algebra $\mathbb{C}P^\vee$, and we have the PBW decomposition

$$H\!\!H_{q,t} = \mathbb{C}[X^{\pm 1}] \otimes \mathcal{H}_t(W) \otimes \mathbb{C}[Y^{\pm 1}] = A_x \otimes \mathbb{C}[Y^{\pm 1}] = \mathbb{C}[X^{\pm 1}] \otimes A_y$$

(the multiplication map is a linear isomorphism). This allows us to define the polynomial representation $\mathbf{P}$ of $H\!\!H_{q,t}$ by $\mathbf{P} := H\!\!H_{q,t} \otimes_{A_y} \mathbb{C}$, where $\mathbb{C}$ is the trivial representation of the affine Hecke algebra A_y, and we have a linear isomorphism $\mathbf{P} \cong \mathbb{C}[X^{\pm 1}]$.

Moreover, the center of A_x is $\mathbb{C}[X^{\pm 1}]^W$ and the center of A_y is $\mathbb{C}[Y^{\pm 1}]^W$.

Proposition (Cherednik)

For generic q, t:
(i) The algebra $\mathbb{C}[Y^{\pm 1}]$ acts with simple spectrum on $\mathbf{P}$.
(ii) The algebra $\mathbb{C}[Y^{\pm 1}]^W$ acts with simple spectrum on $\mathbf{P}^W$ by difference operators.

Macdonald polynomials and operators

Definition

The eigenvectors of $\mathbb{C}[Y^{\pm 1}]^W$ in $\mathbf{P}^W$ are the Macdonald polynomials $P_\lambda^{q,t}$ $(\lambda \in P_+^\vee)$. The difference operators defined by $\mathbb{C}[Y^{\pm 1}]^W$ are the Macdonald operators. The eigenvectors of $\mathbb{C}[Y^{\pm 1}]$ in $\mathbf{P}$ are the nonsymmetric Macdonald polynomials $E_\lambda^{q,t}$ $(\lambda \in P^\vee)$.

Macdonald operators define a quantum integrable system which is a relativistic deformation of the trigonometric Calogero-Moser system. It is called the Macdondald-Ruijsenaars system.

Example (Macdonald)

Macdonald operators for S_n have the form

$$M_r = \sum_{I \subset [1,n]:|I|=r} \prod_{i \in I, j \notin I} \frac{tx_i - x_j}{x_i - x_j} T_I,$$

where T_I replaces x_i with qx_i for $i \in I$.

Remarks

Remark

Cherednik used this structure to prove Macdonald's conjectures about Macdonald polynomials, and this served as a motivation for introducing DAHA.

Remark

In the trigonometric limit the eigenvectors will be (nonsymmetric) Jack polynomials, which are limits of (nonsymmetric) Macdonald polynomials.

Remark

For W of type B_n, there is a DAHA depending on 5 parameters for $n = 1$ and 6 parameters for $n \geq 2$, introduced by Sahi and Stokman. The eigenvectors for the corresponding commutative subalgebras $\mathbb{C}[Y^{\pm 1}]^W$ are Askey-Wilson polynomials for $n = 1$ and Koornwinder polynomials for $n \geq 2$.

The spherical subalgebra and the center

Let $\mathbf{e}$ be the idempotent of the trivial representation of $\mathcal{H}_t(W)$. The algebra $\mathbf{e}H\!\!H_{q,t}\mathbf{e}$ is called the spherical subalgebra.

Theorem (E.-Ginzburg, Oblomkov)

(i) $\mathbf{e}H\!\!H_{1,t}\mathbf{e}$ *is commutative.*
(ii) Let $Z_t \subset H\!\!H_{1,t}$ *be the center of DAHA. Then the map* $Z_t \to \mathbf{e}H\!\!H_{1,t}\mathbf{e}$ *given by* $z \mapsto z\mathbf{e}$ *is an isomorphism.*
(iii) Z_t *is a finitely generated Cohen-Macaulay integral domain, and* $H\!\!H_{1,t}$ *is a Cohen-Macaulay module over* Z_t*.*

Thus, for each t we obtain an irreducible Cohen-Macaulay variety $CM_t = \mathrm{Spec} Z_t$. Moreover, it carries a Poisson structure coming from varying q, which has finitely many symplectic leaves. In particular, CM_t is smooth outside codimension 2, hence normal, and the algebra $H\!\!H_{q,t}$ is Azumaya on the smooth locus of CM_t, of degree $|W|$. The variety CM_t is called the Calogero-Moser space attached to $H\!\!H_{1,t}$. It is a flat deformation of $CM_0 = (T \times T)/W$, where $T := \mathfrak{h}/P^\vee$ is the corresponding torus.

Calogero-Moser spaces

Thus, the algebra $\mathbf{e}H\!\!H_{q,t}\mathbf{e}$ (with $q = e^{\hbar}$) may be viewed as a quantization of the Poisson variety CM_t.

