Lemma. Let $\chi_1, \chi_2 \in P$. The following conditions are equivalent:
(i) $f_{\chi_1} = f_{\chi_2}$;
(ii) $f_{\chi_1}(r_i) = f_{\chi_2}(r_i)$ for $i = 1, \ldots, n$;
(iii) $\bar{\chi}_1 = \bar{\chi}_2$.

Proof. The equivalence of (i) and (ii) follows from 6.1 (a), and the equivalence of (ii) and (iii) follows at once from Definition 5.8.

6.3. Proposition. Let $\mathfrak{g}$ be a classical Lie algebra. Then the restriction $w \mapsto w|_F$ establishes the isomorphism between W and the group of linear transformations of F generated by $r_1|_F, r_2|_F, \ldots, r_n|_F$. We shall always identify W with this group via this isomorphism.

Proof. Obviously, our mapping is an epimorphism. It remains to verify that $w|_F = 1$ implies that $w = 1$. Let $\chi \in P$. By 1.12, there is a $\varphi \in F$ such that $\bar{\chi} = \bar{\varphi}$. Using Lemma 6.2, we see that

$$\chi - w\chi = f_\chi(w) = f_\varphi(w) = \varphi - w\varphi = 0.$$

Since χ is arbitrary, it follows that $w = 1$.

6.4. Corollary. The Weyl group of a classical Lie algebra is finite.

Proof. An element of W is uniquely determined by its action on the finite set R.

6.5. According to 5.9 (a), 5.20 (a) and 6.3, the Weyl group W of a classical Lie algebra can be thought of as a group of orthogonal transformations of the Euclidean vector space $F_{\mathbf{R}}$. This is the most visual way of representing W.

Let us introduce some terminology related to Euclidean geometry. Let E be an Euclidean (real) vector space with the inner product $\langle \, , \rangle$. For any non-zero $\alpha \in E$ we denote by r_α the (orthogonal) reflection of E in the hyperplane orthogonal to α; we put $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. Then $(\alpha^\vee)^\vee = \alpha$, $\langle \alpha, \alpha^\vee \rangle = 2$, $\langle \alpha, \alpha \rangle \cdot \langle \alpha^\vee, \alpha^\vee \rangle = 4$, and $r_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha = x - \langle x, \alpha \rangle \cdot \alpha^\vee$ $(x \in E)$.

Evidently, if $w: E \to E$ is an orthogonal transformation then

$$w \circ r_\alpha \circ w^{-1} = r_{w(\alpha)}. \qquad (*)$$

Now, Definition 5.8 shows that $r_i: F_{\mathbf{R}} \to F_{\mathbf{R}}$ is just the reflection r_{α_i} for $i = 1, \ldots, n$, i.e., W can be defined as a group of orthogonal transformations of F generated by reflections $r_{\alpha_1}, \ldots, r_{\alpha_n}$. More generally:

Proposition. $r_\alpha \in W$ for all $\alpha \in R$.

Proof. By $(*)$, it suffices to prove that each root $\alpha \in R$ is W-conjugate to a simple root. This follows at once from 5.20 (a) and the next general

6.6. Lemma. Let $\mathfrak{g}$ be a Kac-Moody algebra and $\alpha \in R$.
Then the following conditions are equivalent:
(i) α is W-conjugate to a simple root;
(ii) $\langle \alpha, \alpha \rangle > 0$.

The roots satisfying these conditions are called *real* and other roots are called *imaginary*.

Proof. Since $\langle \alpha_i, \alpha_i \rangle > 0$ and $\langle \, , \rangle$ is W-invariant, it follows that (i)⇒(ii). Let now $\langle \alpha, \alpha \rangle > 0$. Without loss of generality we can assume that $\alpha \in R_+$. Choose $w \in W$ so that $w\alpha \in R_+$ and in the expression $w\alpha = \sum_i k_i \alpha_i$ $(k_i \in \mathbf{Z}^+)$ the sum Σk_i is minimal. Then $w\alpha$ is simple. Indeed, if $w\alpha \neq \alpha_i$ for all i, then $r_i w\alpha = w\alpha - \overline{w\alpha}(h_i)\alpha_i$ is positive (see 5.9 (d)), so our choice of w implies that $\langle w\alpha, \alpha_i \rangle = d_i^{-1}\, \overline{w\alpha}(h_i) \leqq 0$ for all i. But then

$$\langle \alpha, \alpha \rangle = \langle w\alpha, w\alpha \rangle = \left\langle w\alpha, \sum_i k_i \alpha_i \right\rangle = \sum_i k_i \langle w\alpha, \alpha_i \rangle \leqq 0,$$

which is a contradiction.

6.7. Let $\mathfrak{g}$ be a classical Lie algebra. We shall rewrite the character and denominator formulas for $\mathfrak{g}$ (see 5.11, 5.12) in more convenient terms.

From now on we shall regard the formal exponentials e^λ $(\lambda \in F)$ as functions on F, defined by $e^\lambda(\varphi) = \exp(-2\pi i \langle \lambda, \varphi \rangle)$. For $\lambda \in F$ we put

$$J_\lambda = \sum_{w \in W} \det w \, e^{w\lambda}$$

(by 6.4, the sum is finite).

Define $\rho_0 \in F$ to be

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

Lemma. $\rho - w\rho = \rho_0 - w\rho_0$ and $\langle \rho, \lambda \rangle = \langle \rho_0, \lambda \rangle$ for all $w \in W$ and $\lambda \in F$.

Proof. By Lemma 6.2, it suffices to verify that $\rho - r_i\rho = \rho_0 - r_i\rho_0$ for $i = 1, \ldots, n$. By definitions 4.4 and 5.8 we have $\rho - r_i\rho = \alpha_i$. On the other hand, $r_i\rho_0 = r_i\left(\frac{\alpha_i}{2} + \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\alpha_i\}} \alpha\right) = -\frac{\alpha_i}{2} + \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\alpha_i\}} \alpha = \rho_0 - \alpha_i$, as required (see 5.9 (d)).

