Morphisms in $\mathcal{O}$ are morphisms of P-graded $\mathfrak{g}$-modules.

Clearly, $\mathcal{O}$ is abelian with finite direct sums, and for any $V \in \mathcal{O}$ each graded submodule of V and each quotient module of V by a graded submodule belong to $\mathcal{O}$.

2.3. Let $V \in \mathcal{O}$. A non-zero vector $v \in V$ is called a *highest weight vector* if it is a weight vector, i.e. belongs to some V_λ, and $uv = 0$ for all $u \in \mathfrak{n}^+$.

Proposition. Any module $V \in \mathcal{O}$ has a highest weight vector.

Proof. The condition (3) from 2.2 implies that there is some $\lambda \in P$ such that $V_\lambda \neq 0$ and $V_{\lambda+\alpha} = 0$ for all $\alpha \in R^+$. Obviously, any $v \in V_\lambda$ is a highest weight vector.

2.4. Now we shall define Verma modules, which are the most important objects of $\mathcal{O}$. Let $\chi \in P$. Denote by $\mathbf{C}_\chi$ the one-dimensional P-graded module $\mathbf{C} \cdot v_\chi$ over the algebra $\mathfrak{h} \oplus \mathfrak{n}^+$ such that the weight of v_χ is χ, $hv_\chi = \bar{\chi}(h)\, v_\chi$ for $h \in \mathfrak{h}$ and $\mathfrak{n}^+ v_\chi = 0$. Denote by $M(\chi)$ the induced $\mathfrak{g}$-module, i.e.

$$M(\chi) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbf{C}_\chi$$

(here $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ (see 1.6); $U(\mathfrak{g})$ is considered as right $U(\mathfrak{h} \oplus \mathfrak{n}^+)$-module, and it acts on $M(\chi)$ by left multiplication). Since $U(\mathfrak{g})$ is obviously Q-graded and $\mathbf{C}_\chi$ is a P-graded module over the Q-graded subalgebra $U(\mathfrak{h} \oplus \mathfrak{n}^+)$, the module $M(\chi)$ also becomes P-graded. The vector $1 \otimes v_\chi \in M(\chi)$ will be denoted simply by v_χ.

Proposition. (a) $M(\chi)$ is a free $U(\mathfrak{n}^-)$-module with one generator v_χ. Therefore, $P(M(\chi)) = D(\chi)$.

(b) $M(\chi) \in \mathcal{O}$ and v_χ is a highest weight vector of weight χ.

(c) If $V \in \mathcal{O}$ then Hom $(M(\chi), V)$ is naturally isomorphic to the subspace of highest weight vectors in V_χ. In particular, this space is finite-dimensional.

Proof. (a) follows from the decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}^+)$ (see 1.6).

(b) By (a), every weight vector of $M(\chi)$ has the form uv_χ where $u \in U(\mathfrak{n}^-)_\alpha$; the corresponding weight is $\chi + \alpha$. The properties (1) and (3) from Definition 2.2 become obvious, and (2) follows from the identity $hu - uh = \bar{\alpha}(h)u$. The remaining part of (b) is obvious.

(c) means that any $A \in$ Hom $(M(\chi), V)$ is uniquely determined by the vector Av_χ which can be chosen as any highest weight vector in V_χ. This follows at once from (a) and (b).

2.5. Proposition. For any $\chi \in P$ the character of $M(\chi)$ is given by

$$\operatorname{ch} M(\chi) = e^\chi \prod_{\alpha \in R^+} (1 - e^{-\alpha})^{-\dim \mathfrak{g}_\alpha}. \qquad (*)$$

The right-hand side has the following meaning. Clearly, each element $(1 - e^{-\alpha})$ is invertible in $\mathscr{E}$ and $(1 - e^{-\alpha})^{-1} = 1 + e^{-\alpha} + e^{-2\alpha} + \ldots$. Substituting this into the right-hand side and expanding the product, we see that each e^λ will occur only finite number of times.

Proof. By Prop. 2.4 (a), $\operatorname{ch} M(\chi)(\chi - \lambda) = \dim U(\mathfrak{n}^-)_{-\lambda}$. Choose a basis $f_1, f_2, \ldots$ of $\mathfrak{n}^-$ consisting of root vectors. Then by the Poincaré–Birkhoff–Witt theorem (see 1.6) the monomials $f_{i_1} f_{i_2} \ldots f_{i_k}$ such that $i_1 \leqq i_2 \leqq \ldots \leqq i_k$ and the sum of corresponding roots equals $(-\lambda)$, form the basis of $U(\mathfrak{n}^-)_{-\lambda}$. Expanding the product in the right-hand side of $(*)$, we see that the coefficient of $e^{\chi - \lambda}$ is just the number of such monomials.

2.6. Non-formally speaking, our next proposition shows that the Verma modules "generate" the category $\mathcal{O}$, at least at the level of the Grothendieck group.

Proposition. For any $V \in \mathcal{O}$ we have $\operatorname{ch} V = \sum_\chi c_\chi \operatorname{ch} M(\chi)$, where $c_\chi \in \mathbf{Z}$ and the sum is over the set $\bigcup_{\lambda \in P(V)} D(\lambda)$ (as above, each e^λ occurs in a finite number of the ch $M(\chi)$).

