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Toward Harmonic Analysis on DAHA (Integral formulas for canonical traces)

A. AHA-decomposition
B. DAHA integrations
C. Canonical traces/forms
D. Rational daha (A_1)
E. Main daha (A_1)
F. Positivity
G. Analytic continuation
H. P-adic limit
I. Jantzen filtrations
J. The rational case

Classical HA	HA on DAHA (facts)
Unitary dual	semisimple (mainly A_n)
Fourier theory	polynom. and delta reps
Trace formulas	general approach and A_n

Fourier transform and the Lie theory:

FT $\leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **is fine for the classical** $F = \int e^{2\lambda x}\{\cdot\}\,dx$ **(an outer automorphism of the Heisenberg algebra) and for the Hankel transform. A similar interpretation holds for** $F_N = \sum_j e^{\frac{2\pi j i}{N}}\{\cdot\}$ **via the Weyl algebra and in arithmetic (Weil, metaplectic representations).**

Problems:

1) Cannot be extended to the spherical and hypergeometric functions.

2) A counterpart of $F(e^{-x^2}) = \sqrt{\pi}e^{+x^2}$ **at roots of unity is** $F_N(e^{x^2}) = \pm\sqrt{N}e^{-x^2}$**;** $\pm$ **is very important. Weyl algebra doesn't help.**

3) Multi-dim theory. There are 3 **"natural" candidates for FT in** $SL(3)$ **(reflections in** $S_3 \subset SL(3)$**); however, FT must be unique:**

FT: Polynomials $\mapsto$ **Delta-functions.**

DAHA: FT^2 **is essentially** id **(addresses (1)); FT comes from the transposition of the periods of** E **with one puncture, which settles (2); any root systems can be managed (3).**

A. AHA-DECOMPOSITION

$R \in \mathbb{R}^n$, W, $W^a = W \ltimes Q$;
$\mathcal{H} =< T_i, 0 \le i \le n > / \{ \, relations \}$:
$\{$**Coxeter,** $(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0 \, \}$.

$T_{\widehat{w}} = T_{i_l} \cdots T_{i_1}, \ \widehat{w} = s_{i_l} \cdots s_{i_1},$
$l = l(\widehat{w})$. $\quad \boldsymbol{T^*_{\widehat{w}} \stackrel{\textbf{def}}{=\!=} T_{\widehat{w}^{-1}}}$.

$< T_{\widehat{w}} >= \ 1(\widehat{w} = \textbf{id}), \ 0$ **otherwise**.
$< f, g > \stackrel{\textbf{def}}{=\!=} < f^* g >= \sum_{\widehat{w} \in W^a} c_{\widehat{w}} d_{\widehat{w}},$

$f = \sum c_{\widehat{w}} T_{\widehat{w}}, g = \sum d_{\widehat{w}} T_{\widehat{w}} \in L^2(\mathbb{R})$.
Dixmier: $< f, g >= \int_{\mathcal{H}^\vee} \textbf{Tr}(\pi(f^* g)) d\nu(\pi)$.
SPH: $f, g \in P_+ \mathcal{H} P_+$, $P_+ = \sum_{w \in W} t^{\frac{l(w)}{2}} T_w$.
Macdonald' case: $\nu_{sph}(\pi)$, $t > 1$.
Arthur, Heckman, Opdam: $0 < t < 1$,
$\int \{\cdot\} \, d\nu^{an}_{sph}(\pi) = \sum C_{s,S} \cdot \int_{s+iS} \{\cdot\} \, d\nu_{s,S},$
summed over $s + S =$ **residual subtori.**

The approach aimed at "reducing" Alg Geometry (Lusztig, ...) in Harmonic Analysis.
DAHA: AHO formula results from "Main thm of q-Calculus".

B. DAHA INTEGRATIONS

imaginary $(\lvert q\rvert \neq 1)$	**real** $(\lvert q\rvert \neq 1)$
$\Downarrow$	$\Downarrow$
constant term $(\forall q)$	**Jackson sums**
$\Uparrow$	$\Downarrow$
the case $\lvert q\rvert = 1 \quad =$	$\Rightarrow$ **roots of unity**

C. CANONICAL TRACES

$R \in \mathbb{R}^n$ **a root system (reduced),**
$W =< s_i, 1 \leq i \leq n >$, $P =$ **weight lattice.**

$\mathcal{H\!H} =< X_b, T_w, Y_b >_{q,t},\ b \in P, w \in W.$
Over $\mathbb{R} \ni q, t,\ \ q = exp(-1/a),\ a > 0.$