Using this Lemma, we rewrite the formula 5.12 as follows:

$$\operatorname{ch} L(\varphi) = J_{\varphi + \rho_0}/J_{\rho_0} \quad \text{for} \quad \varphi \in P^0_+ \quad \text{(see 5.21)}. \qquad (*)$$

By 1.11 $(*)$, $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$, so the formula 5.11 can be rewritten as follows:

$$J_{\rho_0} = e^{\rho_0} \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}). \qquad (**)$$

6.8. Proof of Proposition 5.22. Under the notation of 6.7, we have $\dim L(\varphi) = \operatorname{ch} L(\varphi)(0)$.

Lemma. If $\varphi\in F$ and $t\in\mathbf{C}$ then

$$\frac{J_{\varphi+\rho_0}(t\rho_0)}{J_{\rho_0}(t\rho_0)}=\prod_{\alpha\in R^+}\frac{\sin\pi t\langle\alpha,\varphi+\rho_0\rangle}{\sin\pi t\langle\alpha,\rho_0\rangle}.$$

Proof. The definition of J_λ readily implies that $J_\lambda(t\mu)=J_\mu(t\lambda)$ for all $\lambda,\mu\in F$, $t\in\mathbf{C}$. It follows that

$$J_{\varphi+\rho_0}(t\rho_0)=J_{\rho_0}(t(\varphi+\rho_0))=\prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/2})(t(\varphi+\rho_0))=$$

$$=\prod_{\alpha\in R^+}(-2i\sin\pi t\langle\alpha,\varphi+\rho_0\rangle).$$

Therefore,

$$J_{\varphi+\rho_0}(t\rho_0)/J_{\rho_0}(t\rho_0)=\prod_{\alpha\in R^+}\frac{\sin\pi t\langle\alpha,\varphi+\rho_0\rangle}{\sin\pi t\langle\alpha,\rho_0\rangle},$$

as claimed.

Using this Lemma and Lemma 6.7, we see that

$$\dim L(\varphi)=\lim_{t\to0}\operatorname{ch}L(\varphi)(t\rho_0)=\lim_{t\to0}\prod_{\alpha\in R^+}\frac{\sin\pi t\langle\alpha,\varphi+\rho_0\rangle}{\sin\pi t\langle\alpha,\rho_0\rangle}=$$

$$=\prod_{\alpha\in R^+}\frac{\langle\alpha,\varphi+\rho_0\rangle}{\langle\alpha,\rho_0\rangle}=\prod_{\alpha\in R^+}\frac{\langle\alpha,\varphi+\rho\rangle}{\langle\alpha,\rho\rangle},$$

as required.

6.9. Before treating the case of affine Lie algebras, we collect together some properties of the Euclidean space $F_{\mathbf{R}}$ which will be used later.

Denote by P_0 the lattice of integral weights in F (see 5.6). Clearly, $P_0=\mathbf{Z}\omega_1\oplus\ldots\oplus\mathbf{Z}\omega_n$ where the ω_i are the *fundamental weights* of $\mathfrak{g}$ defined by $\bar\omega_i(h_j)=\delta_{ij}$.

We choose the form $(\,,)$ on $\mathfrak{g}$ to be the Killing form and let $\langle\,,\rangle$ be the corresponding form on F (see 4.1, 4.4). We define the lattice $Q^\vee$ by

$$Q^\vee=\mathbf{Z}\alpha_1^\vee\oplus\mathbf{Z}\alpha_2^\vee\oplus\ldots\oplus\mathbf{Z}\alpha_n^\vee \qquad \text{(see 6.5).}$$

Proposition. (a) $\langle\varphi,\psi\rangle=2\sum_{\alpha\in R^+}\langle\varphi,\alpha\rangle\cdot\langle\psi,\alpha\rangle$ for all $\varphi,\psi\in F$.

(b) $\bar\varphi(h_i)=\langle\varphi,\alpha_i^\vee\rangle$ for $\varphi\in F$, $i=1,\ldots,n$. In particular,

$$P_0=\{\varphi\in F\,|\,\langle\varphi,\alpha_i^\vee\rangle\in\mathbf{Z}\quad\text{for}\quad i=1,\ldots,n\}.$$

(c) Each of the lattices P_0, Q and $Q^\vee$ is W-invariant. P_0 contains each of the lattices Q and $\frac{1}{2}Q^\vee$.

(d) $\alpha^\vee\in Q^\vee$ for any $\alpha\in R$. In particular, if γ is the maximal root then $\gamma^\vee=\sum_i k_i\alpha_i^\vee$, where $k_i\in\mathbf{N}$ for $i=1,\ldots,n$; we put $g=1+\sum_i k_i\in\mathbf{N}$.

(e) If $\alpha\in R_+\setminus\{\gamma\}$ then $\langle\alpha,\gamma^\vee\rangle$ equals either 0, or 1.

(f) $\langle\gamma^\vee,\gamma^\vee\rangle=4g$, $\langle\gamma,\gamma\rangle=1/g$, $\langle\rho_0,\gamma^\vee\rangle=g-1$ and $\langle\rho_0,\gamma\rangle=(g-1)/2g$.

(g) (The "strange" formula of Freudental-de Vries)

$$\langle\rho_0,\rho_0\rangle=\dim\mathfrak{g}/24.$$

Proof. (a) By definitions,

$$\langle\varphi,\psi\rangle=(h_\varphi,h_\psi)=\operatorname{tr}(\operatorname{ad}h_\varphi\circ\operatorname{ad}h_\psi)=2\sum_{\alpha\in R^+}\bar\alpha(h_\varphi)\,\bar\alpha(h_\psi)=$$

$$=2\sum_{\alpha\in R^+}\langle\alpha,\varphi\rangle\cdot\langle\alpha,\psi\rangle.$$

(b) Definitions readily imply that $h_{\alpha_i^\vee}=h_i$ for $i=1,\ldots,n$ (see 4.1, 4.2). This immediately implies our assertion.

(c) By (b), $\langle\alpha_j,\alpha_i^\vee\rangle=a_{ij}\in\mathbf{Z}$, where $A=(a_{ij})$ is the Cartan matrix of $\mathfrak{g}$. Therefore, $Q\subset P_0$.

By (a),

$$\left\langle\frac{1}{2}\alpha_j^\vee,\alpha_i^\vee\right\rangle=\sum_{\alpha\in R^+}\langle\alpha_j^\vee,\alpha\rangle\langle\alpha_i^\vee,\alpha\rangle\in\mathbf{Z},$$

so $\frac{1}{2}Q^\vee\subset P_0$.

The W-invariance of P_0 and Q follows at once from 5.14. By (b), $Q^\vee=\{\varphi\in F\,|\,\langle\varphi,\lambda\rangle\in\mathbf{Z}$ for all $\lambda\in P_0\}$. Since $\langle\,,\rangle$ is W-invariant, it follows that $Q^\vee$ is W-invariant.

(d) By (b), if $\varphi=\sum_i k_i\alpha_i^\vee\in F$ then the coefficients k_i are given by $k_i=\langle\omega_i,\varphi\rangle=\bar\omega_i(h_\varphi)$. Therefore, it suffices to prove that $\bar\omega_i(h_{\alpha^\vee})\in\mathbf{Z}$ for all $\alpha\in R$ and $i=1,\ldots,n$.

