A counterexample to the PBW theorem

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The PBW theorem

One of the most basic theorems about Lie algebras is the Poincaré-Birkhoff-Witt theorem:

Theorem

If $\mathfrak{g}$ is a Lie algebra over a field k then the natural surjective homomorphism $\psi : S\mathfrak{g} \to \mathrm{gr}\, U\mathfrak{g}$ is an isomorphism.

Proof of the PBW theorem

Consider the formal deformation of $S\mathfrak{g}[z]$ over $k[[\hbar]]$ generated by $\mathfrak{g} \oplus kz$ with defining relations

$$[a, z] = 0, a \in \mathfrak{g},$$

and

$$ab - ba = \hbar z[a, b], a, b \in \mathfrak{g}.$$

The PBW theorem is equivalent to the statement that this deformation is flat. It is flat in degree 2 by the skew-symmetry of $[,]$ and in degree 3 by the Jacobi identity. Therefore, since the algebra $S\mathfrak{g}$ (hence $S\mathfrak{g}[z]$) is Koszul (as it is a polynomial algebra), by Drinfeld's Koszul deformation principle, the deformation is flat in all degrees, as desired.

Generalization to symmetric tensor categories

- We want to generalize the PBW theorem to Lie algebras in symmetric tensor categories (STC).
- By a STC over a field k (for simplicity assumed algebraically closed) we mean a symmetric rigid monoidal artinian category with bilinear tensor product, such that $\mathrm{End}(\mathbf{1}) = k$.
- A basic example of a STC is the category of representations of an algebraic group or supergroup, but there are other examples which we will see later.

First we need to define a Lie algebra in a STC $\mathcal{C}$. The first attempt is to say that a Lie algebra in $\mathcal{C}$ is an object $\mathfrak{g}$ of $\mathcal{C}$ with a skew-symmetric morphism $[,]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying the Jacobi identity $[[,],]+\mathrm{cyc}=0$. In other words, a Lie algebra is an algebra in $\mathcal{C}$ over the Lie operad **Lie**.

Lie algebras in symmetric tensor categories, continued

Unfortunately, this definition is problematic already for $\mathcal{C} = \mathrm{Vec}$ if $\mathrm{char} k = 2$. Namely, as we know, we need to add the condition $[x, x] = 0$, which does not follow from the skew-symmetry in characteristic 2. Without this condition, the PBW theorem fails (already for the free "Lie algebra" in one generator!).

There is a similar problem in characteristic 3 for Lie superalgebras: for odd x, we need to add the condition $[[x, x], x] = 0$, which does not follow from the Jacobi identity in characteristic 3; without it, the PBW theorem fails, already for the free "Lie superalgebra" in one odd generator.

So, let us stay away from characteristics 2,3.

Then we have the following
"**Theorem.**" If $\mathrm{char} k \neq 2, 3$ then the PBW theorem holds for any Lie algebra $\mathfrak{g} \in \mathcal{C}$.
The "proof" is the same as above (the condition on the characteristic ensures flatness in degrees 2,3).

Unfortunately, this proof and the theorem are WRONG; they are correct only in characteristic zero (or in categories of representations of groups and supergroups). The error is that in characteristic $p \geq 5$, the algebra $S\mathfrak{g}$ for an object $\mathfrak{g} \in \mathcal{C}$ may not be Koszul.Because of this, we need additional conditions to ensure the PBW theorem, similar to the above in characteristics 2,3. Thus, we will call an algebra $\mathfrak{g}$ in $\mathcal{C}$ over the operad **Lie** an *operadic Lie algebra*.

The Verlinde category

To give a counterexample, we need to define a remarkable symmetric tensor category Ver_p, the Verlinde category, which was defined by Gelfand-Kazhdan and Georgiev-Mathieu.

To do so, call a morphism $f : X \to Y$ in a symmetric tensor category $\mathcal{C}$ *negligible* if for any morphism $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$. It is easy to show that negligible morphisms form a tensor ideal $\mathcal{N}$, so one can define a new symmetric tensor category $\mathcal{C}/\mathcal{N}$, easily seen to be semisimple. It is called the *semisimplification* of $\mathcal{C}$. The simple objects of $\mathcal{C}/\mathcal{N}$ are indecomposable objects of $\mathcal{C}$ of nonzero dimension (in k).

Definition

The Verlinde category Ver_p is the semisimplification of $\mathrm{Rep}(\mathbb{Z}/p, k)$, where $\mathrm{char} k = p$.

Recall that indecomposable representations of $\mathbb{Z}/p$ over k are Jordan blocks of size $1, ..., p$. The block of size p has zero dimension, so it drops out in the semisimplification. Thus, Ver_p has simple objects $L_1 = \mathbf{1}, ..., L_{p-1}$, and it is not hard to show that the fusion in Ver_p is defined by the Verlinde rule, justifying the terminology. In particular, $\mathrm{Ver}_2 = \mathrm{Vec}$ and $\mathrm{Ver}_3 = \mathrm{Supervec}$.

Now define the free operadic Lie algebra generated by an object $V \in \mathcal{C}$:

$$\mathrm{FOLie}(V) := \oplus_{n\geq 1}(V^{\otimes n} \otimes \mathbf{Lie}_n)_{S_n}$$

(where we recall that $\mathbf{Lie}_n$ is the polylinear component of the free Lie algebra in n generators over $\mathbb{Z}$).

