Double affine Hecke algebras and applications

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November 5, 2017
The Calogero-Moser system is a system of \( n \) particles of unit mass on the line with interaction potential \( K/r^2 \), where \( r \) is the distance between two particles. Thus the Hamiltonian is

\[
H_{\text{cl}} = \frac{1}{2} \left( \sum_{i=1}^{n} p_i^2 + \sum_{1 \leq i \neq j \leq n} \frac{K}{(x_i - x_j)^2} \right),
\]

To quantize this system, we are supposed to replace \( p_j \) with \(-i\hbar \partial_j\), so setting \( \hbar = 1 \), we get the quantum Calogero-Moser Hamiltonian

\[
H = \Delta - \sum_{1 \leq i \neq j \leq n} \frac{K}{(x_i - x_j)^2},
\]

where \( \Delta := \sum_{j=1}^{n} \partial_j^2 \) is the Laplace operator (and we drop the physical factor \(-1/2\) for brevity).
The integrability theorem

**Theorem (Calogero-Sutherland-Moser)**

*H defines a quantum integrable system. That is, there are unique differential operators* $L_1, \ldots, L_n$ *such that*

- $L_2 = H$;
- $[L_i, L_j] = 0$ for all $i, j$;
- $L_i$ is homogeneous and the symbol of $L_i$ is $\sum_{j=1}^{n} \partial^i_j$.

**Remark**

Quantum integrability is good because it reduces the stationary Schrödinger equation

$$H \psi = \Lambda \psi$$

to the joint eigenvalue problem

$$L_i \psi = \Lambda_i \psi, \ i = 1, \ldots, n,$$

which is holonomic and therefore reduces to ODE.
Let us prove the existence of $L_i$. Let $K = c(c + 1)$. Define the Dunkl operators on $\mathbb{C}(x_1, ..., x_n)$

$$D_i = \partial_i + c \sum_{j \neq i} \frac{1}{x_i - x_j} s_{ij},$$

where $s_{ij} \in S_n$ is the permutation of $i$ and $j$.

**Lemma (Dunkl)**

$$[D_i, D_j] = 0.$$

The proof is by an easy computation.

Now define $L_i$ to be the restriction of $\sum_{j=1}^{n} D_j^i$ to symmetric functions. A short computation shows that $L_2 = H$, and it is easy to see that the operators $L_i$ have the required properties.
Let $W$ be a finite Coxeter group with reflection representation $\mathfrak{h}$. Then by the Chevalley-Sheppard-Todd theorem, we have $(S\mathfrak{h})^W = \mathbb{C}[P_1, ..., P_r]$ for some homogeneous polynomials $P_1, ..., P_r$. The above constructions can be generalized to this case, and are recovered in the special case $W = S_n$. Namely, assume for simplicity that $W$ is irreducible and let $s \mapsto c_s$ be a $W$-invariant function on the set of reflections of $W$. Then the Hamiltonian of the quantum CM system is the differential operator on $\mathfrak{h}$ given by

$$H = \Delta - \sum_s c_s(c_s + 1)(\alpha_s, \alpha_s)\frac{\alpha_s^2(x)}{\alpha_s^2(x)},$$

where $\alpha_s \in \mathfrak{h}^*$ defines the reflection hyperplane of $s$.

**Theorem (Olshanetsky-Perelomov, Heckman)**

The operator $H$ defines a unique quantum integrable system $L_1, ..., L_r$ such that $L_1 = H$, $[L_i, L_j] = 0$, and $L_i$ is homogeneous with symbol $P_i(\partial)$. 
Generalization to any finite Coxeter group

The proof is again by using Dunkl operators. Namely, to each \( y \in \mathfrak{h} \) we attach the Dunkl operator

\[
D_y := \partial_y + \sum_s c_s \frac{\alpha_s(y)}{\alpha_s} s,
\]

Then by Dunkl’s lemma \([D_y, D_y'] = 0\), and \( L_i \) may be defined as the restrictions of \( P_i(D) \) to \( W \)-invariant functions on \( \mathfrak{h} \).
Since Dunkl operators turned out to be so useful, let us consider the algebra they define.