By Prop. 4.3, $h_{\alpha^\vee}\in[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$, and we have $\bar\alpha(h_{\alpha^\vee})=\langle\alpha,\alpha^\vee\rangle=2$. It follows that if elements $e_\alpha\in\mathfrak{g}_\alpha$, $f_\alpha\in\mathfrak{g}_{-\alpha}$ are chosen so that $[e_\alpha,f_\alpha]=h_{\alpha^\vee}$, then $h_{\alpha^\vee}$, e_α and f_α form the standard basis of the Lie algebra isomorphic to $\mathfrak{sl}_2$. Denote this algebra by $\mathfrak{sl}_2(\alpha)$.

Now consider the finite-dimensional $\mathfrak{g}$-module $L(\omega_i)$ (see 5.21) as a module over $\mathfrak{sl}_2(\alpha)$. Then $\bar\omega_i(h_{\alpha^\vee})$ is its weight, hence belongs to $\mathbf{Z}$, by Cor. 3.4.

(e) Consider $\mathfrak{g}$ as a module over the algebra $\mathfrak{sl}_2(\gamma)$ just constructed. Let $\alpha\in R_+\setminus\{\gamma\}$ and $v\in\mathfrak{g}_\alpha$; then v has a weight $\langle\alpha,\gamma^\vee\rangle\in\mathbf{Z}$ w.r.t. $\mathfrak{sl}_2(\gamma)$. Since γ is maximal, it follows that $\alpha+\gamma$ and $\alpha-2\gamma$ are not roots. Therefore, $\operatorname{ad}e_\gamma(v)=(\operatorname{ad}f_\gamma)^2(v)=0$, and our assertion follows from Corollary 3.4.

(f) The definition of g (see (d)) implies that

$$\langle\rho,\gamma^\vee\rangle=\langle\rho,\sum_i k_i\alpha_i^\vee\rangle=\Sigma k_i=g-1.$$

By Lemma 6.7,

$$g-1=\langle\rho_0,\gamma^\vee\rangle=\frac{1}{2}\sum_{\alpha\in R^+}\langle\alpha,\gamma^\vee\rangle=1+\frac{1}{2}\sum_{\alpha\in R^+\setminus\{\gamma\}}\langle\alpha,\gamma^\vee\rangle,$$

so

$$\sum_{\alpha\in R^+\setminus\{\gamma\}}\langle\alpha,\gamma^\vee\rangle=2(g-2).$$

This formula together with (a) and (e) implies that

$$\langle\gamma^\vee,\gamma^\vee\rangle=2\sum_{\alpha\in R^+}\langle\alpha,\gamma^\vee\rangle^2=8+2\sum_{\alpha\in R^+\setminus\{\gamma\}}\langle\alpha,\gamma^\vee\rangle=8+4(g-2)=4g.$$

Therefore, $\langle\gamma,\gamma\rangle=4/\langle\gamma^\vee,\gamma^\vee\rangle=1/g$, and $\langle\rho_0,\gamma\rangle=2\langle\rho_0,\gamma^\vee\rangle/\langle\gamma^\vee,\gamma^\vee\rangle=(g-1)/2g$, as claimed.

(g) Taking in Lemma 6.8 the second term in the Taylor expansion, one easily obtains $\mathrm{ch}\, L(\varphi)(t\rho_0)=\dim L(\varphi)\left\{1-\frac{\pi^2t^2}{6}\sum_{\alpha\in R^+}(\langle\alpha,\ \varphi+\rho_0\rangle^2-\langle\alpha,\ \rho_0\rangle^2)+o(t^2)\right\}$.
Using (a), this can be rewritten as

$$\mathrm{ch}\, L(\varphi)(t\rho_0)=\dim L(\varphi)\left\{1-\frac{\pi^2t^2}{12}(\langle\varphi+\rho_0,\varphi+\rho_0\rangle-\langle\rho_0,\rho_0\rangle)+o(t^2)\right\}. \quad (*)$$

Consider the case $\varphi=\gamma$, i.e., $L(\varphi)$ is the adjoint representation of $\mathfrak{g}$ (see 2.9). By definition,

$$\mathrm{ch}\,\mathfrak{g}=n\cdot e^0+\sum_{\alpha\in R^+}(e^\alpha+e^{-\alpha}),$$

so

$$\mathrm{ch}\,\mathfrak{g}(t\rho_0)=n+2\sum_{\alpha\in R^+}\cos 2\pi t\langle\alpha,\rho_0\rangle=\dim\mathfrak{g}-4\pi^2t^2\sum_{\alpha\in R^+}\langle\alpha,\rho_0\rangle^2+$$
$$+o(t^2)=\dim\mathfrak{g}-2\pi^2t^2\langle\rho_0,\rho_0\rangle+o(t^2).$$

Comparing this with $(*)$, we have

$$\langle\rho_0,\rho_0\rangle=\frac{\dim\mathfrak{g}}{24}(\langle\gamma+\rho_0,\gamma+\rho_0\rangle-\langle\rho_0,\rho_0\rangle).$$

But $\langle\gamma+\rho_0,\gamma+\rho_0\rangle-\langle\rho_0,\rho_0\rangle=\langle\gamma+2\rho_0,\gamma\rangle=1/g+(g-1)/g=1$ by (f), and our assertion follows.

6.10. Now we shall study the Weyl group of an affine Lie algebra. Let $\tilde{\mathfrak{g}}$ be the affine algebra of rank $n+1$ corresponding to a classical Lie algebra $\mathfrak{g}$ (see 1.14, 1.16).

We retain all our notation related to $\mathfrak{g}$ and use the subscript a for all notions related to $\tilde{\mathfrak{g}}$. Thus, the Weyl group of $\tilde{\mathfrak{g}}$ is denoted by W_a; by definition, it acts on the weight space $P_a=\mathfrak{h}_a^*\oplus F_a$. Recall that $\mathfrak{h}_a=\mathfrak{h}\oplus\mathbf{C}h_0=\mathfrak{h}\oplus\mathbf{C}c$, where

$$h_0=\frac{1}{2}(h_\gamma^\vee,h_\gamma^\vee)\cdot c-h_\gamma^\vee \quad (*)$$

(see 1.16), and $F_a=F\oplus\mathbf{C}\alpha_0$.

We choose the form (,) on $\tilde{\mathfrak{g}}$ as indicated in 4.1. By definition, $(c,h)=0$ for all $h\in\mathfrak{h}_a$, and the restriction of (,) on $\mathfrak{h}$ is just the form on $\mathfrak{h}$ considered above. Recall that this form gives rise to the isomorphism $\alpha\mapsto h_\alpha$ between F_a and $\mathfrak{h}_a$ and to the form $\langle\,,\rangle$ on F_a such that

$$\langle\alpha,\beta\rangle=\bar{\alpha}(h_\beta)=(h_\alpha,h_\beta) \qquad \text{for} \quad \alpha,\beta\in F_a$$

(see 4.2).