Proof (cf. [25]). By definition, $P(V) \subset \bigcup_{i=1}^m D(\lambda_i)$. For $\lambda = \lambda_i - \Sigma k_s \alpha_s \in D(\lambda_i)$ we put $h_i(\lambda) = \Sigma k_s$. For $\lambda \in \bigcup D(\lambda_i)$ we put $h(\lambda) = \Sigma h_i(\lambda)$, where the summation is over those i for which $\lambda \in D(\lambda_i)$. For any $M \in \mathcal{O}$ with $P(M) \subset \bigcup D(\lambda_i)$ we put $h(M) = \min_{\lambda \in P(M)} h(\lambda)$. Let $\mu_1, \ldots, \mu_r$ be all weights of V with $h(\mu_i) = h(V)$ and $\{v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}\}$ a basis of V_{μ_i}. Clearly, each v_{ij} is a highest weight vector. By 2.4 (c), there is a morphism $\varphi: \bigoplus_i M(\mu_i)^{k_i} \to V$, sending the generators v_{μ_i} of $M(\mu_i)$ to vectors v_{ij}. Let M and N be the kernel and co-kernel of φ. From the exact sequence

$$0 \to M \to \oplus M(\mu_i)^{k_i} \to V \to N \to 0$$

it follows that $\operatorname{ch} V = \Sigma k_i \operatorname{ch} M(\mu_i) + \operatorname{ch} N - \operatorname{ch} M$. It is clear that $P(M)$ and $P(N)$ are contained in $\bigcup D(\lambda_i)$ and that $h(M)$ and $h(N)$ are greater than $h(V)$.

Applying this argument to M and N instead of V, repeating the procedure t times and tending t to ∞, we obtain the desired decomposition of ch V.

2.7. Proposition. For any $\chi \in P$ the Verma module $M(\chi)$ has the unique maximal proper graded submodule $I(\chi)$. The modules $L(\chi) = M(\chi)/I(\chi)$ are simple (i.e. have no proper graded submodules), mutually non-isomorphic and any simple object of $\mathcal{O}$ is isomorphic to some $L(\chi)$. We have $\chi \in P(L(\chi)) \subset D(\chi)$ for all $\chi \in P$.

Proof. Clearly, any proper graded submodule of $M(\chi)$ has zero intersection with $\mathbf{C}\, v_\chi$. So the sum of all proper graded submodules of $M(\chi)$ is itself proper; this is just $I(\chi)$. First two statements about $L(\chi)$ are obvious, and the last two follow at once from 2.3 and 2.4 (a), (c).

2.8. Our definition of the weight space P seems rather unnatural. It is justified by the next proposition.

Proposition. For $\chi \in \mathfrak{h}^*$ denote by $\mathcal{O}_\chi$ the full subcategory of $\mathcal{O}$ consisting of modules V with $P(V) \subset \chi + F$.

(a) The category $\mathcal{O}$ naturally decomposes into the direct sum $\bigoplus_{\chi\in\mathfrak{h}^*}\mathcal{O}_\chi$.

(b) If $\chi_1, \chi_2 \in \mathfrak{h}^*$ and $\chi_1-\chi_2=\bar{\alpha}$ for some $\alpha\in F$ then the categories $\mathcal{O}_{\chi_1}$ and $\mathcal{O}_{\chi_2}$ are naturally isomorphic.

(c) Let $\mathfrak{g}=\mathfrak{g}(A)$ with A nonsingular. Then every $\mathcal{O}_\chi$ is naturally isomorphic to $\mathcal{O}_0$. If $V\in\mathcal{O}_0$ and $\lambda\in F$ then

$$V_\lambda=\{v\in V\,|\,h\cdot v=\bar{\lambda}(h)v \quad \text{for all} \quad h\in\mathfrak{h}\} \qquad (*)$$

i.e., the weight decomposition of V coincides with the decomposition into eigenspaces with respect to $\mathfrak{h}$. In the definition of $\mathcal{O}_0$ one can replace conditions 2.2 (1), (2) by the only requirement that V is $\mathfrak{h}$-diagonalizable, i.e., $V=\bigoplus_{\lambda\in F}V_\lambda$ where the V_λ are defined by $(*)$.

In particular, if $\mathfrak{g}$ is classical, we shall consider only $\mathfrak{g}$-modules from $\mathcal{O}_0$, i.e., $\mathfrak{h}$-diagonalizable modules satisfying 2.2(3).

Proof. (a) For $V\in\mathcal{O}$ and $\chi\in\mathfrak{h}^*$ put $V(\chi)=\bigoplus_{\psi\in\chi+F}V_\psi$. Evidently, each $V(\chi)$ is a $\mathfrak{g}$--submodule of V, $V(\chi)\in\mathcal{O}_\chi$ and $V=\bigoplus_{\chi\in\mathfrak{h}^*}V(\chi)$. Moreover, $\mathrm{Hom}(V, V')=0$ if V and V' lie in different $\mathcal{O}_\chi$. This proves our assertion.

(b) Let $V\in\mathcal{O}_{\chi_2}$. Denote by V' the module from $\mathcal{O}$ which coincides with V as a $\mathfrak{g}$--module, but is P-graded as follows: $V'_\chi=V_{\chi-\chi_1+\chi_2+\alpha}$. Clearly, $V'\in\mathcal{O}_{\chi_1}$ and the functor $V\mapsto V'$ establishes the isomorphism of categories $\mathcal{O}_{\chi_2}$ and $\mathcal{O}_{\chi_1}$.