Proposition

For $p \geq 3$, the operadic Lie algebra $\mathrm{FOLie}(L_2)$ in Ver_p fails the PBW theorem. So does its finite dimensional truncation $\mathrm{FOLie}(L_2)_{\leq p}$ (the quotient by degrees $> p$).

Proof

As any associative algebra is in particular a Lie algebra, we have a natural inclusion of operads **Lie** $\hookrightarrow$ **Assoc**. Thus, for any object $V \in \mathcal{C}$, we have a natural map

$$\phi : \mathrm{FOLie}(V) = \oplus_{n \geq 1} (V^{\otimes n} \otimes \mathbf{Lie}_n)_{S_n} \to \oplus_{n \geq 1} (V^{\otimes n} \otimes \mathbf{Assoc}_n)_{S_n} = TV.$$

Let $E(V) = \mathrm{Ker}\phi$. Then $E(V)$ is an obstruction to PBW theorem for $\mathrm{FOLie}(V)$, since any homomorphism from $\mathrm{FOLie}(V)$ to an associative algebra A must factor through ϕ. Finally, one can show that $E_p(L_2) \neq 0$, which implies the statement.

The definition of a Lie algebra

Let $\mathfrak{g}$ be an operadic Lie algebra. Then we have a natural map $\beta : \mathrm{FOLie}(\mathfrak{g}) \to \mathfrak{g}$. A necessary condition for the PBW theorem is that $\beta|_{E(\mathfrak{g})} = 0$, since the composition $\mathrm{FOLie}(\mathfrak{g}) \to T\mathfrak{g} \to U\mathfrak{g}$ factors through β, so the image of $\beta|_{E(\mathfrak{g})}$ is killed by the map $\mathfrak{g} \to U\mathfrak{g}$.

Definition

An operadic Lie algebra $\mathfrak{g}$ is called a *Lie algebra* if $\beta|_{E(\mathfrak{g})} = 0$.

Main Theorem 1

Definition

A symmetric tensor category $\mathcal{C}$ is quasisemisimple if any injection $X \hookrightarrow Y$ in $\mathcal{C}$ gives rise to an injection $SX \hookrightarrow SY$.

Clearly, any semisimple category is quasisemisimple. Moreover, examples of nonquasisemisimple categories are known only in characteristic 2.

Our first main result is:

Theorem

If $\mathcal{C}$ is quasisemisimple then any Lie algebra in $\mathcal{C}$ satisfies the PBW theorem.

This theorem has the drawback that $E(\mathfrak{g})$ is difficult to compute, and the condition $\beta|_{E(\mathfrak{g})} = 0$ is hard to verify. Therefore, we'd like to give a more explicit condition in the case $\mathcal{C} = \mathrm{Ver}_p, p \geq 3$.
To this end, given an operadic Lie algebra $\mathfrak{g} \in \mathrm{Ver}_p$, we will define a twisted linear map $\gamma : \mathrm{Hom}(L_2, \mathfrak{g}) \to \mathrm{Hom}(L_{p-1}, \mathfrak{g})$.
Namely, let R_p be the $p-2$-dimensional irreducible representation of S_p on functions on $[1, p]$ with zero sum of values modulo scalars. Then one can show that $\mathrm{Hom}(L_{p-1}, L_2^{\otimes p}) = R_p$ as an S_p-module, i.e., we have a canonical S_p-invariant inclusion $i : R_p \otimes L_{p-1} \hookrightarrow L_2^{\otimes p}$ as a direct summand.

The p-Jacobi identity, continued

Also, one can show that there is an isomorphism $\eta : k \cong (R_p \otimes \mathbf{Lie}_p)_{S_p}$. Thus, given $x : L_2 \to \mathfrak{g}$, we may define $\gamma(x) : L_{p-1} \to \mathfrak{g}$ by

$$\gamma(x) = \beta \circ (x^{\otimes p} \circ i \otimes \mathrm{Id}_{\mathbf{Lie}_p}) \circ (\mathrm{Id}_{L_{p-1}} \otimes \eta_p).$$

The p-Jacobi identity is then the identity

$$\gamma(x) = 0, x \in \mathrm{Hom}(L_2, \mathfrak{g}).$$

Note that if $p = 3$ (i.e., $\mathrm{Ver}_p = \mathrm{Supervec}$), this identity says that $[[x, x], x] = 0$ for odd x.

Main theorem 2

Our second main result is

Theorem

TFAE for an operadic Lie algebra $\mathfrak{g}$ in Ver_p:
(i) $\mathfrak{g}$ satisfies PBW;
(ii) $\mathfrak{g}$ is a Lie algebra;
(iii) $\mathfrak{g}$ satisfies the p-Jacobi identity $\gamma(x) = 0$.

Generalizations

Note that this theorem can also be used to prove the PBW theorem in categories that admit a tensor functor to Ver_p. It was shown by Ostrik that such a functor exists for any symmetric fusion category, and for $p \geq 3$ it is conjectured to exist for any symmetric tensor category of moderate growth.

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THANK YOU!