The definitions readily imply that $h_\gamma^\vee=h_{\gamma^\vee}$ (see 1.15 and 6.5). By $(*)$, $(h_0,h_0)==(h_\gamma^\vee,h_\gamma^\vee)$, so we can rewrite $(*)$ as

$$c=h_{\alpha_0}+h_\gamma=h_{\alpha_0+\gamma}.$$

We put $\delta=\alpha_0+\gamma$.

Proposition. (a) $\langle\delta,\varphi\rangle=0$ for all $\varphi\in F_a$, and the restriction of $\langle\,,\rangle$ on F is the form on F considered above.

(b) $w\delta=\delta$ for all $w\in W_a$.

Proof. (a) We have $\langle\varphi,\delta\rangle=(h_\varphi,h_\delta)=(h_\varphi,c)=0$.

(b) Follows at once from (a) and Definition 5.8.

6.11. In the next point we shall verify Prop. 6.3 for affine algebras. To do this we give a more convenient realization of F_a, viz., we shall represent elements of F_a as affine-linear functions on F.

We have $F_a=F\oplus\mathbf{C}\cdot\delta$. Consider the dual space $F_a^*=F^*\oplus\mathbf{C}\delta^*$, where $\delta^*|_F=0$ and $\delta^*(\delta)=1$, and let E be the affine hyperplane $F^*+\delta^*$ in F_a^*. Each element of F_a can be thought of as a linear function on F_a^* or as an affine-linear function on E; obviously, every affine-linear function on E has such a form.

The group W_a acts on F_a and hence on F_a^* in such a way that

$$(wf)(\varphi)=f(w^{-1}\varphi) \qquad \text{for} \quad f\in F_a,\quad \varphi\in F_a^*.$$

Proposition. The affine hyperplane $E\subset F_a^*$ is W_a-invariant. We shall identify E with $F^*(\varphi+\delta^*\leftrightarrows\varphi)$ and then F^* with F by means of the form $\langle\,,\rangle$. Under these identifications, W_a acts on F by affine transformations, and this action of the generators r_i is given by

$$(*) \qquad \begin{cases} r_i\varphi=\varphi-\langle\varphi,\alpha_i\rangle\alpha_i^\vee & (i=1,\dots,n),\\ r_0\varphi=\varphi-(\langle\varphi,\gamma\rangle-1)\gamma^\vee, & \text{where} \quad \varphi\in F.\end{cases}$$

Proof. The fact that E is W_a-invariant, follows at once from 6.10 (b). The proof of $(*)$ is straightforward, and we leave it to the reader.

6.12. Proposition. The homomorphism of W_a to the group of affine transformations of F constructed above, is an imbedding.

Proof. Let $\chi \in P_a = \mathfrak{h}_a^* \oplus F_a$ and $f_\chi \in C^1(W_a, F_a)$ the corresponding 1-cocycle (see 6.2). As in 6.11, we identify F_a with the space of affine-linear functions on F. We must prove that $w|_F = 1$ implies that $w\chi = \chi$, i.e., that $f_\chi(w)(\varphi) = 0$ for all $\varphi \in F$ (cf. 6.3). This follows at once from the next

6.13. Lemma. For any $\chi \in P_a$, $w \in W_a$ and $\varphi \in F$ the value of $f_\chi(w)$ at φ is equal to $B(\varphi) - B(w^{-1}\varphi)$, where B is the quadratic function on F defined by

$$B(\varphi) = -\frac{\bar{\chi}(c)}{2}\langle \varphi, \varphi \rangle + \langle \chi, \varphi \rangle.$$

Proof. It easily follows from 6.11 $(*)$ that the linear part of the affine transformation $w: F \to F$ is orthogonal for any $w \in W_a$. Therefore, $B(\varphi)$ and $B(w^{-1}\varphi)$ have the same quadratic parts, hence $(B(\varphi) - B(w^{-1}\varphi))$ is an affine-linear function of φ, i.e. belongs to F_a. By 6.1 (b), the function $f_B: W_a \to F_a$ defined by

$$f_B(w)(\varphi) = B(\varphi) - B(w^{-1}\varphi)$$

is a 1-cocycle. Using 6.1 (a), it remains only to verify that $f_\chi(r_i) = f_B(r_i)$ for $i = 0, 1, \ldots, n$.

By definition, $f_\chi(r_i) = \bar{\chi}(h_i)\alpha_i$ for $i = 0, 1, \ldots, n$.

Remembering 6.10 $(*)$, we see that

$$f_\chi(r_i)(\varphi) = \langle \chi, \alpha_i^\vee \rangle \cdot \langle \varphi, \alpha_i \rangle \qquad \text{for} \quad i = 1, \ldots, n,$$

and

$$f_\chi(r_0)(\varphi) = \left(\frac{\bar{\chi}(c)}{2}\langle \gamma^\vee, \gamma^\vee \rangle - \langle \chi, \gamma^\vee \rangle\right)(1 - \langle \gamma, \varphi \rangle).$$

On the other hand, if $i = 1, \ldots, n$ then by 6.11 $(*)$,

$$f_B(r_i)(\varphi) = B(\varphi) - B(r_i\varphi) = \langle \chi, \varphi \rangle - \langle \chi, r_i\varphi \rangle =$$
$$= \langle \varphi, \alpha_i \rangle \cdot \langle \chi, \alpha_i^\vee \rangle = f_\chi(r_i)(\varphi).$$

For $i = 0$ we have

$$f_B(r_0)(\varphi) = B(\varphi) - B(r_0\varphi) = \frac{\bar{\chi}(c)}{2}(\|\varphi - (\langle \varphi, \gamma \rangle - 1)\gamma^\vee\|^2 -$$
$$- \|\varphi\|^2) - \langle \chi, \gamma^\vee \rangle (1 - \langle \gamma, \varphi \rangle) = (1 - \langle \gamma, \varphi \rangle) \cdot$$
$$\cdot \left(\frac{\bar{\chi}(c)}{2}\{2\langle \gamma^\vee, \varphi \rangle + (1 - \langle \gamma, \varphi \rangle) \cdot \langle \gamma^\vee, \gamma^\vee \rangle\} - \langle \chi, \gamma^\vee \rangle\right).$$

Since $\gamma = 2\gamma^\vee / \langle \gamma^\vee, \gamma^\vee \rangle$, it follows that the coefficient of $\bar{\chi}(c)/2$ is equal to $\langle \gamma^\vee, \gamma^\vee \rangle$, so $f_B(r_0)(\varphi) = f_\chi(r_0)(\varphi)$, as required.

6.14. According to Prop. 6.12, we shall identify W_a with the group of affine transformations of F generated by $r_0, r_1, \ldots, r_n$ given by 6.11 $(*)$.