**Definition (Drinfeld, Cherednik, E.-Ginzburg)**

The rational Cherednik algebra (rational DAHA) $H_c(W, \mathfrak{h})$ is the algebra generated inside $\text{End}_C C(\mathfrak{h})$ by $W$, $\mathfrak{h}^*$, and the Dunkl operators $D_y, y \in \mathfrak{h}$.

It turns out that this algebra can also be defined by generators and relations.

**Proposition (Drinfeld, E.-Ginzburg)**

The algebra $H_c(W, \mathfrak{h})$ is the quotient of $\mathbb{C} W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \ [y, y'] = 0, \ [y, x] = (y, x) + \sum_s c_s (y, (s - 1)x)s,$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$. 7
Adding the Planck constant

In fact, it is useful to introduce one more parameter $\hbar$ (the Planck constant), replacing the last relation by

$$[y, x] = \hbar(y, x) - \sum_s c_s(y, (s - 1)x)s.$$ 

This defines the most general rational Cherednik algebra, $H_{\hbar, c}(W, \mathfrak{h})$, and the above theory extends to this setting (namely, in the Dunkl operators one needs to replace $\partial_y$ by $\hbar \partial_y$). In particular, it is interesting to set $\hbar = 0$, replacing $\hbar \partial_y$ by classical momenta $p_y$; then the above construction gives the integrals of the classical Calogero-Moser system.

**Example**

If $W = \mathbb{Z}/2$ and $\mathfrak{h}$ is the sign representation of $W$ then $H_{\hbar, c}(W, \mathfrak{h})$ is generated by $x, y, s$ with relations

$$sx = -xs, \quad sy = -ys, \quad s^2 = 1, \quad [y, x] = \hbar - 2cs.$$
The algebra $H_{\hbar, c}(S_n, \mathbb{C}^n)$ is generated by $S_n, x_1, \ldots, x_n, y_1, \ldots, y_n$, with relations

$$sx_i = x_{s(i)}s, \ sy_i = y_{s(i)}s, \ [x_i, x_j] = 0, \ [y_i, y_j] = 0,$$

$$[y_i, x_j] = cs_{ij}, \ [y_i, x_i] = \hbar - c \sum_{j \neq i} s_{ij},$$

where $s \in S_n$ and $i \neq j$.

$H_{0,0}(W, \hbar) = \mathbb{C}W \ltimes S(\hbar \oplus \hbar^*)$ and $H_{1,0}(W, \hbar) = W \ltimes D(\hbar)$. The algebra $H_{\hbar, c}$ has a filtration with $\deg \hbar^* = \deg W = 0, \deg \hbar = 1$, and $\text{gr}H_{\hbar, c} = H_{0,0}$. 
### Corollary (Drinfeld, Cherednik, E. Ginzburg)

*(the PBW theorem)* The multiplication map $S\mathfrak{h}^* \otimes \mathbb{C} W \otimes S\mathfrak{h} \to H_c(W, \mathfrak{h})$ is a linear isomorphism.

### Theorem (Losev)

If $W$ is irreducible then $H_{\hbar, c}(W, \mathfrak{h})$ is the universal filtered deformation of $\mathbb{C} W \ltimes S(\mathfrak{h} \oplus \mathfrak{h}^*)$.

### Theorem (E.-Ginzburg)

If $W$ is irreducible then $H_{1, c}(W, \mathfrak{h})$ for formal $c$ is the universal formal deformation of $H_{1, 0}(W, \mathfrak{h}) = \mathbb{C} W \ltimes D(\mathfrak{h})$.

### Remark

These results extend verbatim to the case when $W$ is a finite complex reflection group, except the corresponding integrable systems will not have a quadratic Hamiltonian, since $\deg P_1 > 2$. 

If $W$ is crystallographic (i.e., a Weyl group) and corresponds to a reduced root system $R$, then the CM system attached to $W$ admits a integrable trigonometric deformation

$$H = \Delta - \sum_s c_s (c_s + 1)(\alpha_s, \alpha_s) \frac{1}{4 \sinh^2 \frac{1}{2} \alpha_s(x)},$$

where $\alpha_s$ is the positive root corresponding to $s$. This deformation is integrated using the trigonometric Dunkl-Cherednik operators

$$D_y^{\text{trig}} := \partial_y + \sum_s c_s \frac{\alpha_s(y)}{e^{\alpha_s} - 1} s,$$

which give rise to the trigonometric Cherednik algebra $H_{\hbar, c}^{\text{trig}}(W, \hbar, R)$, a deformation of $H_{\hbar, c}(W, \hbar)$. 
Trigonometric deformation