Generic anti-involution $\varkappa$ of $\mathcal{H\!H}$:
$T_w^{\varkappa} = T_{w^{-1}}$ **and PBW holds:**
$\mathcal{H\!H} \ni H = \sum c_{awb} Y_a^{\varkappa} T_w Y_b.$

DEF. Let $\{H\}^{\varrho}_{\varkappa} = \sum c_{awb}\varrho(Y_a)\varrho(T_w)\varrho(Y_b)$ **as** $\varrho : \mathbb{R}[Y] \to \mathbb{R}($**or** $\mathbb{C})$ **is a character, and** $\varrho : \{T_w\} \to \mathbb{R}($**or** $\mathbb{C})$ **is a trace. Then:**

$$\{A, B\} \stackrel{\mathbf{def}}{=\!=} \{A^{\varkappa} B\}^{\varrho}_{\varkappa} = \{B, A\}.$$

EX. $\{H\}_{\varkappa}^{sph}$ **for the character** $\varrho : Y_{\omega_i}, T_i \mapsto t^{1/2}$. **Then** $\{A, B\}$ **acts through** $\mathbb{R}[X] \times \mathbb{R}[X]$.

EX. $\varkappa : X_b \leftrightarrow Y_b^{-1}, T_w \to T_{w^{-1}}$ **and** $\{H\}_{\varkappa}^{sph}$; **it serves the evaluation, duality formulas.**

D. RATIONAL DAHA

$\mathcal{H}'' \overset{\textbf{def}}{=\!=} \langle x, y, s\rangle /$**relations :**
$[y, x] = 1/2 + ks,\ \ s^2 = 1,$
$sxs = -x,\ \ sys = -y.$

Polynom rep $\mathbb{R}[x]$**:**
$s(x) = -x,\ \ x =$ **mult by** $x,\ y \mapsto D/2,$
$D = \frac{d}{dx} + \frac{k}{x}(1 - s)$ **(Dunkl).**

REM. $h := xy + yx$ **(Euler operator),**
$[h, x] = x,\ \ [h, y] = -y\ \ :\ \ osp(2|1).$
$e := x^2, f := -y^2\ \ :\ \ sl(2).$

$x^* = x, y^* = -y, s^* = s$ **is special,**
serves $\int f(x)g(x)|x|^{2k}$ **: diverges;**

$\mathbb{R}[x]$ **has no** $*$**-form** $(k \not\in -1/2 - \mathbb{Z}_+)$**:**
$\{1, y(x^{p+1})\}\ =\ 0\ =\ c_{p+1}\{1, x^p\},$
$c_{2p} = p,\ \ c_{2p+1} = p + 1/2 + k.$

Replace y **by** $x+y$, $(x+y)^*=x-y$.
PBW: $h=\sum c_{a\epsilon b}((y+x)^*)^a s^\epsilon (y+x)^b$.
$\{f\}=\sum c_{o\epsilon o}$, $\{f,g\}=\{f^*g\}$ **is**
through $\mathbb{R}[x]e^{-x^2}\times\mathbb{R}[x]e^{-x^2}$**, indeed:**
$\mathbb{R}[x]e^{-x^2}=\mathcal{H}/(\mathcal{H}(x+y),\mathcal{H}(s-1))$.

Let $a,b\in Z_+$, $p=\frac{a+b}{2}$, **then**
$\{x^ae^{-x^2},x^be^{-x^2}\}=$
$=(\frac{1}{2})^p(\frac{1}{2}+k)\cdots(\frac{1}{2}+k+p-1)$.

Integral formula: $\{fe^{-x^2},ge^{-x^2}\}=$
$\mathbf{C}\frac{1}{2i}(\int_{-i\epsilon+\mathbb{R}}+\int_{i\epsilon+\mathbb{R}})\,(fge^{-2x^2}(-x^2)^k)dx$,
$\mathbf{C}=\Gamma(1/2-k)2^{-1/2+k}$, $k\in\mathbb{C}$.

Case $k>-1/2$. $<f,g>=$
$=\frac{1}{\Gamma(k+1/2)}\int_{\mathbb{R}}fge^{-2x^2}|x|^{2k}dx$.

Case $k=-1/2-m$. $\{f,g\}=$
$=$ **const** $\mathbf{Res}_0\,(fge^{-2x^2}x^{-2m-1}dx)$.