In 6.11, we identified F_a with the space of affine-linear functions on F. Recall from 1.16 that the set of roots of $\tilde{\mathfrak{g}}$ is $\{\alpha + k\delta \mid \alpha \in R \cup \{0\}, k \in \mathbf{Z}, (\alpha, k) \neq (0, 0)\}$; roots of $\tilde{\mathfrak{g}}$ will be called *affine roots*. Clearly, $\alpha + k\delta$ is the following function on F:

$$\varphi \mapsto \langle \alpha, \varphi \rangle + k.$$

Now we shall give a geometrical description of W_a similar to that of W given in 6.5. Evidently, W_a preserves the subspace $F_{\mathbf{R}} \subset F$, and each root has real values on $F_{\mathbf{R}}$. For any affine-linear function $a = \alpha + k\delta$, where $\alpha \in F_{\mathbf{R}}$ is non-zero and $k \in \mathbf{R}$, denote by r_a the (orthogonal) reflection of $F_{\mathbf{R}}$ in the affine hyperplane $a(\varphi) = 0$. Clearly,

$$r_a(\varphi) = \varphi - a(\varphi)\alpha^\vee = \varphi - (\langle \alpha, \varphi \rangle + k)\alpha^\vee$$

(cf. 6.5).

Evidently, if w: $F_{\mathbf{R}} \to F_{\mathbf{R}}$ is an affine isometry then

$$w \circ r_a \circ w^{-1} = r_{w(a)} \tag{$*$}$$

(cf. 6.5 $(*)$).

Formulas 6.11 $(*)$ show that $r_i: F_{\mathbf{R}} \to F_{\mathbf{R}}$ is just the reflection r_{α_i} for $i = 0, 1, \ldots, n$, i.e., W_a can be defined as a group of affine isometries of $F_{\mathbf{R}}$ generated by reflections $r_{\alpha_0}, r_{\alpha_1}, \ldots, r_{\alpha_n}$. Similarly to Prop. 6.5, we have:

Proposition. If $a = \alpha + k\delta$ is an affine root and $\alpha \neq 0$ then $r_a \in W_a$.

Proof. By 6.10 (a), $\langle a, a \rangle = \langle \alpha, \alpha \rangle$. Therefore, a root $\alpha + k\delta$ is real if and only if $\alpha \neq 0$ (see 6.6). By Lemma 6.6, every real root a is W_a-conjugate to a simple root, so our assertion follows from $(*)$.

6.15. For any $\lambda \in F_{\mathbf{R}}$ we denote by $t(\lambda)$ the corresponding translation of $F_{\mathbf{R}}$, i.e., $t(\lambda)(\mu) = \mu + \lambda$. Any affine isometry of $F_{\mathbf{R}}$ uniquely decomposes as $u \circ t(\lambda)$, where u is an orthogonal transformation of $F_{\mathbf{R}}$ and $\lambda \in F_{\mathbf{R}}$.

Proposition. $W_a = \{w_0 \circ t(\lambda) \mid w_0 \in W, \lambda \in Q^\vee\}$. Thus, W_a is the semidirect product of a finite group W with the lattice $Q^\vee$.

Proof. Clearly, the elements $r_1, \ldots, r_n$ are the canonical generators of W, so $W \subset W_a$. By Prop. 6.14,

$$W_a \ni r_{\alpha_i - \delta} \circ r_{\alpha_i} = t(\alpha_i^\vee) \circ r_{\alpha_i} \circ r_{\alpha_i} = t(\alpha_i^\vee)$$

for $i = 1, \ldots, n$. Therefore, $W_a \supset Q^\vee$.

On the other hand,

$$r_0=r_{\alpha_0}=r_{\gamma-\delta}=t(\gamma^\vee)\circ r_\gamma=r_\gamma t(-\gamma^\vee)\in WQ^\vee$$

by 6.5 and 6.9 (d), so $W_a\subset W\cdot Q^\vee$.

6.16. Now we shall rewrite the denominator identity 5.11 for an affine Lie algebra $\tilde{\mathfrak{g}}$ of rank $n+1$. Recall that each element $l\in F_a$ has the form $l=\lambda+k\delta$, where $\lambda\in F$, $k\in\mathbb{C}$, so $e^l=e^{k\delta}e^\lambda$. We put $e^{-\delta}=q$ and consider e^λ as a function on F (see 6.7). Thus, both sides of 5.11 will be power series in q, whose coefficients are functions on F.

The following form of the identity 5.11 is due to I. G. Macdonald [39]:

Theorem.

$$J_{\rho_0}\cdot\prod_{k=1}^{\infty}\left\{(1-q^k)^n\prod_{\alpha\in R}(1-q^ke^\alpha)\right\}=\sum_{\mu\in\frac{1}{2}Q^\vee}q^{\langle\mu+2\rho_0,\mu\rangle}\cdot J_{\mu+\rho_0}$$

(see 6.7).

Proof. (1) The explicit form of the root decomposition of $\tilde{\mathfrak{g}}$ given in 1.16, immediately implies that the left-hand side of 5.11 for $\tilde{\mathfrak{g}}$ is equal to

$$e^{-\rho_0}J_{\rho_0}\prod_{k=1}^{\infty}\left\{(1-q^k)^n\prod_{\alpha\in R}(1-q^ke^\alpha)\right\}$$

(see 6.7, 6.14).

(2) Now let us deal with the right-hand side of 5.11. By 6.15, if we represent W_a as the group of affine transformations of F, then each element w of W_a has the form $w=w_0\circ t(\lambda)$, where $w_0\in W$, $\lambda\in Q^\vee$. The proof of Prop. 6.15 shows that $t(\lambda)$ is the product of an even number of reflections in W_a, hence $\det w$ in 5.11 is equal to $\det w_0$.

(3) It remains to calculate $w\rho-\rho$. As in 6.11, $(w\rho-\rho)$ can be thought of as an affine-linear function on F. By Lemma 6.13, for any $\varphi\in F$ we have

$$(w\rho-\rho)(\varphi)=B(w^{-1}\varphi)-B(\varphi),$$

where

$$B(\varphi)=-\frac{\bar\rho(c)}{2}\langle\varphi,\varphi\rangle+\langle\rho,\varphi\rangle.$$

(4) By Lemma 6.7, $\langle\rho,\varphi\rangle=\langle\rho_0,\varphi\rangle$ (there is some ambiguity in the notation, since the element ρ here corresponds to $\tilde{\mathfrak{g}}$ and so, differs from the element ρ considered in 6.7; but this does not matter, because $\langle\rho,\varphi\rangle$ depends only on the restriction $\bar\rho|\mathfrak{h}$, which is the same for both elements).