The trigonometric Cherednik algebra is generated by two subalgebras $A_x, A_y$ which deform $\mathbb{C}W \ltimes S\mathfrak{h}^*$ and $\mathbb{C}W \ltimes S\mathfrak{h}$, respectively. The first subalgebra $A_x$ is generated by $W$ and commuting elements $X_i = e^{\omega_i}$ corresponding to fundamental weights, and is simply the group algebra $\mathbb{C}[\hat{W}]$ of the extended affine Weyl group $\hat{W}$. The second subalgebra $A_y$ is the degenerate (or graded) affine Hecke algebra studied by Drinfeld and Lusztig. It is generated by $W$ and $y \in \mathfrak{h}$ with relations

$$s_i(y)s_i - s_iy = c_{si}\alpha_i(y), \quad [y, y'] = 0,$$

where $s_i$ are the simple reflections, $\alpha_i$ the corresponding positive simple roots, and $y, y' \in \mathfrak{h}$.

There is also an elliptic Cherednik algebra, leading to the integrable elliptic deformation of the CM system, with Hamiltonian

$$H = \Delta - \sum_s c_s(c_s + 1)(\alpha_s, \alpha_s)\wp(\alpha_s(x), \tau).$$
The trigonometric Cherednik algebra $H_{\hbar, c}^{\text{trig}}$ has a $q$-deformation with parameters $q = e^{\varepsilon \hbar}$ and $t_s = e^{\varepsilon c_s}$, which degenerates to the trigonometric algebra as $\varepsilon \to 0$. This deformation is Cherednik’s double affine Hecke algebra (DAHA) $HH_{q,t}(W, \hbar, R)$. To define it, it is best to use the topological language. Namely, recall that the usual Hecke algebra $\mathcal{H}_t(W)$ is the quotient of the group algebra of the braid group $B_W$ by the Hecke relations

$$(T_i - t_i)(T_i + t_i^{-1}) = 0.$$ 

This is a special case of the following general definition of the orbifold Hecke algebra, which in particular also includes affine and double affine Hecke algebras.
Let $X$ be a simply connected complex manifold and $G$ be a discrete group of automorphisms of $X$ acting on $X$ properly discontinuously (i.e., we have a complex orbifold $X/G$). Let $X^\circ \subset X$ be the set of points with trivial stabilizer. The braid group of $X$, $G$ is the group $B := \pi_1(X^\circ/G)$. We have a surjective homomorphism $\phi : B \to G$ corresponding to gluing back points with nontrivial stabilizer. Let us describe the kernel of $\phi$. To this end, define a reflection hypersurface to be a codimension 1 connected component of $X^g$ for some $g \in G$. For every reflection hypersurface $Y \subset X$, we have a small counterclockwise loop $T_Y \subset X^\circ/G$ around the image of $Y$ in $X/G$, which defines a conjugacy class in $B$. Then the Seifert-van Kampen theorem yields

**Proposition**

$\ker \phi$ is generated by the relations $T_Y^{n_Y} = 1$, where $n_Y$ is the order of the stabilizer $G_Y$ of a generic point of $Y$. 
Orbifold Hecke algebras

Now to each reflection hypersurface $Y$ attach nonzero parameters $t_j(Y)$, $j = 1, \ldots, n_Y$ which are the same for conjugate reflection hypersurfaces $Y$.

**Definition (E.)**

The orbifold Hecke algebra $\mathcal{H}_t(G, X)$ is the quotient of $\mathbb{C}B$ by the relations

$$\prod_{j=1}^{n_Y} (T_Y - t_j(Y)) = 0.$$

This is clearly a deformation of the group algebra $\mathbb{C}G$ of $G$, which, by the above proposition, is recovered when $t_j(Y) = e^{2\pi ij/n_Y}$.

**Theorem (E.)**

*(Formal PBW theorem)* If $H^2(X, \mathbb{C}) = 0$ then this deformation is flat in the formal sense (i.e., when $t_j(Y) = e^{\tau_j,Y} e^{2\pi ij/n_Y}$, where $\tau_{j,Y}$ are formal parameters).*
Example
Let $X = \mathfrak{h}$.
1. If $G = W$ then $\mathcal{H}_t(G, X)$ is the usual Hecke algebra $\mathcal{H}_t(W)$.
2. If $G = \hat{W} = W \ltimes P^\vee$, where $P^\vee$ is the coweight lattice (in the crystallographic case) then $\mathcal{H}_t(G, X)$ is the affine Hecke algebra $\mathcal{H}_t(\hat{W})$.

