Topology of $\overline{M}_{0,n}(\mathbb{R})$

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1. The moduli space of stable curves

Let k be a field, and $n \geq 3$ an integer.

Definition 1.1. A stable curve of genus 0 with n labeled points over k is a finite union C of projective lines $C_1, ..., C_p$ over k, together with labeled distinct points $z_1, ..., z_n \in C$ such that the following conditions are satisfied:

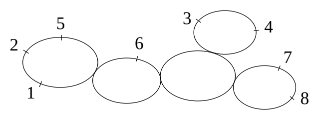
1) each z_i belongs to a unique C_j;

2) $C_i \cap C_j$ is either empty or consists of one point;

3) The graph of components (whose vertices are the lines C_i and whose edges correspond to pairs of intersecting lines) is a tree;

4) The total number of special points (i.e. marked points or intersection points) that belong to a given component C_i is at least 3.

Here is an example of such a curve over $\mathbb{R}$:



A stable curve with 8 marked points.

Denote the set of equivalence classes of stable curves of genus 0 with n labeled points over k by $\overline{M}_{0,n}(k)$. It is a classical fact, due to Deligne-Mumford-Knudsen, that there exists a smooth irreducible projective variety $\overline{M}_{0,n}$ of dimension $n-3$ defined over $\mathbb{Z}$, whose set of k-points is $\overline{M}_{0,n}(k)$. This variety is called the moduli space of stable curves of genus 0 with n labeled points. It is a natural compactification of the moduli space $M_{0,n}$ of n-tuples of distinct points on $\mathbb{P}^1$.

Examples. 1. $\overline{M}_{0,3}$ is a point, since a stable curve with 3 labeled points must be a single $\mathbb{P}^1$, and any two triples of points are equivalent by a unique fractional linear transformation.

2. $\overline{M}_{0,4} = \mathbb{P}^1$. Indeed, $\overline{M}_{0,4}$ consists of $M_{0,4}$ and three special points corresponding to singular curves, with labeled points arranged as follows: (12)(34), (13)(24), and (14)(23). A point of $M_{0,4}$ is completely determined by the cross ratio of the four points, which can take any value except for $0, 1, \infty$. Thus

$M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The three special points fill the holes at $0, 1, \infty$, so we get a complete $\mathbb{P}^1$.

3. It can be shown that $\overline{M}_{0,5}$ is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points.

2. The topology of the complex moduli space

The result of Deligne-Mumford-Knudsen implies that the space $\overline{M}_{0,n}(\mathbb{C})$ is a compact connected complex manifold. Similarly, $\overline{M}_{0,n}(\mathbb{R})$ is a connected compact real manifold. It is interesting to study the topology of these manifolds.

The topology of $\overline{M}_{0,n}(\mathbb{C})$ is very well understood by now, thanks to the work of Keel, Kontsevich-Manin, Getzler, and others. In particular, in

1992 S. Keel computed the cohomology ring and the Betti numbers of $\overline{M}_{0,n}(\mathbb{C})$.

Theorem 2.1. *(Keel) The commutative ring $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$ is generated by elements (of degree 2) D_S, one for each subset $S \subset \{0, 1, 2, \ldots, n\}$ with $2 \leq |S| \leq n-1$, subject to the following relations.*

1. $D_S = D_{\{0,1,\ldots,n\} \setminus S}$.
2. *For distinct elements $i, j, k, l \in \{0, 1, \ldots, n\}$,*

$$\sum_{\substack{i,j \in S \\ k,l \notin S}} D_S = \sum_{\substack{i,k \in S \\ j,l \notin S}} D_S.$$

3. *If $S \cap T \notin \{\emptyset, S, T\}$ and $S \cup T \neq \{0, 1, \ldots, n\}$, then $D_S D_T = 0$.*

The exponential generating function

$$A(u,t) := \sum_{n\geq 2}\sum_{k=0}^{n-2} \mathrm{rk}(H^{2k}(M^{\mathbb{C}}_{n+1},\mathbb{Z}))t^k \frac{u^n}{n!}$$

satisfies the differential equation

$$\frac{\partial A}{\partial u} = \frac{u+(1+t)A}{1-tA}. \tag{1}$$

This differential equation allows one to compute the Betti numbers recursively.