(5) Now we prove that $\bar\rho(c)=1/2$. Indeed, applying $\bar\rho$ to both sides of 6.10 (*), we obtain:

$$1=\frac{1}{2}(h_\gamma^\vee,h_\gamma^\vee)\bar\rho(c)-\bar\rho(h_\gamma^\vee)=\frac{\langle\gamma^\vee,\gamma^\vee\rangle}{2}\bar\rho(c)-\langle\rho_0,\gamma^\vee\rangle=$$

$$=2g\cdot\bar\rho(c)-(g-1),\ \text{by 6.9 (f).}$$

Therefore, $\bar\rho(c)=1/2$.

(6) We shall write down a typical element $w\in W_a$ as $w=w_0\circ t(2\mu)$, where $w_0\in W$, $\mu\in\frac{1}{2}Q^\vee$. Then $w^{-1}\varphi=w_0^{-1}\varphi-2\mu$. Using (3), (4) and (5), we obtain

$$(w\rho-\rho)(\varphi)=\frac{1}{4}(\langle\varphi,\varphi\rangle-\langle w_0^{-1}\varphi-2\mu,w_0^{-1}\varphi-2\mu\rangle)+$$

$$+\langle\rho_0,w_0^{-1}\varphi-2\mu-\varphi\rangle=-\langle\mu,\mu\rangle+\langle\mu,w_0^{-1}\varphi\rangle+$$

$$+\langle w_0\rho_0-\rho_0,\varphi\rangle-\langle 2\mu,\rho_0\rangle=-\langle\mu+2\rho_0,\mu\rangle+$$

$$+\langle w_0(\mu+\rho_0)-\rho_0,\varphi\rangle.$$

It follows that

$$w\rho-\rho=w_0(\mu+\rho_0)-\rho_0-\langle\mu+2\rho_0,\mu\rangle\delta,$$

hence

$$e^{w\rho-\rho}=q^{\langle\mu+2\rho_0,\mu\rangle}e^{-\rho_0}e^{w_0(\mu+\rho_0)}.$$

Remembering (2), we see that the right-hand side of 5.11 becomes

$$e^{-\rho_0}\sum_{\mu\in\frac{1}{2}Q^\vee}q^{\langle\mu+2\rho_0,\mu\rangle}J_{\mu+\rho_0}.$$

Comparing this with (1), we obtain our assertion.

6.17. Now we give the beautiful reformulation of the Macdonald's identity 6.16 due to B. Kostant [34]. The Kostant's idea is to transform the sum over a lattice in 6.16 into the sum over the finite-dimensional simple $\mathfrak{g}$-modules, i.e., over the set P_+^0 (see 5.21).

Theorem. $\prod_{k=1}^{\infty}\left\{(1-q^k)^n\prod_{\alpha\in R}(1-q^ke^\alpha)\right\}=\sum_{\varphi\in P_+^0}\varepsilon(\varphi)q^{\langle\varphi+2\rho_0,\varphi\rangle}\cdot\operatorname{ch}L(\varphi)$, where $\varepsilon(\varphi)=0$, $+1$ or -1 for $\varphi\in P_+^0$. The precise value of $\varepsilon(\varphi)$ is

$$\varepsilon(\varphi)=\operatorname{ch}L(\varphi)(2\rho_0)=\prod_{\alpha\in R^+}\frac{\sin 2\pi\langle\alpha,\varphi+\rho_0\rangle}{\sin 2\pi\langle\alpha,\rho_0\rangle}\qquad\text{(see 6.8)}.$$

6.18. To prove Theorem 6.17 we need some information about the action of W_a on $F_{\mathbb{R}}$.

Definition. A point $\lambda\in F_{\mathbb{R}}$ is called *singular* if $a(\lambda)=0$ for some affine root a, and *regular* otherwise (see 6.14). The set

$$\bar C_a=\{\lambda\in F_{\mathbb{R}}\mid\alpha_i(\lambda)\geqq 0\quad\text{for}\quad i=0,1,\dots,n\}$$

is called the *positive affine Weyl chamber*. We denote by C_a the interior of $\bar C_a$, i.e., $C_a=\{\lambda\in F_{\mathbb{R}}\mid\alpha_i(\lambda)>0$ for $i=0,\dots,n\}$.

In other words, $\lambda \in F_{\mathbb{R}}$ is regular if and only if $\langle \alpha, \lambda \rangle \notin \mathbf{Z}$ for all $\alpha \in R$. We have

$$\bar{C}_a = \{\lambda \in F_{\mathbb{R}} | \langle \alpha_i, \lambda \rangle \geqq 0 \quad \text{for} \quad i = 1, \dots, n, \quad \text{and} \quad \langle \gamma, \lambda \rangle \leqq 1\}$$

(similarly for C_a). Evidently, C_a is a (non-empty) open simplex in $F_{\mathbb{R}}$.

Proposition. (a) Each of the sets of regular and singular points in $F_{\mathbb{R}}$ is W_a-invariant.

(b) Every W_a-orbit in $F_{\mathbb{R}}$ has non-zero intersection with $\bar{C}_a$.

(c) If $\varphi \in C_a$, $\psi \in \bar{C}_a$ and $w\varphi = \psi$ for some $w \in W_a$, then $\varphi = \psi$ and $w = 1$.

(d) Let $\lambda \in F_{\mathbb{R}}$. The following conditions are equivalent:

(i) λ is regular;

(ii) λ is W_a-conjugate to some point of C_a;

(iii) $w\lambda = \lambda$ implies that $w = 1$;

(e) each of the lattices $Q^\vee$ and $2P_0$ is W_a-invariant;

(f) a point of $2P_0$ is regular if and only if it is W_a-conjugate to $2\rho_0$.

Proof. (a) is evident, because W_a permutes affine roots.

(b) Choose a point $p \in C_a$. Let Ω be a W_a-orbit in $F_{\mathbb{R}}$. By 6.15, Ω is the union of a finite number of lattices, hence has no limit points in $F_{\mathbb{R}}$. Therefore, there is $q \in \Omega$ with the minimal distance from p. Then $q \in \bar{C}_a$. Indeed, if $\alpha_i(q) < 0$ for some i, then $|r_{\alpha_i}(q) - p| < |q - p|$, which contradicts our choice of q.

(c) The proof coincides with that of Prop. 5.16.

(d) (i)$\Rightarrow$(ii) follows from (a), (b) and the evident fact that each point of $\bar{C}_a \setminus C_a$ is singular.

(ii)$\Rightarrow$(iii) follows from (c).

(iii)$\Rightarrow$(i) follows from Proposition 6.14.

(e) follows from 6.15 and 6.9 (c).