Definition (Cherednik)
If $G = W \ltimes (P^\vee \oplus \tau P^\vee)$, where $\text{Im} \tau > 0$, then $\mathcal{H}_t(G, X)$ is the double affine Hecke algebra $\mathcal{H}_{q,t}(W, \mathfrak{h}, R)$.

From this description it is easy to see that as the trigonometric Cherednik algebra is $q$-deformed, the two subalgebras $A_x, A_y$ deform into two copies of the affine Hecke algebra $\mathcal{H}_t(\hat{W})$. 

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Let us describe by explicit generators and relations the DAHA for \( W = S_n, \mathfrak{h} = \mathbb{C}^n \) (i.e., \( G = S_n \rtimes \mathbb{Z}^{2n} \)), which we call \( \mathcal{H}_{q,t}(n) \).

**Proposition (Cherednik)**

\( \mathcal{H}_{q,t}(n) \) is generated by invertible elements \( X_i, Y_i, i = 1, \ldots, n \), and \( T_i, i = 1, \ldots, n-1 \), with relations

\[
\begin{align*}
(T_i - t^{1/2})(T_i + t^{-1/2}) &= 0, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad T_i T_j &= T_j T_i \quad (|i - j| \geq 2), \\
T_i X_i T_i &= X_{i+1}, \quad T_i X_j &= X_j T_i \quad (j \neq i, i + 1), \quad [X_i, X_j] = 0, \\
T_i Y_i T_i &= Y_{i+1}, \quad T_i Y_j &= Y_j T_i \quad (j \neq i, i + 1), \quad [Y_i, Y_j] = 0, \\
X_1^{-1} Y_2^{-1} X_1 Y_2 &= T_1^2, \quad Y_i \tilde{X} = q \tilde{X} Y_i, \quad X_i \tilde{Y} = q^{-1} \tilde{Y} X_i,
\end{align*}
\]

where \( \tilde{X} := \prod_i X_i \) and \( \tilde{Y} = \prod_i Y_i \).
Polynomial representation of DAHA

The DAHA contains two Laurent polynomial subalgebras \( \mathbb{C}[X^{\pm 1}] \subset A_x \) and \( \mathbb{C}[Y^{\pm 1}] \subset A_y \) isomorphic to the group algebra \( \mathbb{C} P^\vee \), and we have the PBW decomposition

\[
\mathcal{H}_q,t = \mathbb{C}[X^{\pm 1}] \otimes \mathcal{H}_t(W) \otimes \mathbb{C}[Y^{\pm 1}] = A_x \otimes \mathbb{C}[Y^{\pm 1}] = \mathbb{C}[X^{\pm 1}] \otimes A_y
\]

(the multiplication map is a linear isomorphism). This allows us to define the polynomial representation \( \mathbf{P} \) of \( \mathcal{H}_q,t \) by

\[
\mathbf{P} := \mathcal{H}_q,t \otimes_{A_y} \mathbb{C}
\]

where \( \mathbb{C} \) is the trivial representation of the affine Hecke algebra \( A_y \), and we have a linear isomorphism \( \mathbf{P} \cong \mathbb{C}[X^{\pm 1}] \).

Moreover, the center of \( A_x \) is \( \mathbb{C}[X^{\pm 1}]^W \) and the center of \( A_y \) is \( \mathbb{C}[Y^{\pm 1}]^W \).

**Proposition (Cherednik)**

For generic \( q, t \):

(i) The algebra \( \mathbb{C}[Y^{\pm 1}] \) acts with simple spectrum on \( \mathbf{P} \).

(ii) The algebra \( \mathbb{C}[Y^{\pm 1}]^W \) acts with simple spectrum on \( \mathbf{P}^W \) by difference operators.
Macdonald polynomials and operators

**Definition**

The eigenvectors of $\mathbb{C}[Y^{\pm1}]^W$ in $P^W$ are the Macdonald polynomials $P_{\lambda}^{q,t} (\lambda \in P^\vee_+)$. The difference operators defined by $\mathbb{C}[Y^{\pm1}]^W$ are the Macdonald operators. The eigenvectors of $\mathbb{C}[Y^{\pm1}]$ in $P$ are the nonsymmetric Macdonald polynomials $E_{\lambda}^{q,t} (\lambda \in P^\vee)$.