(c) Since A is nonsingular, the mapping $\alpha\mapsto\bar{\alpha}$ is an isomorphism $F\xrightarrow{\sim}\mathfrak{h}^*$ (see 1.8). This readily implies all our assertions.

2.9. Example. If $\mathfrak{g}$ is a classical Lie algebra then $\mathfrak{g}$ viewed as a $\mathfrak{g}$-module under the adjoint action is a simple module belonging to $\mathcal{O}_0$ (see 1.12 (b)). By Prop. 2.7., $\mathfrak{g}=L(\chi)$ for some $\chi\in F$. By definition, $P(\mathfrak{g})=R$, so χ is just the maximal root γ (see 1.15), i.e., $R\subset D(\gamma)$. Thus, we have obtained a conceptual proof of the existence of γ.

If $\mathfrak{g}$ is an infinite-dimensional contragredient Lie algebra, then the set of roots of $\mathfrak{g}$ is infinite, and $\mathfrak{g}$ as a $\mathfrak{g}$-module does not belong to $\mathcal{O}$, since the function $\lambda\mapsto\dim\mathfrak{g}_\lambda$ does not belong to $\mathcal{E}$ (see 2.2 (3)).

2.10. Remark. Verma modules for semisimple Lie algebras were introduced by D. N. Verma [49] and, independently, by J. N. Bernstein, I. M. Gelfand and S. I. Gelfand [3], as algebraic counterparts of the principal series representations of semisimple Lie groups. The category $\mathcal{O}$ is the "smallest" abelian category containing all Verma modules.

3. REPRESENTATIONS OF $\mathfrak{sl}_2$

3.1. In this Section we describe representations of the simplest contragredient Lie algebra $\mathfrak{g}=\mathfrak{sl}_2$. This algebra has a basis $\{f, h, e\}$ such that $[h, e]=2e$, $[h, f]=-2f$ and $[e, f]=h$, viz.,

$$f=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad e=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

The relations between f, h and e, and the induction yield the following relations in $U(\mathfrak{g})$:

$$\begin{cases}[h, f^k]=-2kf^k, \quad [h, e^k]=2ke^k,\\ [e, f^k]=kf^{k-1}(h-(k-1))\end{cases} \qquad (k\in\mathbf{N}). \qquad (*)$$

It is easy to verify that the element

$$\Delta=4fe+h^2+2h$$

belongs to the center of $U(\mathfrak{g})$; Δ is called the *Casimir element* of $\mathfrak{g}$.

3.2. Let us describe Verma modules $M(\chi)$ and simple modules $L(\chi)$ for $\mathfrak{g}=\mathfrak{sl}_2$. According to 2.8 (c), we shall consider only modules whose weights are in F. This simply means that all our modules have the weight decomposition $V=\bigoplus_{\lambda\in\mathbf{C}}V_\lambda$, where $V_\lambda=\{v\in V\,|\,hv=\lambda v\}$. The definition of a Verma module can be rewritten as follows.

For any $l\in\mathbf{C}$ the Verma module $M(l)$ is generated by one vector v_0 such that $ev_0=0$, $hv_0=lv_0$ and the vectors $v_k=f^kv$ $(k=0, 1, 2, \ldots)$ form the basis of $M(l)$. The relations $(*)$ from 3.1 imply at once that

$$fv_k=v_{k+1}, \quad ev_k=k(l-k+1)v_{k-1} \quad \text{and} \quad hv_k=(l-2k)v_k. \qquad (**)$$

Proposition. (a) $M(l)$ is simple if and only if $l\notin\mathbf{Z}^+$. If $l\in\mathbf{Z}^+$ then $M(l)$ has the unique graded submodule $I(l)$, which is generated by the highest weight vector v_{l+1} and is naturally isomorphic to $M(-l-2)$. Hence for $l\in\mathbf{Z}^+$ the simple module $L(l)$ is $(l+1)$--dimensional and has the basis $\{v_0, v_1, \ldots, v_l\}$, in which the action of $\mathfrak{g}$ is given by $(**)$, where one must put $v_{l+1}=0$.

(b) The Casimir element Δ acts on $M(l)$ (and hence, on each its subquotient) as multiplication by $l(l+2)$.

Proof. (a) follows at once from $(**)$ (recall that simplicity means the absence of proper *graded* submodules).

(b) Evidently, $\Delta v_0=l(l+2)v_0$. If $v\in M(l)$ then $v=xv_0$ for some $x\in U(\mathfrak{g})$. Hence, $\Delta v=\Delta xv_0=x\Delta v_0=l(l+2)v$.

3.3. Now we shall describe all finite-dimensional modules from $\mathcal{O}$.

Proposition. Any finite-dimensional graded $\mathfrak{sl}_2$-module V is isomorphic to a direct sum of modules of the type $L(l)$ $(l \in \mathbf{Z}^+)$.

Proof. (1) For $s \in \mathbf{C}$ put $V(s) = \{v \in V | (\Delta - s)^N v = 0 \text{ for some } N\}$. Since Δ commutes with the action of $\mathfrak{g}$ and preserves all weight subspaces V_λ, it follows that for any s $V(s)$ is a graded submodule of V. By standard linear algebra, $V = \bigoplus_{s \in \mathbf{C}} V(s)$, so it suffices to treat the case when $V = V(s)$ for some s.