(f) In view of (a) and (d), our assertion is equivalent to the fact that $C_a \cap 2P_0 = \{2\rho_0\}$. Let $\lambda = \sum_{i=1}^{n} m_i \omega_i \in F_{\mathbb{R}}$, where ω_i are the fundamental weights of $\mathfrak{g}$ (see 6.9). Then $\lambda \in 2P_0$ if and only if $m_i \in 2\mathbf{Z}$ for all i. On the other hand, we have

$$\langle \alpha_i, \lambda \rangle = \frac{2\langle \alpha_i^\vee, \lambda \rangle}{\langle \alpha_i^\vee, \alpha_i^\vee \rangle} = \frac{2m_i}{\langle \alpha_i^\vee, \alpha_i^\vee \rangle}$$

and

$$\langle \gamma, \lambda \rangle = \frac{2\langle \gamma^\vee, \lambda \rangle}{\langle \gamma^\vee, \gamma^\vee \rangle} = \frac{1}{2g} \sum_{i=1}^{n} m_i k_i$$

(see 6.9 (d), (f)). It follows that $\lambda \in C_a$ if and only if $m_i > 0$ for all i, and $\sum_{i=1}^{n} m_i k_i < 2g$. Since $g = 1 + \Sigma k_i$, the latter inequality can be rewritten as $\sum_{i=1}^{n} (m_i - 2) k_i < 2$.

It follows that $C_a \cap 2P_0$ is the unique point $2\sum_{i=1}^{n} \omega_i$. By Lemma 6.7, this point is just $2\rho_0$.

6.19. Remark. The statements similar to 6.18 (a)–(d) with the same proof can be given for finite Weyl groups. We leave this to the reader.

6.20. Put $P' = \{\varphi \in P_+^0 | 2(\varphi + \rho_0) \text{ is regular}\}$.

Lemma. For any $\varphi \in P'$ there are unique $w_0 \in W$ and $\mu \in \frac{1}{2} Q^\vee$ such that

$$w_0(\varphi + \rho_0) = \mu + \rho_0. \tag{$*$}$$

The correspondence $\varphi \mapsto \mu$ is a bijection between P' and $\frac{1}{2} Q^\vee$. We have

$$\det w_0 = \operatorname{ch} L(\varphi)(2\rho_0).$$

Before proving this Lemma we shall show that it implies Theorem 6.17.

Clearly, $(*)$ implies that

$$\langle \mu + 2\rho_0, \mu \rangle = \langle \mu + \rho_0, \mu + \rho_0 \rangle - \langle \rho_0, \rho_0 \rangle =$$
$$= \langle \varphi + \rho_0, \varphi + \rho_0 \rangle - \langle \rho_0, \rho_0 \rangle = \langle \varphi + 2\rho_0, \varphi \rangle$$

and

$$J_{\mu + \rho_0} = \det w_0 \cdot J_{\varphi + \rho_0} = \operatorname{ch} L(\varphi)(2\rho_0) \cdot J_{\varphi + \rho_0}.$$

Thus, the bijection $\varphi \leftrightarrows \mu$ given by $(*)$, allows us to rewrite the right-hand side of 6.16 as

$$\sum_{\varphi \in P'} \varepsilon(\varphi) \cdot q^{\langle \varphi + 2\rho_0, \varphi \rangle} \cdot J_{\varphi + \rho_0},$$

where $\varepsilon(\varphi)$ is defined as in 6.17; the equality $\varepsilon(\varphi) = \det w_0$ shows that $\varepsilon(\varphi) = \pm 1$ for $\varphi \in P'$.

It remains only to observe that $\varepsilon(\varphi) = 0$ for any $\varphi \in P_+^0 \setminus P'$. Indeed, in this case $2(\varphi + \rho_0)$ is singular, hence $2\pi \langle \alpha, \varphi + \rho_0 \rangle \in \pi\mathbf{Z}$ for some $\alpha \in R_+$, and our assertion follows from Lemma 6.8.

6.21. Proof of Lemma 6.20. (1) We rewrite $(*)$ as

$$2(\varphi + \rho_0) = (w_0^{-1} \circ t(2\mu))(2\rho_0).$$

Now, the existence and uniqueness of w_0 and μ follow from 6.18 (d), (f). Therefore, the correspondence $\varphi \mapsto \mu$ is well-defined.

(2) Let us show that $\varphi \mapsto \mu$ is injective. Suppose that $w_0(\varphi + \rho_0) = \mu + \rho_0 = w_0'(\varphi' + \rho_0)$ for some $\varphi, \varphi' \in P'$ and $w_0, w_0' \in W$. Then, by Proposition 5.16, $\varphi = \varphi'$ and $w_0 = w_0'$, as required.

(3) Let us show that $\varphi \mapsto \mu$ is surjective. Let $\mu \in \frac{1}{2} Q^\vee$. By 6.9 (c), $\mu + \rho_0 \in P_0$. Since W is finite, Prop. 5.14 (b) implies that $w_0^{-1}(\mu+\rho_0) \in P_0^+$ for some $w_0 \in W$ (this follows also from the counterpart of 6.18 (b) for the finite Weyl group). This means that $\langle w_0^{-1}(\mu+\rho_0), \alpha_i^\vee \rangle \in \mathbf{Z}^+$ for $i=1, \ldots, n$. But 6.18 (f) shows that $2w_0^{-1}(\mu+\rho_0)=(w_0^{-1} \circ t(2\mu))(2\rho_0)$ is regular, so $\langle w_0^{-1}(\mu+\rho_0), \alpha_i^\vee \rangle \neq 0$ for all i. It follows that $\varphi = w_0^{-1}(\mu+\rho_0)-\rho_0$ belongs to P', which implies our assertion.

(4) It remains to calculate det w_0. Clearly, 6.20 (*) implies that

$$\operatorname{ch} L(\varphi)= \frac{J_{\varphi+\rho_0}}{J_{\rho_0}} = \det w_0 \frac{J_{\mu+\rho_0}}{J_{\rho_0}}.$$

By 6.9 (c), $\langle \alpha, \mu \rangle \in \frac{1}{2} \mathbf{Z}$ for all $\alpha \in Q$, $\mu \in \frac{1}{2} Q^\vee$. Therefore, Lemma 6.8 implies that

$$\frac{J_{\mu+\rho_0}(2\rho_0)}{J_{\rho_0}(2\rho_0)} = 1.$$

It follows that det $w_0 = \operatorname{ch} L(\varphi)(2\rho_0)$, as claimed.

6.22. The power series $\varphi(q)$ and $\eta(q)$ are defined by

$$\varphi(q)= \prod_{k=1}^{\infty} (1-q^k) \qquad \text{and} \qquad \eta(q)=q^{1/24} \cdot \varphi(q);$$

$\varphi(q)$ has numerous applications in combinatorics, and $\eta(q)$ naturally arises in the theory of modular forms (cf. e.g., [21], [48]).

