Topology of $\overline{M}_{0,n}(\mathbb{R})$

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1. The moduli space of stable curves

Let $k$ be a field, and $n \geq 3$ an integer.

Definition 1.1. A stable curve of genus 0 with $n$ labeled points over $k$ is a finite union $C$ of projective lines $C_1, \ldots, C_p$ over $k$, together with labeled distinct points $z_1, \ldots, z_n \in C$ such that the following conditions are satisfied:

1) each $z_i$ belongs to a unique $C_j$;
2) $C_i \cap C_j$ is either empty or consists of one point;
3) The graph of components (whose vertices are the lines $C_i$ and whose edges correspond to pairs of intersecting lines) is a tree;
4) The total number of special points (i.e. marked points or intersection points) that belong to a given component $C_i$ is at least 3.

Here is an example of such a curve over $\mathbb{R}$:

A stable curve with 8 marked points.

Denote the set of equivalence classes of stable curves of genus 0 with $n$ labeled points over $k$ by $\overline{M}_{0,n}(k)$. It is a classical fact, due to Deligne-Mumford-Knudsen, that there exists a smooth irreducible projective variety $\overline{M}_{0,n}$ of dimension $n - 3$ defined over $\mathbb{Z}$, whose set of $k$-points is $\overline{M}_{0,n}(k)$. This variety is called the moduli space of stable curves of
genus 0 with $n$ labeled points. It is a natural compactification of the moduli space $M_{0,n}$ of $n$-tuples of distinct points on $\mathbb{P}^1$.

**Examples.** 1. $\overline{M}_{0,3}$ is a point, since a stable curve with 3 labeled points must be a single $\mathbb{P}^1$, and any two triples of points are equivalent by a unique fractional linear transformation.

2. $\overline{M}_{0,4} = \mathbb{P}^1$. Indeed, $\overline{M}_{0,4}$ consists of $M_{0,4}$ and three special points corresponding to singular curves, with labeled points arranged as follows: (12)(34), (13)(24), and (14)(23). A point of $M_{0,4}$ is completely determined by the cross ratio of the four points, which can take any value except for 0, 1, $\infty$. Thus
\[ M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\} \]. The three special points fill the holes at 0, 1, \( \infty \), so we get a complete \( \mathbb{P}^1 \).

3. It can be shown that \( \overline{M}_{0,5} \) is the blowup of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at three points.

2. **The topology of the complex moduli space**

   The result of Deligne-Mumford-Knudsen implies that the space \( \overline{M}_{0,n}(\mathbb{C}) \) is a compact connected complex manifold. Similarly, \( \overline{M}_{0,n}(\mathbb{R}) \) is a connected compact real manifold. It is interesting to study the topology of these manifolds.

   The topology of \( \overline{M}_{0,n}(\mathbb{C}) \) is very well understood by now, thanks to the work of Keel, Kontsevich-Manin, Getzler, and others. In particular, in
1992 S. Keel computed the cohomology ring and the Betti numbers of $M_{0,n}(\mathbb{C})$.

**Theorem 2.1. (Keel)** The commutative ring $H^*(M_{n+1}^\mathbb{C}, \mathbb{Z})$ is generated by elements (of degree 2) $D_S$, one for each subset $S \subset \{0,1,2,\ldots,n\}$ with $2 \leq |S| \leq n-1$, subject to the following relations.

1. $D_S = D_{\{0,1,\ldots,n\}\setminus S}$.
2. For distinct elements $i, j, k, l \in \{0,1,\ldots,n\}$,
   \[
   \sum_{i,j \in S} D_S = \sum_{i,k \in S} D_S.
   \]
3. If $S \cap T \not\in \{\emptyset, S, T\}$ and $S \cup T \not\in \{0,1,\ldots,n\}$, then $D_SD_T = 0$. 

The exponential generating function

\[ A(u, t) := \sum_{n \geq 2} \sum_{k=0}^{n-2} \text{rk}(H^{2k}(M_{n+1}^{\mathbb{C}}, \mathbb{Z})) t^k u^n n! \]

satisfies the differential equation

\[ \frac{\partial A}{\partial u} = \frac{u + (1 + t) A}{1 - tA}. \]

This differential equation allows one to compute the Betti numbers recursively.