(2) Since $\dim V < \infty$, there exists a chain $0 = V_0 \subset V_1 \subset \ldots \subset V_k = V$ of graded submodules of V such that V_i/V_{i-1} is simple for each i. By 3.2, $V_i/V_{i-1} = L(l_i)$, where $l_i \in \mathbf{Z}^+$. The operator Δ preserves each of V_i and therefore acts on V_i/V_{i-1}. Since $(\Delta - s)^N$ vanishes on V for large N, it vanishes on all V_i/V_{i-1}, so $s = l_i(l_i + 2)$. It follows that all modules V_i/V_{i-1} are isomorphic to one simple module $L(l)$ for a certain $l \in \mathbf{Z}^+$.

(3) Since $\operatorname{ch} V = k \operatorname{ch} L(l)$ and V is graded, the weight subspace $V_l = \{v \in V | hv = lv\}$ is k-dimensional. Clearly, for any $v \in V_l$ one has $ev = 0$, so Prop. 3.2 implies that the $\mathfrak{g}$-submodule generated by v is isomorphic to $L(l)$ (cf. 2.4 (c)). Choose a basis $\{v_1, \ldots, v_k\}$ of V_l and let V^i be the $\mathfrak{g}$-submodule generated by v_i. It is easy to see that the sum of V^i is direct. Since $\operatorname{ch} V = k \operatorname{ch} L(l) = \Sigma \operatorname{ch} V^i$, it follows that $V = V^1 \oplus \ldots \oplus V^k$, as desired.

3.4. Corollary. Any finite-dimensional graded $\mathfrak{sl}_2$-module V has the weight decomposition $V = \bigoplus_{k \in \mathbf{Z}} V_k$, and for each $k \in \mathbf{Z}^+$ the operators e^k and f^k induce isomorphisms $V_k \rightleftarrows V_{-k}$.

3.5. Our next result is that Prop. 3.3 gives in fact the description of all finite-dimensional $\mathfrak{sl}_2$-modules.

Proposition. Any finite dimensional $\mathfrak{sl}_2$-module V is h-diagonalizable, i.e., $V = \bigoplus_{\lambda \in \mathbf{C}} V_\lambda$, where $V_\lambda = \{v \in V | hv = \lambda v\}$.

Proof. (1) The commutation relations in $\mathfrak{g} = \mathfrak{sl}_2$ imply that $eV_\lambda \subset V_{\lambda+2}$, $hV_\lambda \subset V_\lambda$ and $fV_\lambda \subset V_{\lambda-2}$, so the subspace $V^{\text{diag}} = \bigoplus_{\lambda \in \mathbf{C}} V_\lambda$ is a $\mathfrak{g}$-submodule of V. In particular, any irreducible finite-dimensional $\mathfrak{g}$-module is h-diagonalizable, i.e. is isomorphic to some $L(l)$, where $l \in \mathbf{Z}^+$. Using arguments of steps (1) and (2) of the proof of Prop. 3.3, one can assume that there exists $l \in \mathbf{Z}^+$ such that each irreducible subquotient of V is isomorphic to $L(l)$.

By Prop. 3.3, $V^{\text{diag}} \simeq L(l) \oplus L(l) \oplus \ldots \oplus L(l)$. It follows that the set of weights of V is $\{-l, -l+2, \ldots, l\}$, and for any weight $\lambda = l - 2k$ the operators e^k: $V_\lambda \to V_l$ and f^{l-k}: $V_\lambda \to V_{-l}$ are isomorphisms.

(2) Suppose that h is not diagonalizable on V. Reducing h to the Jordan normal form, we see that there exist $\lambda \in \mathbf{C}$ and non-zero vectors v and u in V such that $hv = \lambda v$ and $hu = \lambda u + v$. By relations 3.1 (∗), for any $k \in \mathbf{Z}^+$ one has

$$h(e^k u) = (\lambda + 2k) e^k u + e^k v \qquad (*)$$

and

$$h(f^k u) = (\lambda - 2k) f^k u + f^k v.$$

By (1), $\lambda = l - 2k$ for some $k \in \mathbf{Z}^+$ and $e^k v \neq 0$. Replacing v by $e^k v$ and u by $e^k u$, we can assume that $\lambda = l$, i.e., $v \in V_l$.

Since $ev \in V_{l+2}$ and $f^{l+1} v \in V_{-l-2}$, it follows that $ev = f^{l+1} v = 0$. So by (∗) one has $eu \in V_{l+2}$ and $f^{l+1} u \in V_{-l-2}$ hence $eu = f^{l+1} u = 0$. Using 3.1 (∗), we obtain that

$$0 = ef^{l+1} u - f^{l+1} eu = [e, f^{l+1}] u = (l+1) f^l (h - l) u = (l+1) f^l v .$$

But this contradicts the fact that f^l: $V_l \to V_{-l}$ is an isomorphism.

3.6. Corollary. Any finite-dimensional module V over a classical Lie algebra $\mathfrak{g}$ is $\mathfrak{h}$-diagonalizable, i.e. belongs to $\mathcal{O}$.

Proof. Let $\{h_1, \ldots, h_n\}$ be the standard basis of $\mathfrak{h}$. Since the h_i-s generate $\mathfrak{h}$ and commute, it suffices to verify that V is h_i-diagonalizable. But this follows from Prop. 3.4, since V is a finite-dimensional module over the Lie algebra $\mathfrak{g}_i$ generated by f_i, h_i and e_i and $\mathfrak{g}_i \simeq \mathfrak{sl}_2$.

3.7. Corollary. Every classical Lie algebra (except $\mathfrak{o}_2$ and $\mathfrak{o}_4$) is simple, i.e. has no proper ideals.