Macdonald operators define a quantum integrable system which is a relativistic deformation of the trigonometric Calogero-Moser system. It is called the Macdonald-Ruijsenaars system.

**Example (Macdonald)**

Macdonald operators for $S_n$ have the form

$$M_r = \sum_{l \subset [1,n]: |l| = r} \prod_{i \in l, j \notin l} \frac{tx_i - x_j}{x_i - x_j} T_l,$$

where $T_l$ replaces $x_i$ with $qx_i$ for $i \in l$. 
### Remark

Cherednik used this structure to prove Macdonald’s conjectures about Macdonald polynomials, and this served as a motivation for introducing DAHA.

### Remark

In the trigonometric limit the eigenvectors will be (nonsymmetric) Jack polynomials, which are limits of (nonsymmetric) Macdonald polynomials.

### Remark

For $W$ of type $B_n$, there is a DAHA depending on 5 parameters for $n = 1$ and 6 parameters for $n \geq 2$, introduced by Sahi and Stokman. The eigenvectors for the corresponding commutative subalgebras $\mathbb{C}[Y^{\pm 1}]^W$ are Askey-Wilson polynomials for $n = 1$ and Koornwinder polynomials for $n \geq 2$. 
The spherical subalgebra and the center

Let \( e \) be the idempotent of the trivial representation of \( \mathcal{H}_t(W) \). The algebra \( e \mathcal{H}_{q,t} e \) is called the **spherical subalgebra**.

**Theorem (E.-Ginzburg, Oblomkov)**

(i) \( e \mathcal{H}_{1,t} e \) is commutative.
(ii) Let \( Z_t \subset \mathcal{H}_{1,t} \) be the center of DAHA. Then the map \( Z_t \to e \mathcal{H}_{1,t} e \) given by \( z \mapsto ze \) is an isomorphism.
(iii) \( Z_t \) is a finitely generated Cohen-Macaulay integral domain, and \( \mathcal{H}_{1,t} \) is a Cohen-Macaulay module over \( Z_t \).

Thus, for each \( t \) we obtain an irreducible Cohen-Macaulay variety \( CM_t = \text{Spec} Z_t \). Moreover, it carries a Poisson structure coming from varying \( q \), which has finitely many symplectic leaves. In particular, \( CM_t \) is smooth outside codimension 2, hence normal, and the algebra \( \mathcal{H}_{q,t} \) is Azumaya on the smooth locus of \( CM_t \), of degree \( |W| \). The variety \( CM_t \) is called the **Calogero-Moser space** attached to \( \mathcal{H}_{1,t} \). It is a flat deformation of \( CM_0 = (T \times T)/W \), where \( T := \mathfrak{h}/P^\vee \) is the corresponding torus.
Calogero-Moser spaces

Thus, the algebra $eH_{q,t}e$ (with $q = e^\hbar$) may be viewed as a quantization of the Poisson variety $CM_t$.

The same theory applies in the trigonometric and rational limits, producing Calogero-Moser spaces $CM^\text{trig}_c$, $CM^\text{rat}_c$ and their quantizations given by the spherical subalgebras of trigonometric and rational Cherednik algebras.

Moreover, for $W = S_n$ these spaces admit a quiver-theoretic description. Namely, we have (for generic $t \in \mathbb{C}^*$)

<table>
<thead>
<tr>
<th>Theorem (E.-Ginzburg, Oblomkov)</th>
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<tbody>
<tr>
<td>(i) $CM^\text{rat}_1 = {(X, Y) \in gl_n \times gl_n : \text{rk}(XY - YX + \text{Id}) = 1}/GL_n;$</td>
</tr>
<tr>
<td>(ii) $CM^\text{trig}_1 = {(X, Y) \in GL_n \times gl_n : \text{rk}(XY - YX + X) = 1}/GL_n;$</td>
</tr>
<tr>
<td>(iii) $CM_t = {(X, Y) \in GL_n \times GL_n : \text{rk}(tXY - YX) = 1}/GL_n.$</td>
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</table>