**Remark 2.2.** The class \( D_S \) has a geometrical interpretation as the class of the divisor of \( M_{n+1}^{\mathbb{C}} \) consisting of singular genus 0 curves in which the removal of a singular point separates the points in \( S \) from the points not in \( S \).
3. The topology of the real moduli space

In this talk I will deal with topological questions about $\overline{M}_{0,n}(\mathbb{R})$, which I will denote by $M_n$ for brevity. First of all, we have the following result.

**Proposition 3.1.** (i) $M_n$ is not orientable for $n \geq 5$.

(ii) (Davis-Januskiewicz-Scott) $M_n$ is aspherical.

**Example.** $M_5$ is the blowup of $S^1 \times S^1$ at three points, so it is a closed nonorientable surface of Euler characteristic $-3$.

The main result I want to discuss is the determination of the Betti numbers and the cohomology ring of $M_n$ (or, equivalently, of its fundamental
group $\Gamma_n$). Before giving the answer, we should do a bit of algebra.

4. The algebra $\Lambda_n$

Definition 4.1. $\Lambda_n$ is the skew-commutative algebra generated over $\mathbb{Z}$ by elements $\omega_{ijkl}$, $1 \leq i, j, k, l \leq n$, which are antisymmetric in $ijkl$, with defining relations

\begin{align*}
(2) \quad \omega_{ijkl} + \omega_{jklm} + \omega_{klmi} + \omega_{lmij} + \omega_{mijk} &= 0, \\
(3) \quad \omega_{ijkl} \omega_{ijkm} &= 0, \\
(4) \quad \omega_{ijkl} \omega_{lmpi} + \omega_{klmp} \omega_{pijk} + \omega_{mpi} \omega_{jklm} &= 0
\end{align*}

for any distinct $i, j, k, l, m, p$.

In particular, $\Lambda_n$ is a quadratic algebra.

We will also consider the algebras $\Lambda_n \otimes R$ for commutative rings $R$. 
They are defined over $R$ by the same generators and relations.

**Remark 4.2.** One can show that $2$ times (4) is in the ideal generated by (2) and (3). So this relation becomes redundant if $1/2 \in R$.

The algebra $\Lambda_n$ has a natural action of $S_n$.

**Proposition 4.3.** One has $\Lambda_n[1] = \wedge^3 \mathfrak{h}_n$, as $S_n$-modules, where $\mathfrak{h}_n$ is the $n - 1$-dimensional submodule of the permutation representation, consisting of vectors with zero sum of coordinates (in particular, $\Lambda_n[1]$ is free over $\mathbb{Z}$ of rank $(n - 1)(n - 2)(n - 3)/6$).

It is convenient to use another presentation of $\Lambda_n$, in which only the
$S_{n-1}$-symmetry, rather than the full $S_n$-symmetry, is apparent, but which contains only quadratic relations. This presentation is given by the following proposition.

**Proposition 4.4.** The algebra $\Lambda_n$ is isomorphic to the skew-commutative algebra generated by $\nu_{ijk}$, $1 \leq i, j, k \leq n - 1$ (antisymmetric in $ijk$) with defining relations

$$\nu_{ijk}\nu_{ijl} = 0,$$

and

$$\nu_{ijk}\nu_{klm} + \nu_{jkl}\nu_{lmi} + \nu_{klm}\nu_{mij} + \nu_{lmi}\nu_{ijk} + \nu_{mij}\nu_{jkl} = 0.$$

The identification of the two presentations is defined by the formula $\nu_{ijk} \rightarrow \omega_{ijkn}$.
Theorem 4.5. For each $n$, $\Lambda_n$ is a free $\mathbb{Z}$-module with Poincaré polynomial

$$P_n(t) = \prod_{0 \leq k < (n-3)/2} (1+(n-3-2k)^2 t).$$

This theorem is proved by constructing a homogeneous basis of $\Lambda_n$ and counting the number of elements in this basis. Namely, define a triangle graph on vertices $1, \ldots, n-1$ to be a collection of triangles on these vertices. To every triangle graph with $m$ triangles we can attach an element of $\Lambda_n$ of degree $m$ defined up to sign, by taking the product of $\nu_{ijk}$ over all triangles $ijk$ in the graph.