Radical$=(x^{2m+1}e^{-x^2})$ **is unitary**
w.r.t $\int_{\mathbb{R}}fge^{-2x^2}|x|^{-2m-1}dx$;
$\mathbb{R}[x]/(x^{2m+1})$ **: module over** $osp(2|1)$**.**

E. DAHA

$\mathcal{B}_q \stackrel{\textbf{def}}{=\!=} \langle T, X, Y, q^{1/4}\rangle /$ **relations:**

$$\boldsymbol{TXT = X^{-1},\ TY^{-1}T = Y,}$$
$$\boldsymbol{Y^{-1}X^{-1}YXT^2 \ = \ q^{-1/2}.}$$

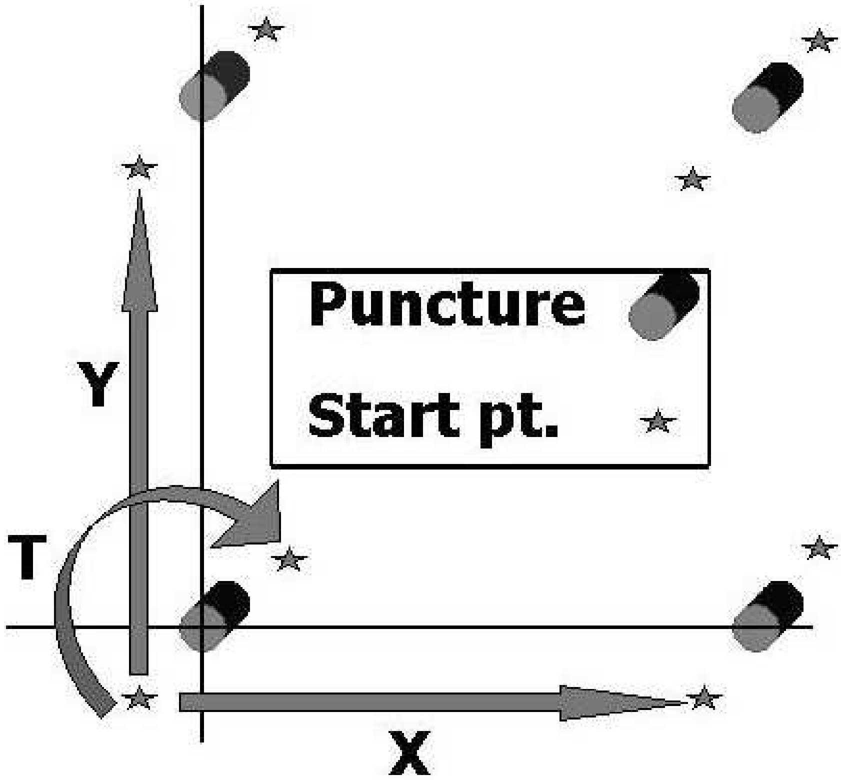
$\mathcal{B}_1 = \pi_1^{\text{orb}}(\{E \setminus 0\}/\mathbf{S}_2) \cong$
$\pi_1(\{E \times E \setminus \mathbf{diag}\}/\mathbf{S}_2)$, $E =$**elliptic curve.**

$\mathcal{H} \stackrel{\textbf{def}}{=\!=} \mathbb{R}[\boldsymbol{\mathcal{B}_q}]/((\boldsymbol{T - t^{1/2}})(\boldsymbol{T + t^{-1/2}}))$,
$q = \exp(-1/a),\ a > 0,\ \ t = q^k,\ k \in \mathbb{R}.$