The same theory applies in the trigonometric and rational limits, producing Calogero-Moser spaces $CM_c^{\rm trig}$, $CM_c^{\rm rat}$ and their quantizations given by the spherical subalgebras of trigonometric and rational Cherednik algebras.

Moreover, for $W = S_n$ these spaces admit a quiver-theoretic description. Namely, we have (for generic $t \in \mathbb{C}^*$)

Theorem (E.-Ginzburg, Oblomkov)

(i) $CM_1^{\rm rat} = \{(X,Y) \in \mathfrak{gl}_n \times \mathfrak{gl}_n : \mathrm{rk}(XY - YX + \mathrm{Id}) = 1\}/GL_n$;
(ii) $CM_1^{\rm trig} = \{(X,Y) \in GL_n \times \mathfrak{gl}_n : \mathrm{rk}(XY - YX + X) = 1\}/GL_n$;
(iii) $CM_t = \{(X,Y) \in GL_n \times GL_n : \mathrm{rk}(tXY - YX) = 1\}/GL_n$.

These varieties are smooth, and are topologically trivial deformations of the Hilbert scheme $\mathrm{Hilb}_n(S)$, where $S = \mathbb{C}^2$ in (i), $S = \mathbb{C}^* \times \mathbb{C}$ in (ii), and $S = \mathbb{C}^* \times \mathbb{C}^*$ in (iii). The space in (i) was introduced by Kazhdan, Kostant and Sternberg in 1978.

As follows from the above discussion, in the case $\hbar = 0$ the representation theory of DAHA and their degenerations is controlled by the geometry of the Calogero-Moser spaces. For example, we have:

Theorem (E.-Ginzburg, Oblomkov)

For $W = S_n$ all irreducible representations of $H_{0,1}^{\mathrm{rat}}$, $H_{0,1}^{\mathrm{trig}}$, $H\!\!H_{1,t}$ have dimension $n!$ and are parametrized by the space CM_1^{rat}, CM_1^{trig} and CM_t, respectively. (I.e., DAHA are Azumaya algebras).

Let us now turn to the quantum case $\hbar \neq 0$. For simplicity consider the rational case (so we may assume $\hbar = 1$). Recall that we have the factorization $H_c(W) = S\mathfrak{h}^* \otimes \mathbb{C}W \otimes S\mathfrak{h}$, analogous to the factorization $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$ for a semisimple Lie algebra $\mathfrak{g}$. This makes the representation theory of $H_c(W)$ similar to the representation theory of $\mathfrak{g}$.

Category O_c

Definition (Ginzburg, Guay, Opdam, Rouquier)

The category O_c of $H_c(W)$-modules consists of finitely generated modules on which $\mathfrak{h}$ acts locally nilpotently.

Theorem (Ginzburg, Guay, Opdam, Rouquier)

The category O_c is a highest weight category with highest weights being the irreducible representations τ of W. Its standard objects are the Verma modules $\Delta_c(\tau) := H_c(W) \otimes_{\mathbb{C}W \ltimes S\mathfrak{h}} \tau$, where $\mathfrak{h}$ acts on τ by zero.

This raises the question about multiplicities of simples in standards in O_c; recall that such multiplicities for Lie algebras are given by the values at $q = 1$ of Kazhdan-Lusztig polynomials.

Theorem (Rouquier, Losev)

If $W = S_n$ then multiplicities in O_c are given by the values at $q = 1$ of certain affine parabolic Kazhdan-Lusztig polynomials.

For a general Coxeter group W, the story is even more complicated, but there has been significant progress recently. The problem is also solved in the case $W = S_n \ltimes (Z/r)^n$ using the theory of Kac-Moody categorification. However, for a general complex reflection group W the problem is still open.

Other connections

- DAHA arise in harmonic analysis on p-adic loop groups (Kapranov, Braverman, Kazhdan, Gaitsgory, Matnaik).
- DAHA are used to study quasiinvariants of reflection groups (Chalykh, Veselov, Feigin, Ginzburg, E., Rains)
- DAHA act on the equivariant K-theory of affine Springer fibers. Trigonometric DAHA act on their equivariant cohomology. Rational DAHA act on the associated graded of this cohomology (Varagnolo, Vasserot, Oblomkov, Yun).

Other connections

- Khovanov-Rozansky homology of torus links and Poincare polynomials of compactified Jacobians of plane curve singularities express via characters of representations of rational Cherednik algebras for S_n (Gorsky, Oblomkov, Rasmussen, Shende, E., Losev).
- Representation categories of $H_c(S_n \ltimes (\mathbb{Z}/r)^n)$ can be described in terms of parabolic categories $\mathcal{O}$ for affine Lie algebras and provide examples of categorical $\widehat{\mathfrak{sl}}_e$-actions (Varagnolo, Vasserot, Shan, Rouquier, Losev).
- As $n \to \infty$, spherical DAHA of S_n stabilize to toroidal quantum groups, or elliptic Hall algebras (Varagnolo, Vasserot, Schiffmann).

Thank you!