We will say that a triangle graph is a triangle forest if there is no cycle whose all edges are contained in
different triangles (in particular, no two triangles have a common edge). If in addition it is connected, it is called a **triangle tree**.

**Definition 4.6.** We define **basic** triangle trees and forests as follows, by induction on the number of triangles.

1. A single point is a basic triangle tree.
2. A nontrivial triangle tree is basic iff the two smallest vertices are on a common triangle, and each of the three components after removing that triangle is basic.
3. A triangle forest is basic iff each component is basic.
Theorem 4.7. The algebra $\Lambda_n$ is freely spanned by the monomials associated to basic triangle forests.

Conjecture 4.8. The algebra $\Lambda_n \otimes \mathbb{Q}$ is a Koszul quadratic algebra.

5. The cohomology of the real moduli space

For any ordered $m$-element subset $S = \{s_1, \ldots, s_m\}$ of $\{1, \ldots, n\}$ we have a natural map $\phi_S : M_n \to M_m$, forgetting the points with labels outside $S$. More precisely, given a stable curve $C$ with labeled points $z_1, \ldots, z_n$, $\phi_S(C)$ is $C$ with labeled points $z_{s_1}, \ldots, z_{s_m}$, in which the components that have fewer than 3 special points have been collapsed in an obvious way.
Thus for any commutative ring \( R \) we have a homomorphism of algebras \( \phi_s^*: H^*(M_m, R) \to H^*(M_n, R) \).

For \( m = 4 \), \( M_m = \mathbb{RP}^1 \) is a circle, and we denote by \( \omega_s \) the image of the standard generator of \( H^1(M_4, R) \) under \( \phi_s^* \).

**Proposition 5.1.** Over any ring \( R \) in which 2 is invertible, the elements \( \omega_s \) satisfy the relations (2), (3).

**Proof.** Let us give a proof when \( R = \mathbb{Q} \). The skew-symmetry of \( \omega_s \) is obvious. Next, we check the quadratic relations (3). By considering the maps \( \phi_s \) for \( |S| = 5 \), it suffices to check this relation on \( M_5 \). But \( H^2(M_5, \mathbb{Q}) = 0 \) because \( M_5 \) is non-orientable.
The 5-term linear relation (2) may also be checked on $M_5$. It is easy to see that as an $S_5$-module, $H^1(M_5, \mathbb{Q})$ is the tensor product of the permutation and sign representations. In particular, the 5-cycle has no invariants in this representation, and hence the 5-term relation holds. \hfill \Box

**Corollary 5.2.** For any ring $R$ in which 2 is invertible, we have a homomorphism of algebras

$$f_n^R : \Lambda_n \otimes R \to H^*(M_n, R),$$

which maps $\omega_S$ to $\omega_S$.

Our main result is the following theorem.

**Theorem 5.3.** $f_n^\mathbb{Q}$ is an isomorphism.
It then follows from Theorem 4.5 that the Poincaré polynomial of $M_n$ is $P_n(t)$.

We also have the following result.

**Theorem 5.4.** $H^*(M_n, \mathbb{Z})$ does not have 4-torsion.

**Theorem 5.5.** (E. Rains) $H^*(M_n, \mathbb{Z})$ does not have odd torsion.

The cohomology of $M_n$ does, however, have 2-torsion. It is determined by the following theorem.

**Theorem 5.6.** There is a natural isomorphism of algebras $H^{2*}(\overline{M}_{0,n}(\mathbb{C}), \mathbb{F}_2) \cong H^*(\overline{M}_{0,n}(\mathbb{R}), \mathbb{F}_2)$.

Thus, the integral cohomology groups of $M_n$ are

$$H^m(M_n, \mathbb{Z}) = \mathbb{Z}^{b_m^\mathbb{R}} \oplus \mathbb{F}_2^{b_m^\mathbb{C} - b_m^\mathbb{R}}.$$
where \( b^\mathbb{R}_m, b^\mathbb{C}_m \) are the Betti numbers of the manifolds \( \overline{M}_{0,n}(\mathbb{R}), \overline{M}_{0,n}(\mathbb{C}) \), whose generating functions are given above.