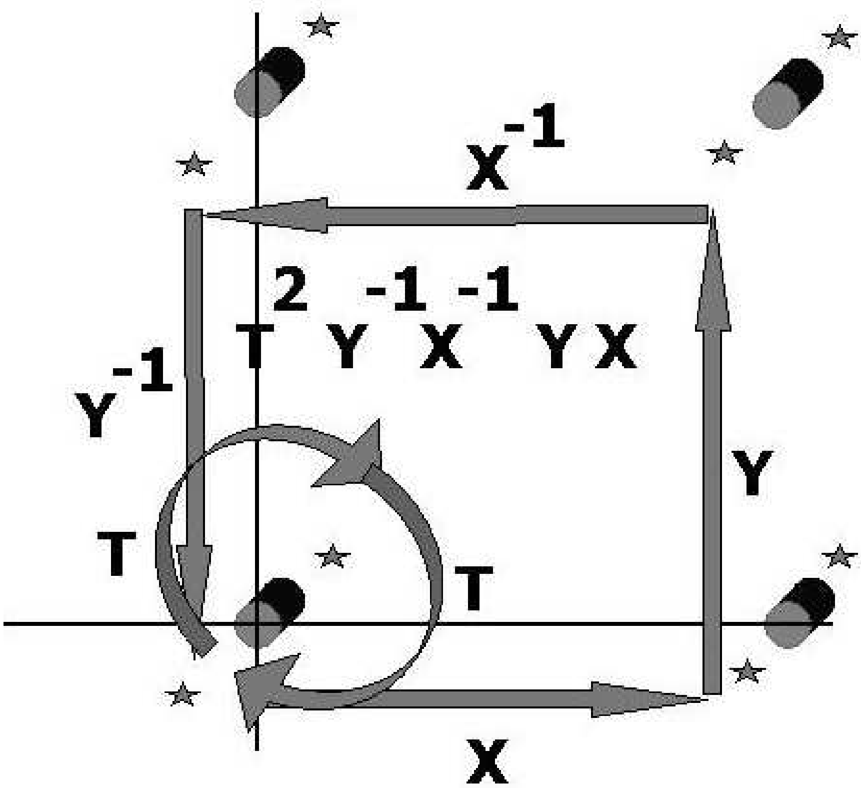
If $t = 1$, $\mathcal{H} =$ **Weyl algebra** $\rtimes \mathbf{S}_2 : T \to s :$

$$sXs = X^{-1},\ sYs = Y^{-1},$$
$$Y^{-1}X^{-1}YX \ = \ q^{-1/2}.$$

Fourier transform $\sim$ **automorphism:**
$X \mapsto Y^{-1},\ \ Y \mapsto T^{-1}X^{-1}T,\ \ T \mapsto T$;
transposes the periods of E.



Relation: $Y^{-1}X^{-1}YXT^2 \;=\; 1.$



$PSL_2(\mathbf{Z})$ **acts projectively**

$$\begin{pmatrix}1 1\\ 0 1\end{pmatrix} : Y \mapsto q^{-1/4}XY,\ X \mapsto X,\ T \mapsto T,$$

$$\begin{pmatrix}1 0\\ 1 1\end{pmatrix} : X \mapsto q^{1/4}YX,\ \ Y \mapsto Y,\ T \mapsto T.$$

$\boldsymbol{X = q^x}$. **The first: conjugation by** $\boldsymbol{q^{x^2}}$.

Polynom. rep. $= Ind(\varrho^{sph})_{\mathcal{H}_Y}^{\mathcal{H}}$ **(PBW):**
$\boldsymbol{\mathcal{L}}$ **= Laurent polynomials of** $\boldsymbol{X = q^x}$**,**

$$\boldsymbol{T \mapsto t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{q^{2x} - 1}(s-1),}$$

$$\boldsymbol{Y \mapsto \pi T, \pi = sp, sf(x) = f(-x),}$$

$$\boldsymbol{pf(x) = f(x+1/2),\ t = q^k.}$$

Y **is the difference Dunkl Operator.**

$Y + Y^{-1}$ **preserves** $\mathcal{L}_{sym} \overset{\textbf{def}}{=\!=}$
sym (even) Laurent polynomials.
$Y + Y^{-1}\,|_{sym}$ **is the** q**-radial part.**

F. INNER PRODUCTS

Macdonald Truncated θ-function: $\mu(x) =$
$= \prod_{i=0}^{\infty} \frac{(1-q^{i+2x})(1-q^{i+1-2x})}{(1-q^{i+k+2x})(1-q^{i+k+1-2x})}$,

$< f, g >_o \overset{\textbf{def}}{=\!=} \frac{1}{i} \int_{1/4+P} f(x)\, T(g)(x)\mu(x)dx,$
$P = [-\pi i a, \pi i a], \;\; q = \exp(-1/a).$

THM. As $k > -1/2$**, it serves** $T^* = T$, $Y^* = Y$, $X^* = TXT^{-1}$, **which is an anti-involution; therefore, symmetric and positive on** $\mathbb{R}[X^{\pm 1}]$**.**