Proof. In view of 1.12 (b), it suffices to prove that each ideal I of $\mathfrak{g}$ is Q-graded. This follows at once from Cor. 3.6 applied to the $\mathfrak{g}$-module I (cf. also 2.8 (c)).

4. THE INVARIANT FORM AND THE CASIMIR OPERATOR FOR CONTRAGREDIENT LIE ALGEBRAS

Throughout this Section $\mathfrak{g} = \mathfrak{g}(A)$, where the matrix $A = (a_{ij})$, $i, j = 1, \ldots, n$, is *symmetrizable*, i.e., $a_{ij} d_j = a_{ji} d_i$ for some non-zero numbers $d_1, \ldots, d_n$ (cf. 1.8, 1.9 (b)). We fix such numbers $d_1, \ldots, d_n$; in the case when $\mathfrak{g}$ is a Kac–Moody algebra, the d_i will always be chosen to be positive rational numbers. We shall use the notation of 1.8.

In this Section we shall construct for $\mathfrak{g}$ the counterparts of the Killing form and the Casimir operator.

4.1. We want to construct an invariant symmetric bilinear form on $\mathfrak{g}$ (cf. 1.13). Suppose that $(,)$ is such a form. Then we have

$$(h_j, h_i) = (h_j, [e_i, f_i]) = ([h_j, e_i], f_i) = a_{ji}(e_i, f_i)$$

for all $i, j = 1, \ldots, n$. Since $(,)$ is symmetric, we see that

$$a_{ji}(e_i, f_i) = a_{ij}(e_j, f_j) \quad \text{for all } i \text{ and } j.$$

This suggests to put

$$(e_i, f_i) = d_i \quad \text{and} \quad (h_i, h_j) = d_i a_{ji} = d_j a_{ij} \quad \text{for all} \quad i, j. \qquad (*)$$

Proposition. There is the unique invariant symmetric bilinear form $(,)$ on $\mathfrak{g}$ satisfying $(*)$ and such that $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$, where $\alpha, \beta \in R \cup \{0\}$.

The uniqueness is readily implied by the invariance of $(,)$. Indeed, bilinear forms on $\mathfrak{g}$ naturally correspond to linear operators $\mathfrak{g} \to \mathfrak{g}^*$. Clearly, a form $(,)$ is invariant if and only if the corresponding operator $T: \mathfrak{g} \to \mathfrak{g}^*$ is a homomorphism of $\mathfrak{g}$-modules, where $\mathfrak{g}$ is viewed as a $\mathfrak{g}$-module under the adjoint representation and $\mathfrak{g}^*$ is the dual module. By definitions, $\mathfrak{g}$ is generated as a $\mathfrak{g}$-module by $\{h_i, e_i, f_i \mid i = 1, \ldots, n\}$. Since the values $T(h_i)$, $T(e_i)$ and $T(f_i)$ are determined by conditions of the proposition, it follows that T and hence the form $(,)$ is unique.

The proof of the existence is rather tedious. We shall give it in 4.7.

Note that for classical and affine Lie algebras the form $(,)$ can be constructed explicitly. Namely, when $\mathfrak{g}$ is classical, we put $(x, y) = B(x, y)$, and when $\mathfrak{g} = \tilde{\mathfrak{g}}_0$ is an affine Lie algebra corresponding to a classical algebra $\mathfrak{g}_0$, we put

$$(uc + xt^k, vc + yt^{-l}) = B(x, y)\delta_{k,l}$$

(here $u, v \in \mathbf{C}$, $x, y \in \mathfrak{g}_0$ and B is the Killing form on $\mathfrak{g}_0$; see 1.13, 1.14). By 1.13, these forms satisfy the conditions of our proposition with the appropriate positive rational numbers $d_1, \ldots, d_n$.

4.2. Let $(,)$ be the form on $\mathfrak{g}$ given by Prop. 4.1. This form gives rise to the operator $T: \mathfrak{h} \to \mathfrak{h}^*$ acting by $Th(h') = (h, h')$ for $h, h' \in \mathfrak{h}$. The formula 4.1 $(*)$ shows that T is an isomorphism (or, equivalently, the restriction of $(,)$ on $\mathfrak{h}$ is non-degenerate) if and only if the matrix A is nonsingular.

On the other hand, the linear operator $F \to \mathfrak{h}^* (\alpha \mapsto \bar{\alpha})$ is also an isomorphism if and only if A is nonsingular (see 1.8).

Thus, in the case when A is nonsingular there is the unique isomorphism $\alpha \mapsto h_\alpha$ between F and $\mathfrak{h}$ such that $\bar{\alpha}(h) = (h_\alpha, h)$ for $\alpha \in F$, $h \in \mathfrak{h}$.

In fact, such an isomorphism exists even when A is singular:

Proposition. Let $\alpha \mapsto h_\alpha$ be the isomorphism between F and $\mathfrak{h}$ sending each α_i to $d_i^{-1} h_i$. Then

$$\bar{\alpha}(h) = (h_\alpha, h) \quad \text{for all} \quad \alpha \in F, \ h \in \mathfrak{h}.$$

This follows at once from 1.8 and 4.1 $(*)$.

4.3. The next proposition clarifies the structure of a Lie algebra $\mathfrak{g}$.

Proposition. (a) For any root $\alpha \in R$ and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ we have $[x, y] = (x, y)h_\alpha$.