Theorem. For any classical Lie algebra $\mathfrak{g}$ we have

$$\varphi(q)^{\dim \mathfrak{g}} = \sum_{\mu \in \frac{1}{2} Q^\vee} q^{\langle \mu + 2\rho_0, \mu \rangle} \prod_{\alpha \in R^+} \frac{\langle \alpha, \mu+\rho_0 \rangle}{\langle \alpha, \rho_0 \rangle} =$$

$$= \sum_{\varphi \in P_0^+} \varepsilon(\varphi) \dim L(\varphi)\, q^{\langle \varphi + 2\rho_0, \varphi \rangle};$$

$$\eta(q)^{\dim \mathfrak{g}} = \sum_{\mu \in \frac{1}{2} Q^\vee} q^{\langle \mu + \rho_0, \mu+\rho_0 \rangle} \cdot \prod_{\alpha \in R^+} \frac{\langle \alpha, \mu+\rho_0 \rangle}{\langle \alpha, \rho_0 \rangle} =$$

$$= \sum_{\varphi \in P_0^+} \varepsilon(\varphi) \dim L(\varphi)\, q^{\langle \varphi + \rho_0, \varphi+\rho_0 \rangle}.$$

Proof. The identities for $\varphi(q)$ follow at once by evaluating the identities 6.16 and 6.17 at the point $0 \in F$ (see 6.16, 6.8). Using the "strange" formula 6.9 (g), we obtain the identities for $\eta(q)$.

6.23. Example. $\mathfrak{g} = \mathfrak{sl}_2$. In this case $n = \operatorname{rank} \mathfrak{g} = 1$, and there is the unique positive root α. Putting $e^{-\alpha}=z$, one can readily rewrite 6.16 as

$$(1-z) \prod_{k=1}^{\infty} (1-q^k)(1-q^k z)(1-q^k z^{-1}) = \sum_{m \in \mathbf{Z}} q^{m(2m+1)}(z^{-2m}-z^{2m+1}).$$

Replacing q by q^2 and z by qz, we obtain the famous Gauss–Jacobi identity:

$$\prod_{k=1}^{\infty} (1-q^{2k})(1-q^{2k-1}z)(1-q^{2k-1}z^{-1}) = \sum_{m \in \mathbf{Z}} (-1)^m q^{m^2} z^m.$$

The Kostant's form of the identity 6.22 is the Jacobi identity:

$$\varphi(q)^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1)\, q^{\frac{k(k+1)}{2}}.$$

7. REVIEW OF SOME RECENT RESULTS RELATED TO CONTRAGREDIENT LIE ALGEBRAS

In the last few years many papers connected with the subject of this paper have appeared. We shall briefly comment some of them.

1) The structure of the category $\mathcal{O}$

a) In [3] the complete description of homomorphisms between the Verma modules for a semisimple Lie algebra is obtained. This implies a criterion for irreducibility of a Verma module. In [29] such a criterion is found for contragredient Lie algebras corresponding to arbitrary symmetrizable matrices. Namely, $M(\chi)$ is irreducible if and only if $2\langle \chi+\rho, \alpha \rangle \neq n \cdot \langle \alpha, \alpha \rangle$ for $\alpha \in R_+$, $n \in \mathbf{N}$. In [29] there is also obtained some partial information about homomorphisms of Verma modules.

b) In [4] the category $\mathcal{O}$ for a semisimple Lie algebra $\mathfrak{g}$ is decomposed into the direct sum of its subcategories corresponding to various characters of the center of $U(\mathfrak{g})$. In [5] it was shown that all categories corresponding to regular characters of the center of $U(\mathfrak{g})$ are equivalent. Some analogues of these statements for contragredient Lie algebras are obtained in [10].

c) In [4] there is constructed a resolution of a finite-dimensional irreducible module for a semisimple Lie algebra $\mathfrak{g}$, whose terms are direct sums of Verma modules. This result includes as special cases the Weyl's character formula and the Bott's theorem on the cohomology of the maximal nilpotent subalgebra of $\mathfrak{g}$. In [20] and [44] such a resolution is constructed for any Kac-Moody algebra.

d) Let χ and ψ be weights of a semisimple Lie algebra. In [4] there are found the necessary and sufficient conditions for the irreducible module $L(\psi)$ to be a composition factor of $M(\chi)$. The multiplicity of $L(\psi)$ in the composition series of $M(\chi)$ is computed in [33], [7]. The result of [4] is generalized in [29] for contragredient algebras corresponding to arbitrary symmetrizable matrices. Some results in this direction for Kac-Moody algebras and a conjecture concerning multiplicities are given in [10].

2) The Macdonald's identities, combinatorics and modular forms

a) Other proofs of the Macdonald's identities can be found in [6], [38], [39], [14] and [17]. The first three proofs are based on the ideas of the theory of modular forms. The two last proofs use the interpretation of these identities in terms of the heat equation ([34]).

b) Various specializations of the denominator formula for affine Lie algebras are treated in [39] and [25] (see also [35]).

c) In [37] a Lie-theoretic proof of the famous Rogers–Ramanujan identities is obtained using the character formula 5.12 for some non-trivial representations. Another combinatorial application of the character formula is given in [36]. In [31] some characters are computed explicitly by means of the theory of modular forms.

3) Basic representations

There is a number of papers where for some special affine algebras are obtained various explicit realizations of so-called basic representations, which are in some sense the simplest nontrivial irreducible representations. The first group of realizations [30], [18], [45] gives explicit formulas for generators of the algebra, expressing them as differential operators in infinite number of indeterminates. These formulas are inspired by the theory of relativistic string.

The second group of realizations [16], [32], [9], [46] is based on the following idea. Certain affine Lie algebras are imbedded into the algebra $g\ell(\infty)$ of infinite "Toeplitz" matrices or into its orthogonal subalgebra $o(\infty)$. Then the basic representation is realized as restriction to $\mathfrak{g}$ of a certain universal irreducible representation of $g\ell(\infty)$ or $o(\infty)$.

The relationships between these two types of realizations are discussed in [16], [46]. It is shown in [16] that the isomorphism of these realizations is just the boson-fermion correspondence in field theory.

In [9] there are found some connections of these realizations with the theory of non-linear differential equations.

4) In [17] the characters of locally finite simple modules for affine algebras are expressed as orbital integrals.

5) In [15] the cohomology of loop algebras and affine algebras with coefficients in the trivial module is computed. Some related results appear also in [35].

6) In [1], [2], [12] and [43] there are obtained some applications of affine algebras to Hamiltonian mechanics.

7) Most of the results on contragredient Lie algebras can be generalized to Lie superalgebras; see [25].

There appear also to be some interesting but as yet little understood connections of contragredient Lie algebras with representations of quivers [27] and the "Monster" simple group [8], [26], [28].

As remarked by I. G. Macdonald, "the range of these applications, all of which are in a stage of active development, continues to increase at an alarming rate".

Added in proof

The account of new results on contragredient Lie algebras and their applications is given in the book by *V. G. Kac* "Infinite dimensional Lie algebras", Birkhäuser, 1983. This book contains complete bibliography.

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