These varieties are smooth, and are topologically trivial deformations of the Hilbert scheme $\text{Hilb}_n(S)$, where $S = \mathbb{C}^2$ in (i), $S = \mathbb{C}^* \times \mathbb{C}$ in (ii), and $S = \mathbb{C}^* \times \mathbb{C}^*$ in (iii). The space in (i) was introduced by Kazhdan, Kostant and Sternberg in 1978.
As follows from the above discussion, in the case $\hbar = 0$ the representation theory of DAHA and their degenerations is controlled by the geometry of the Calogero-Moser spaces. For example, we have:

**Theorem (E.-Ginzburg, Oblomkov)**

For $W = S_n$ all irreducible representations of $H_{0,1}^{\text{rat}}$, $H_{0,1}^{\text{trig}}$, $\mathcal{H}_{1,t}$ have dimension $n!$ and are parametrized by the space $\text{CM}_{1}^{\text{rat}}$, $\text{CM}_{1}^{\text{trig}}$ and $\text{CM}_t$, respectively. (i.e., DAHA are Azumaya algebras).

Let us now turn to the quantum case $\hbar \neq 0$. For simplicity consider the rational case (so we may assume $\hbar = 1$). Recall that we have the factorization $H_{c}(W) = S_{\mathfrak{h}}^{\ast} \otimes \mathbb{C}W \otimes S_{\mathfrak{h}}$, analogous to the factorization $U(\mathfrak{g}) = U(\mathfrak{n}_{-}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_{+})$ for a semisimple Lie algebra $\mathfrak{g}$. This makes the representation theory of $H_{c}(W)$ similar to the representation theory of $\mathfrak{g}$. 

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**Category $O_c$**

**Definition (Ginzburg, Guay, Opdam, Rouquier)**

The category $O_c$ of $H_c(W)$-modules consists of finitely generated modules on which $\mathfrak{h}$ acts locally nilpotently.

**Theorem (Ginzburg, Guay, Opdam, Rouquier)**

*The category $O_c$ is a highest weight category with highest weights being the irreducible representations $\tau$ of $W$. Its standard objects are the Verma modules $\Delta_c(\tau) := H_c(W) \otimes_{C[W \ltimes S_{\mathfrak{h}}]} \tau$, where $\mathfrak{h}$ acts on $\tau$ by zero.*

This raises the question about multiplicities of simples in standards in $O_c$; recall that such multiplicities for Lie algebras are given by the values at $q = 1$ of Kazhdan-Lusztig polynomials.

**Theorem (Rouquier, Losev)**

*If $W = S_n$ then multiplicities in $O_c$ are given by the values at $q = 1$ of certain affine parabolic Kazhdan-Lusztig polynomials.*
For a general Coxeter group $W$, the story is even more complicated, but there has been significant progress recently. The problem is also solved in the case $W = S_n \ltimes (\mathbb{Z}/r)^n$ using the theory of Kac-Moody categorification. However, for a general complex reflection group $W$ the problem is still open.
DAHA arise in harmonic analysis on p-adic loop groups (Kapranov, Braverman, Kazhdan, Gaitsgory, Matnaik).

DAHA are used to study quasiinvariants of reflection groups (Chalykh, Veselov, Feigin, Ginzburg, E., Rains)

DAHA act on the equivariant K-theory of affine Springer fibers. Trigonometric DAHA act on their equivariant cohomology. Rational DAHA act on the associated graded of this cohomology (Varagnolo, Vasserot, Oblomkov, Yun).
Khovanov-Rozansky homology of torus links and Poincare polynomials of compactified Jacobians of plane curve singularities express via characters of representations of rational Cherednik algebras for $S_n$ (Gorsky, Oblomkov, Rasmussen, Shende, E., Losev).

Representation categories of $H_c(S_n \ltimes (\mathbb{Z}/r)^n)$ can be described in terms of parabolic categories $O$ for affine Lie algebras and provide examples of categorical $\hat{\mathfrak{sl}}_e$-actions (Varagnolo, Vasserot, Shan, Rouquier, Losev).

As $n \to \infty$, spherical DAHA of $S_n$ stabilize to toroidal quantum groups, or elliptic Hall algebras (Varagnolo, Vasserot, Schiffmann).
Thank you!