(b) The kernel of $(,)$ is the center of $\mathfrak{g}$. In particular, for any $\alpha \in R$ the form $(,)$ induces a non-degenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{C}$ (see Prop. 1.10 (a)).

Proof. (a) For any $\alpha = \Sigma k_i \alpha_i \in R \cup \{0\}$ put $|\alpha| = |\Sigma k_i| = \Sigma |k_i|$. We proceed by induction on $|\alpha|$. For $|\alpha| = 1$ our claim follows at once from 4.1 $(*)$. Now, let $|\alpha| = k + 1 > 1$ and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. It suffices to consider the case when $x = [a, b]$, where $a \in \mathfrak{g}_\beta$, $b \in \mathfrak{g}_\gamma$, $\alpha = \beta + \gamma$ and $|\beta|, |\gamma| \leq k$. Using the inductive assumption and the Jacobi identity, we obtain that

$$[x, y] = [[ab], y] = [[ay], b] + [a, [by]] = ([ay], b)h_\gamma + (a, [by])h_\beta.$$

But $([ay], b) = (a, [by]) = ([ab], y) = (x, y)$ since the form $(,)$ is invariant. Therefore,

$$[x, y] = (x, y)(h_\gamma + h_\beta) = (x, y)h_\alpha,$$

as required.

(b) Using 4.1 $(*)$, we see that $h \in \mathfrak{h}$ belongs to $\operatorname{Ker}(,)$ if and only if h commutes with all e_i and f_i, i.e. belongs to the center of $\mathfrak{g}$ (see 1.10 (a)). It remains to prove that $\mathfrak{g}_\alpha \not\subset \operatorname{Ker}(,)$ for $\alpha \in R^+$. The condition 1.8 (c) readily implies that $\operatorname{ad} f_i(\mathfrak{g}_\alpha) \neq 0$ for some i. Choose $x \in \mathfrak{g}_\alpha$ so that $[x, f_i] \neq 0$. Using the induction on $|\alpha|$, we can find $y \in \mathfrak{g}$ such that $([x, f_i], y) \neq 0$. But then $(x, [f_i, y]) = ([x, f_i], y) \neq 0$, as required.

4.4. Now we define the "inner product" in the weight space $P = \mathfrak{h}^* \oplus F$ (see 2.1). For $\varphi, \psi \in \mathfrak{h}^*$ and $\alpha, \beta \in F$ we put

$$\langle \varphi + \alpha, \psi + \beta \rangle = \varphi(h_\beta) + \psi(h_\alpha) + (h_\alpha, h_\beta).$$

Clearly, $\langle , \rangle$ is symmetric, has zero restriction on $\mathfrak{h}^* \subset P$ and

$$\langle \chi, \alpha \rangle = \bar{\chi}(h_\alpha) \quad \text{for all} \quad \chi \in P, \quad \alpha \in F$$

(see 2.1, 4.2).

We define the weight $\rho \in \mathfrak{h}^* \subset P$ letting

$$\rho(h_i) = a_{ii}/2 \quad \text{for} \quad i = 1, \ldots, n.$$

Evidently,

$$\langle \rho, \rho \rangle = 0 \quad \text{and} \quad \langle \rho, \alpha_i \rangle = 1/2 \ \langle \alpha_i, \alpha_i \rangle \quad \text{for all} \quad i.$$

4.5. The following lemma will play the key role in the next Section.

Lemma. Let $\lambda \in P$ be such that $\langle \lambda, \alpha_i \rangle > 0$ for $i = 1, \ldots, n$. Suppose that $\varphi \in D(\lambda)$ (see 2.1) is such that $\langle \varphi, \varphi \rangle = \langle \lambda, \lambda \rangle$ and $\langle \varphi, \alpha_i \rangle \geq 0$ for $i = 1, \ldots, n$. Then $\varphi = \lambda$.

Proof. By definition, $\varphi = \lambda - \beta$, where $\beta = \Sigma k_i \alpha_i$ with $k_i \in \mathbf{Z}^+$. Suppose that $\beta \neq 0$, i.e. $k_i > 0$ for some i. We have

$$\langle \lambda, \lambda \rangle = \langle \varphi, \varphi \rangle = \langle \lambda - \beta, \lambda - \beta \rangle = \langle \lambda, \lambda \rangle - 2\langle \lambda, \beta \rangle + \langle \beta, \beta \rangle,$$

so

$$\langle \beta, \beta \rangle = 2\langle \lambda, \beta \rangle = 2\Sigma k_i \langle \lambda, \alpha_i \rangle > 0.$$

On the other hand,

$$\langle\varphi,\varphi\rangle=\langle\lambda,\lambda\rangle=\langle\varphi+\beta,\varphi+\beta\rangle=\langle\varphi,\varphi\rangle+2\langle\varphi,\beta\rangle+\langle\beta,\beta\rangle,$$

so

$$\langle\beta,\beta\rangle=-2\langle\varphi,\beta\rangle=-2\Sigma k_i\langle\varphi,\alpha_i\rangle\leqq 0,$$

which is a contradiction.

4.6. Now we shall define the Casimir element Δ. For $\mathfrak{g}=\mathfrak{sl}_2$, Δ was an element of the center of $U(\mathfrak{g})$ (see 3.1). Unfortunately, such a definition makes sense only for finite-dimensional algebras. In general case, we shall define Δ as an operator acting on each $\mathfrak{g}$-module $V\in\mathcal{O}$ and commuting with the action of $\mathfrak{g}$.