Proof. **a)** **As** $k > -1/2$, μ **has no poles between** $\pm\frac{1}{4} + P$**; this gives** $T^* = T$ **(directly, for the path** $0 + P$**), and also** $Y^* = (\pi T)^* = Y$ **due to** $\pi(\mu) = \mu$ **and therefore** $\pi^* = T^{-1}\pi T$**.**

b) $Y(E_n) = q^{-n_\sharp} E_n, \;\; E_n = X^n + (l.t.),$

$$n_\sharp = \frac{n-k}{2} \;\; \textbf{as} \;\; n \le 0, \;\; = \frac{n+k}{2} \;\; \textbf{as} \;\; n > 0.$$

c) $< E_n, E_m >_o = C_n \delta_{nm}$ **due to** $Y^* = Y$**.**

d) $C_n = q^{-n_\sharp} \int_{1/4+P} E_n \overline{E_n} \mu(x)dx > 0$, **since** $\pi(x) = \overline{x}$ **and** $\mu(x) > 0$ **at** $1/4+P$**; use** $T(E_n) = \pi Y(E_n) = q^{-n_\sharp}\pi(E_n) = q^{-n_\sharp}\overline{E_n}$.$\square$

Imaginary Integration. **On** $\mathbb{R}[X^{\pm 1}]$,

$$< f, g >^{\gamma}_{\infty} \overset{\textbf{def}}{=\!=} \frac{1}{i}\int_{\frac{1}{4}+i\mathbb{R}} fT(g)q^{-x^2}\mu(x)dx$$

$$= \frac{1}{i}\int_{\frac{1}{4}+P} fT(g)\sum_{j=-\infty}^{\infty} q^{n^2/4+nx}\mu(x)dx$$

is a symmetric and positive form, serving

$T^{\varkappa} = T$, $X^{\varkappa} = TXT^{-1}$, $Y^{\varkappa} = q^{-1/4}XY$.

G. ANALYTIC CONTINUATION

The anti-involution $\varkappa$ is generic:

$$\varrho\Big(\sum_{a,b\in\mathbb{Z}}^{\epsilon=0,1} c_{a\epsilon b}(Y^{\varkappa})^a\, T^{\epsilon}\, Y^b\Big) \overset{\textbf{def}}{=\!=} \sum c_{a\epsilon b} t^{\frac{a+\epsilon+b}{2}},$$

$$\{A, B\}^{\varrho}_{\varkappa} = \varrho(A^{\varkappa}B) = \{B, A\}^{\varrho}_{\varkappa} \text{ on } \mathcal{H}.$$

It acts via $\mathcal{L}\times\mathcal{L}$, $\mathcal{L} = \mathbb{R}[X^{\pm 1}]$, **and** $\{1, 1\} = 1$.

This form is analytic for all $k \in \mathbb{C}$.

THM. $G(k)\{f, g\}^{\varrho}_{\varkappa} =< f, g >^{\gamma}_{\infty}$,

$$G(k) = \sqrt{\pi a}\prod_{j=1}^{\infty}\frac{1-q^{k+j}}{1-q^{2k+j}},\quad \Re k > -1/2.$$

Proof. Generally, let $C = \{\epsilon + i\mathbb{R}\}$ **and**

$$\Phi^k_{\epsilon}(f, g) \overset{\textbf{def}}{=\!=} \frac{1}{i}\int_{\epsilon+i\mathbb{R}} fT(g)q^{-x^2}\mu(x)dx.$$

Bad $k = \{\, 2C - 1 - \mathbb{Z}_+,\ -2C - \mathbb{Z}_+ \,\}$;

so $\{\Re k > -1/2\}$ **are good as** $\epsilon = -1/4$.

Case $\epsilon = 0$. $\Phi_0^k(f,g)$ **gives** $G(k)\{f,g\}_\varkappa^\varrho$ **for** $\Re k > 0$ **only!**
However it is symmetric for all k and $T^\varkappa = T,\ X^\varkappa = TXT^{-1}$.

Relation: $\Phi_{1/4}^k = \Phi_0^k + A(-k)\,\Pi(-k)\,F(-k)$,

$$A(\widetilde{k}) = \frac{\sqrt{\pi a}}{2}\sum_{m=-\infty}^{\infty} q^{m^2+m\widetilde{k}},$$

$$\Pi(\widetilde{k}) = \prod_{j=0}^{\infty}\frac{(1-q^{\widetilde{k}+j})(1-q^{-\widetilde{k}+j+1})}{(1-q^{1+j})(1-q^{-2\widetilde{k}+j+1})},$$

$$F(\widetilde{k}) = fT(g)(x \mapsto \widetilde{k}/2).$$

Here $\Phi_0^k + A(-k)\Pi(-k)F(-k)$ is analytic and therefore coincides with $G(k)\{f,g\}_\varkappa^\varrho$ as $\Re k > -1$. It is also symmetric for ALL k:

$$fT(g)(-\tfrac{k}{2}) = t^{1/2}fg(-\tfrac{k}{2}) = T(f)g(-\tfrac{k}{2}).$$

since $T = \frac{q^{2x+k/2}-q^{-k/2}}{q^{2x}-1}\, s - \frac{q^{k/2}-q^{-k/2}}{q^{2x}-1}$,
where $(q^{2x+k/2} - q^{-k/2})(x \mapsto -k/2) = 0$.