Remark 2.2. The class D_S has a geometrical interpretation as the class of the divisor of $M^{\mathbb{C}}_{n+1}$ consisting of singular genus 0 curves in which the removal of a singular point separates the points in S from the points not in S.

3. The topology of the real moduli space

In this talk I will deal with topological questions about $\overline{M}_{0,n}(\mathbb{R})$, which I will denote by M_n for brevity. First of all, we have the following result.

Proposition 3.1. *(i) M_n is not orientable for $n \geq 5$.*

(ii) (Davis-Januskiewicz-Scott) M_n is aspherical.

Example. M_5 is the blowup of $S^1 \times S^1$ at three points, so it is a closed nonorientable surface of Euler characteristic -3.

The main result I want to discuss is the determination of the Betti numbers and the cohomology ring of M_n (or, equivalently, of its fundamental group Γ_n). Before giving the answer, we should do a bit of algebra.

4. The algebra Λ_n

Definition 4.1. Λ_n is the skew-commutative algebra generated over $\mathbb{Z}$ by elements ω_{ijkl}, $1 \leq i,j,k,l \leq n$, which are antisymmetric in $ijkl$, with defining relations

$$\omega_{ijkl}+\omega_{jklm}+\omega_{klmi}+\omega_{lmij}+\omega_{mijk} = 0, \tag{2}$$

$$\omega_{ijkl}\,\omega_{ijkm} = 0, \tag{3}$$

$$\omega_{ijkl}\,\omega_{lmpi}+\omega_{klmp}\,\omega_{pijk}+\omega_{mpij}\,\omega_{jklm} = 0 \tag{4}$$

for any distinct i,j,k,l,m,p,

In particular, Λ_n is a quadratic algebra.

We will also consider the algebras $\Lambda_n \otimes R$ for commutative rings R.

They are defined over R by the same generators and relations.

Remark 4.2. One can show that 2 times (4) is in the ideal generated by (2) and (3). So this relation becomes redundant if $1/2 \in R$.

The algebra Λ_n has a natural action of S_n.

Proposition 4.3. *One has* $\Lambda_n[1] = \wedge^3 \mathfrak{h}_n$, *as* S_n*-modules, where* $\mathfrak{h}_n$ *is the* $n-1$*-dimensional submodule of the permutation representation, consisting of vectors with zero sum of coordinates (in particular,* $\Lambda_n[1]$ *is free over* $\mathbb{Z}$ *of rank* $(n-1)(n-2)(n-3)/6$*).*

It is convenient to use another presentation of Λ_n, in which only the

S_{n-1}-symmetry, rather than the full S_n-symmetry, is apparent, but which contains only quadratic relations. This presentation is given by the following proposition.

Proposition 4.4. *The algebra Λ_n is isomorphic to the skew-commutative algebra generated by ν_{ijk}, $1 \leq i, j, k \leq n-1$ (antisymmetric in ijk) with defining relations*

$$\nu_{ijk}\nu_{ijl} = 0,$$

and

$$\nu_{ijk}\nu_{klm} + \nu_{jkl}\nu_{lmi} + \nu_{klm}\nu_{mij} + \nu_{lmi}\nu_{ijk} + \nu_{mij}\nu_{jkl} = 0.$$

The identification of the two presentations is defined by the formula $\nu_{ijk} \to \omega_{ijkn}$.

Theorem 4.5. *For each n, Λ_n is a free $\mathbb{Z}$-module with Poincaré polynomial*

$$P_n(t) = \prod_{0\le k<(n-3)/2} (1+(n-3-2k)^2 t).$$

This theorem is proved by constructing a homogeneous basis of Λ_n and counting the number of elements in this basis. Namely, define a **triangle graph** on vertices $1, ..., n-1$ to be a collection of triangles on these vertices. To every triangle graph with m triangles we can attach an element of Λ_n of degree m defined up to sign, by taking the product of ν_{ijk} over all triangles ijk in the graph.