For any positive root α let $\{e_\alpha^{(i)}\}$ and $\{f_\alpha^{(i)}\}$ be dual bases of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ with respect to the form (,), where $i=1,2,\ldots,\dim\mathfrak{g}_\alpha$. We put $\Delta_1=2\sum_{\alpha\in R^+}\sum_i f_\alpha^{(i)}e_\alpha^{(i)}$. Then for any $V\in\mathcal{O}$ the expression Δ_1 defines the operator $\Delta_1\colon V\to V$, since for any $v\in V$ the sum $\sum_{\alpha,i} f_\alpha^{(i)}e_\alpha^{(i)}v$ has only finite number of non-zero terms. Define the operator $\Delta_2\colon V\to V$ setting $\Delta_2(v)=\langle\lambda+2\rho,\lambda\rangle v$ for $v\in V_\lambda$. We define the *Casimir operator* $\Delta\colon V\to V$ by $\Delta=\Delta_1+\Delta_2$.

Proposition. (a) For any $V\in\mathcal{O}$ the operator $\Delta\colon V\to V$ is a morphism in $\mathcal{O}$, i.e. $\Delta(V_\lambda)\subset V_\lambda$ for all $\lambda\in P$, and Δ commutes with the action of $\mathfrak{g}$.

(b) Δ commutes with morphisms in $\mathcal{O}$, i.e. for any morphism $\varphi\colon V_1\to V_2$ the following diagram is commutative:

$$\begin{array}{ccc} V_1 & \xrightarrow{\ \Delta\ } & V_1 \\ \varphi\downarrow & & \downarrow\varphi \\ V_2 & \xrightarrow{\ \Delta\ } & V_2 \end{array}$$

(c) Δ acts on the Verma module $M(\chi)$ (and hence on each its subquotient) as scalar multiplication by $\langle\chi+2\rho,\chi\rangle$.

Proof. The only non-trivial statement is that Δ commutes with the action of $\mathfrak{g}$. It suffices to verify that $[\Delta,e_k]v=[\Delta,f_k]v=0$ for $v\in V_\lambda\subset V$ and $k=1,\ldots,n$. First we shall prove that

$$[\Delta_1,e_k]=-2h_{\alpha_k}e_k.$$

Indeed,

$$[\Delta_1,e_k]=2\sum_{\alpha,i}(f_\alpha^{(i)}e_\alpha^{(i)}e_k-e_kf_\alpha^{(i)}e_\alpha^{(i)})=$$

$$=2\sum_{\alpha,i}f_\alpha^{(i)}[e_\alpha^{(i)}e_k]-2\sum_{\alpha,i}[e_kf_\alpha^{(i)}]e_\alpha^{(i)}. \qquad (*)$$

It is easy to see that the term in the second sum corresponding to $\alpha=\alpha_k$, equals $(-2h_{\alpha_k}e_k)$. For all other terms we have

$$[e_\alpha^{(i)}e_k]=\sum_j([e_\alpha^{(i)}e_k],f_{\alpha+\alpha_k}^{(j)})e_{\alpha+\alpha_k}^{(j)}$$

and

$$[e_kf_\alpha^{(i)}]=\sum_j([e_kf_\alpha^{(i)}],e_{\alpha-\alpha_k}^{(j)})f_{\alpha-\alpha_k}^{(j)}.$$

Now we substitute these expressions into (*) and change the indices in the second sum, putting $i'=j$, $j'=i$ and $\alpha'=\alpha-\alpha_k$. Since (,) is invariant, we see that all terms except $-2h_{\alpha_k}e_k$, cancel out, as claimed.

Now, let $v\in V_\lambda$. We have $[\Delta_1,e_k]v=-2h_{\alpha_k}e_kv=-2\langle\lambda+\alpha_k,\alpha_k\rangle e_kv$. On the other hand,

$$[\Delta_2,e_k]v=(\langle\lambda+2\rho+\alpha_k,\lambda+\alpha_k\rangle-\langle\lambda+2\rho,\lambda\rangle)e_kv=(2\langle\lambda+\alpha_k,\alpha_k\rangle+$$

$$+2\langle\rho,\alpha_k\rangle-\langle\alpha_k,\alpha_k\rangle)e_kv=2\langle\lambda+\alpha_k,\alpha_k\rangle e_kv$$

(see 4.4). Therefore, $[\Delta,e_k]v=0$. The same argument shows that $[\Delta,f_k]=0$.

4.7. It remains to prove the existence of a form (,) in Prop. 4.1. For $k\geqq 1$ put $\mathfrak{g}^k=\bigoplus_{|\alpha|\leqq k}\mathfrak{g}_\alpha$ (see the proof of Prop. 4.3 (a)). We shall construct the form (,) successively on subspaces $\mathfrak{g}^k$.

We define the symmetric bilinear form (,) on $\mathfrak{g}^1=\mathfrak{h}\oplus\Sigma\mathbf{C}e_i\oplus\Sigma\mathbf{C}f_i$ by 4.1 (*) (the values of (,) on remaining pairs of generators are zero). We must verify that this form on $\mathfrak{g}^1$ is invariant, i.e., that $([xy],z)=(x,[yz])$ for all $x,y,z\in\mathfrak{g}$ such that all elements $[x,y]$, z, x and $[y,z]$ lie in $\mathfrak{g}^1$. Clearly, it suffices to verify that

$$([h_j,e_i],f_i)=(h_j,[e_if_i]) \quad\text{and}\quad ([h_j,f_i],e_i)=(h_j,[f_ie_i])$$

for all i,j. This follows at once from the commutation relations in $\mathfrak{g}$ and 4.1 (*).