6. **The homology operad of \( M_n \)**

The spaces \( M_n \) form a topological operad (the mosaic operad, introduced by Devadoss). To define this operad, it is convenient to agree that each of the undefined moduli spaces \( M_1 \) and \( M_2 \) consists of one point. Then we will define a topological operad with set of \( n \)-ary operations \( M(n) := M_{n+1} \) (we think of a point of \( M_{n+1} \) as an \( n \)-ary operation where the inputs sit at points \( 1, \ldots, n \) and the output is \( n + 1 \)). Namely, given \( p, q \geq 0 \) and \( 1 \leq j \leq p \), we have a “substitution”
map $\gamma_{ipq} : M_{p+1} \times M_{q+1} \to M_{p+q}$
given by attaching a curve $C_1$ with $p + 1$ marked points to a curve $C_2$
with $q + 1$ marked points by identifying the point $i$ on the first curve with
the point $q + 1$ on the second curve, and then adding $q - 1$ to the labels
$i + 1, \ldots, p + 1$ on $C_1$ and adding $i - 1$ to the labels of the points $1, \ldots, q$
on $C_2$. The operad structure is obtained by iterating such maps.

Since $M(\bullet)$ is a topological operad,
the spaces $O(n) := H_*(M_{n+1}, \mathbb{Q}) = (\Lambda_{n+1} \otimes \mathbb{Q})^*$ form an operad in the
category of $\mathbb{Z}$-graded supervector spaces.
The following result determines the structure of this operad.

**Theorem 6.1.** The operad $O(n)$
is the operad of unital 2-Gerstenhaber
algebras. More specifically, it is generated by $1 \in O(0)$, $\mu \in O(2)$, and $\tau \in O(3)$, such that

1) $\mu$ is a commutative associative product of degree 0 with unit 1;

2) $\tau$ is an skew-symmetric ternary operation of degree $-1$, which is a derivation in each variable with respect to the product $\mu$.

3) $\tau$ satisfies the Jacobi identity: $\Alt(\tau \circ (\tau \otimes \Id \otimes \Id)) = 0$, where the alternator is over $S_5$ (As usual, the alternator is understood in the supersense).

**Corollary 6.2.** Consider the sub-operad $O' \subset O$, with $O'(2k) = 0$, $O'(2k + 1) = H_k(M_{2k+2}, \mathbb{Q})$. Then $O'$ is the Hanlon-Wachs operad of
Lie 2-algebras, generated by a skew-symmetric ternary operation \( \tau \), satisfying the Jacobi identity
\[ \text{Alt}(\tau \circ (\tau \otimes \text{Id} \otimes \text{Id})) = 0. \]

For comparison let us say that the homology operad of the complex moduli spaces \( \overline{M}_{0,n}(\mathbb{C}) \) is the operad of hypercommutative algebras, studied by Kontsevich and Manin. It is generated by infinitely many “multiproduct” operations.

7. The fundamental group of \( M_n \).

Let \( \Gamma_n \) be the fundamental group of \( M_n \). To understand this group, one should consider another group \( J_n \) which is the orbifold fundamental group of the orbifold \( M_{n+1}/S_n \).
(the group $S_n$ leaves the point $n + 1$ fixed). One has a short exact sequence

$$1 \to \Gamma_{n+1} \to J_n \to S_n \to 1.$$ 

**Theorem 7.1.** (Devadoss, Davis-Januskiewicz-Scott, Henriques-Kamnitzer) The group $J_n$ has the following presentation: it is generated by elements $s_{p,q}$, $1 \leq p < q \leq n$, with defining relations

1) $s_{p,q}^2 = 1$;
2) $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$ if $[p, q] \cap [m, r] = \emptyset$;
3) $s_{p,q}s_{m,r} = s_{p+q-r-p+q-m}s_{p,q}$ if $[m, r] \subset [p, q]$.

The above map $J_n \to S_n$ is then defined by sending $s_{p,q}$ to the involution that reverses the interval $[p, q]$.