We will say that a triangle graph is a **triangle forest** if there is no cycle whose all edges are contained in different triangles (in particular, no two triangles have a common edge). If in addition it is connected, it is called a **triangle tree**.

Definition 4.6. We define **basic** triangle trees and forests as follows, by induction on the number of triangles.

1. A single point is a basic triangle tree.
2. A nontrivial triangle tree is basic iff the two smallest vertices are on a common triangle, and each of the three components after removing that triangle is basic.
3. A triangle forest is basic iff each component is basic.

Theorem 4.7. *The algebra Λ_n is freely spanned by the monomials associated to basic triangle forests.*

Conjecture 4.8. The algebra $\Lambda_n \otimes \mathbb{Q}$ is a Koszul quadratic algebra.

5. The cohomology of the real moduli space

For any ordered m-element subset $S = \{s_1, ..., s_m\}$ of $\{1, ..., n\}$ we have a natural map $\phi_S : M_n \to M_m$, forgetting the points with labels outside S. More precisely, given a stable curve C with labeled points $z_1, ..., z_n$, $\phi_S(C)$ is C with labeled points $z_{s_1}, ..., z_{s_m}$, in which the components that have fewer than 3 special points have been collapsed in an obvious way.

Thus for any commutative ring R we have a homomorphism of algebras $\phi_S^* : H^*(M_m, R) \to H^*(M_n, R)$.

For $m = 4$, $M_m = \mathbb{RP}^1$ is a circle, and we denote by ω_S the image of the standard generator of $H^1(M_4, R)$ under ϕ_S^*.

Proposition 5.1. *Over any ring R in which 2 is invertible, the elements ω_S satisfy the relations (2), (3).*

Proof. Let us give a proof when $R = \mathbb{Q}$. The skew-symmetry of ω_S is obvious. Next, we check the quadratic relations (3). By considering the maps ϕ_S for $|S| = 5$, it suffices to check this relation on M_5. But $H^2(M_5, \mathbb{Q}) = 0$ because M_5 is non-orientable.

The 5-term linear relation (2) may also be checked on M_5. It is easy to see that as an S_5-module, $H^1(M_5, \mathbb{Q})$ is the tensor product of the permutation and sign representations. In particular, the 5-cycle has no invariants in this representation, and hence the 5-term relation holds. $\square$

Corollary 5.2. *For any ring R in which 2 is invertible, we have a homomorphism of algebras*

$$f_n^R : \Lambda_n \otimes R \to H^*(M_n, R), \tag{5}$$

which maps ω_S to ω_S.

Our main result is the following theorem.

Theorem 5.3. *$f_n^{\mathbb{Q}}$ is an isomorphism.*

It then follows from Theorem 4.5 that the Poincaré polynomial of M_n is $P_n(t)$.

We also have the following result.

Theorem 5.4. *$H^*(M_n, \mathbb{Z})$ does not have 4-torsion.*

Theorem 5.5. *(E. Rains) $H^*(M_n, \mathbb{Z})$ does not have odd torsion.*

The cohomology of M_n does, however, have 2-torsion. It is determined by the following theorem.

Theorem 5.6. *There is a natural isomorphism of algebras $H^{2*}(\overline{M}_{0,n}(\mathbb{C}), \mathbb{F}_2) \cong H^*(\overline{M}_{0,n}(\mathbb{R}), \mathbb{F}_2)$.*

Thus, the integral cohomology groups of M_n are

$$H^m(M_n, \mathbb{Z}) = \mathbb{Z}^{b_m^{\mathbb{R}}} \oplus \mathbb{F}_2^{b_m^{\mathbb{C}} - b_m^{\mathbb{R}}},$$

where $b_m^{\mathbb{R}}, b_m^{\mathbb{C}}$ are the Betti numbers of the manifolds $\overline{M}_{0,n}(\mathbb{R}), \overline{M}_{0,n}(\mathbb{C})$, whose generating functions are given above.