MAIN THM. $G(k)\{f,g\}_\varkappa^\varrho = \Phi_0^k +$
$+\sum_{\widetilde{k}\in\widetilde{K}} A(\widetilde{k})\,\Pi(\widetilde{k})\,F(\widetilde{k})$ **as** $\Re k < 0$,
$\widetilde{K} = \{-k\} \cup \{-k-j, k+j\}$ **for** $j = 1,\ldots,[\Re(-k)]$; $[\cdot]$ =**integer part.**

Note. $\widetilde{K}$ **must be symmetric but at** $-k$**.**

COR. $\{f,g\}^{\varrho}_{\varkappa}$ **is degenerate exactly at the poles of** $G(k)\ :\ k=-1/2-m,\ m\in\mathbb{Z}_+$**.** **Then** $\widetilde{K/2}$ **is** π**-invariant and Funct**$(\widetilde{K/2})\ =\ \mathcal{L}/$**Radical**$\{\ ,\ \}$ **is an** $\mathcal{H\!H}$**-module of dim**$=2m+1$.

The radical is unitary with respect to Φ^k_0**.**

Rational Limit: $q=e^h,\ \ h\to 0$,
$Y=e^{-\sqrt{h}y},\ \ X=e^{\sqrt{h}x}$.
Rational DAHA$=\lim \mathcal{H\!H}$.

H. P-ADIC LIMIT

GENERAL THEOREM. For generic $\varkappa$**, the canonical trace/form is a sum of integrals over affine residual subtori.**

As $q\to\infty$ **(renoramlization is needed), the Fourier transform of** $\mathcal{L}$ **tends to the spherical part of the affine module** $\mathcal{H}$**.**
Only non-affine residual subtori contribute.
The radical filtration (small $\Re k<0$**) becomes a filtration of** $\mathcal{H}$**-modules (!).**

I. JANTZEN FILTRATIONS

INGREDIENTS: a) a PBW-generic anti-involution $\varkappa$ of $\mathcal{HH}$ (w.r.t. the subalgebra Y),

b) the canonical trace associated with $\varkappa$ (requires a character ϱ on Y and the trace of the non-affine Hecke algebra),

c) the "Shapovalov form", a combination of (a) and (b) with $< 1, 1 >= 1$ (k-analytic).

PROBLEM: "expand" the canonical trace and the Shapovalov form $\{f, g\}_{\varkappa}^{\varrho}$ as a sum of integrals over the affine residual subtori.

Analytic Jantzen firtration of $\mathcal{L}$ (AHA (!) but not DAHA modules) for $\Re k < 0$. The top/first term is the sum over the smallest tori, the bottom/last term is the integration over $i\mathbb{R}^n$.

Algebraic Jantzen filtration of $\mathcal{L}$ is in terms of DAHA (!) modules at singular k_o w.r.t. $\widetilde{k} = k - k_o$. The top form is the coefficient of $\widetilde{k}^0$, the bottom term is (in known examples) the straight integration over $i\mathbb{R}^n$.

Now restrict the m-th form to the radical of the $(m-1)$-th form and consider its radical, continue. The construction gives the total irreducible decomposition of $\mathcal{L}$ for A_n ("Kasatani conjecture" proven via the rational limit).

J. RATIONAL CASE

Represent $\{f,g\}^{\varrho}_{\varkappa}$ as an integral over the boundary of the tube neighborhood of the resolution of the cross $\prod x_\alpha = 0$ extended to ∞, e.g., over $\pm i\epsilon + \mathbb{R}$ for A_1. Resolution: Ch, de Concini - Procesi, Beilinson-Ginzburg.

Conjecturally: singular k_o are the k when this integral can be reduced to integrations "over smaller subtori" (e.g., over a compact integration counter in case of the point).

Conjecturally: for singular $k_o = s/d_i$, the bottom module is "semi-simple" (à la Suzuki for A_n). It may be always unitary for $s=1$ with respect to the "pure" $\int_{\mathbb{R}^n}\{\cdot\}e^{-x^2}\,dx$ (as for A_1; Etingof and his students for A_n).

A relation to singularities à la Shokurov.