6. The homology operad of M_n

The spaces M_n form a topological operad (the mosaic operad, introduced by Devadoss). To define this operad, it is convenient to agree that each of the undefined moduli spaces M_1 and M_2 consists of one point. Then we will define a topological operad with set of n-ary operations $M(n) := M_{n+1}$ (we think of a point of M_{n+1} as an n-ary operation where the inputs sit at points $1, ..., n$ and the output is $n+1$). Namely, given $p, q \geq 0$ and $1 \leq j \leq p$, we have a "substitution" map $\gamma_{ipq} : M_{p+1} \times M_{q+1} \to M_{p+q}$ given by attaching a curve C_1 with $p+1$ marked points to a curve C_2 with $q+1$ marked points by identifying the point i on the first curve with the point $q+1$ on the second curve, and then adding $q-1$ to the labels $i+1, \ldots, p+1$ on C_1 and adding $i-1$ to the labels of the points 1, ..., q on C_2. The operad structure is obtained by iterating such maps.

Since $M(\bullet)$ is a topological operad, the spaces $O(n) := H_*(M_{n+1}, \mathbb{Q}) = (\Lambda_{n+1} \otimes \mathbb{Q})^*$ form an operad in the category of $\mathbb{Z}$-graded supervector spaces. The following result determines the structure of this operad.

Theorem 6.1. *The operad $O(n)$ is the operad of unital 2-Gerstenhaber* *algebras. More specifically, it is generated by* $1 \in O(0)$, $\mu \in O(2)$, *and* $\tau \in O(3)$, *such that*

1) μ *is a commutative associative product of degree 0 with unit* 1*;*

2) τ *is an skew-symmetric ternary operation of degree* -1*, which is a derivation in each variable with respect to the product* μ*.*

3) τ *satisfies the Jacobi identity:* $\mathrm{Alt}(\tau \circ (\tau \otimes \mathrm{Id} \otimes \mathrm{Id})) = 0$*, where the alternator is over* S_5 *(As usual, the alternator is understood in the supersense).*

Corollary 6.2. *Consider the suboperad* $O' \subset O$*, with* $O'(2k) = 0$, $O'(2k+1) = H_k(M_{2k+2}, \mathbb{Q})$*. Then* O' *is the Hanlon-Wachs operad of*

Lie 2-algebras, generated by a skew-symmetric ternary operation τ, satisfying the Jacobi identity $\mathrm{Alt}(\tau \circ (\tau \otimes \mathrm{Id} \otimes \mathrm{Id})) = 0$.

For comparison let us say that the homology operad of the complex moduli spaces $\overline{M}_{0,n}(\mathbb{C})$ is the operad of hypercommutative algebras, studied by Kontsevich and Manin. It is generated by infinitely many "multiproduct" operations.

7. The fundamental group of M_n.

Let Γ_n be the fundamental group of M_n. To understand this group, one should consider another group J_n which is the orbifold fundamental group of the orbifold M_{n+1}/S_n

(the group S_n leaves the point $n+1$ fixed). One has a short exact sequence

$$1 \to \Gamma_{n+1} \to J_n \to S_n \to 1.$$

Theorem 7.1. *(Devadoss, Davis-Januskiewicz-Scott, Henriques-Kamnitzer) The group J_n has the following presentation: it is generated by elements $s_{p,q}$, $1 \le p < q \le n$, with defining relations*

1) $s_{p,q}^2 = 1$;

2) $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$ if $[p,q] \cap [m,r] = \emptyset$;

3) $s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}$ if $[m,r] \subset [p,q]$.

The above map $J_n \to S_n$ is then defined by sending $s_{p,q}$ to the involution that reverses the interval $[p,q]$.