Now, we shall proceed by induction on k. Suppose that the invariant symmetric form (,) is already constructed on $\mathfrak{g}^k$ for some $k\geqq 1$. We must extend it to $\mathfrak{g}^{k+1}$. Let α be a positive root, $|\alpha|=k+1$, $x\in\mathfrak{g}_\alpha$ and $y\in\mathfrak{g}_{-\alpha}$. By definition, x has the form $\Sigma[a_i,b_i]$, where $a_i,b_i\in\mathfrak{g}^k$. We put $(x,y)=(y,x)=\Sigma(a_i,[b_iy])$. To prove that (x,y) is well-defined we must verify that the equality $\Sigma[a_ib_i]=0$ implies $\Sigma(a_i,[b_iy])=0$. It suffices to consider the case when $y=[u,v]$, $u,v\in\mathfrak{g}^k$.

Using the Jacobi identity and the fact that (,) is invariant on $\mathfrak{g}^k$, we have

$$(a_i,[b_iy])=(a_i,[b_i[uv]])=(a_i,[[b_iu]v])+(a_i,[u[b_iv]])=$$

$$=([a_i[b_iu]],v)+([[b_iv]a_i],u)=(v,[a_i[b_iu]])+$$

$$+([b_iv],[a_iu])=(v,[a_i[b_iu]])+(v,[[a_iu]b_i])=$$

$$=(v,[[a_ib_i]u]).$$

Therefore, $\sum_i (a_i, [b_i y]) = \left(v, \left[\sum_i [a_i b_i], u\right]\right) = 0$, as desired.

It remains to verify that $(\,,)$ is invariant on $\mathfrak{g}^{k+1}$. Let α be a positive root with $|\alpha| = k+1$. We must prove two identities:

$$\text{(1)} \qquad ([ve], y) = (v, [ey]), \quad \text{where} \quad y \in \mathfrak{g}_{-\alpha}, \quad e \in \mathfrak{g}_\beta, \quad v \in \mathfrak{g}_{\alpha-\beta},$$

and β is a positive root with $|\beta| \leqq k$. This follows at once from our definition.

$$\text{(2)} \qquad ([xf], u) = (x, [fu]), \quad \text{where} \quad x \in \mathfrak{g}_\alpha, \quad f \in \mathfrak{g}_{-\beta}, \quad u \in \mathfrak{g}_{\beta-\alpha}$$

and β is a positive root with $|\beta| \leqq k+1$. As above, it suffices to consider the case when $x = [a, b]$, where a and b lie in $\mathfrak{g}^k$. Now the desired equality $([[ab]f], u) = ([ab], [fu])$ is proved by the same arguments as above.

5. LOCALLY FINITE MODULES AND THEIR CHARACTERS

In this Section we develop the representation theory of Kac-Moody algebras. Throughout this Section $\mathfrak{g}$ will be a Kac–Moody algebra with standard generators $f_1, \ldots, f_n, h_1, \ldots, h_n$ and $e_1, \ldots, e_n$. For $i = 1, \ldots, n$ denote by $\mathfrak{g}_i$ the subalgebra of $\mathfrak{g}$ generated by f_i, h_i and e_i. Each $\mathfrak{g}_i$ is naturally isomorphic to $\mathfrak{sl}_2$.

5.1. We begin with some preliminary results.

Lemma. Let V be a P-graded $\mathfrak{g}$-module (not necessarily belonging to $\mathcal{O}$) and $v \in V_\lambda$ a highest weight vector of V (see 2.2, 2.3). Suppose that $\bar{\lambda}(h_i) = k \in \mathbf{Z}^+$ for some $i \in \{1, \ldots, n\}$. Then either $f_i^{k+1} v = 0$, or $f_i^{k+1} v$ is a highest weight vector.

Proof. We must verify that $e_j(f_i^{k+1} v) = 0$ for all j. For $j \neq i$ this is obvious and for $j = i$ follows at once from the relations 3.1 $(*)$.

5.2. Corollary. Let $\chi \in P$ be such that $\bar{\chi}(h_i) = k \in \mathbf{Z}^+$ for some i. Put $\chi' = \chi - (k+1)\alpha_i$. There is a natural imbedding $M(\chi') \subsetneq M(\chi)$, sending the generator $v_{\chi'}$ to $f_i^{k+1} v_\chi$.

This follows at once from Lemma 5.1 and Prop. 2.4.

5.3. Corollary. The relations $(\operatorname{ad} f_i)^{-a_{ij}+1}(f_j) = (\operatorname{ad} e_i)^{-a_{ij}+1}(e_j) = 0$ hold in any Kac–Moody algebra.

Proof. Using the antiautomorphism ι (see 1.8), it suffices to prove that $(\operatorname{ad} f_i)^{-a_{ij}+1}(f_j) = 0$. Denote by $\mathfrak{g}_{\hat{j}}$ the subalgebra of $\mathfrak{g}$ generated by all e_k, f_k and h_k with $k \neq j$, and consider $\mathfrak{g}$ as a $\mathfrak{g}_{\hat{j}}$-module under the adjoint representation. Put $f = (\operatorname{ad} f_i)^{-a_{ij}+1}(f_j)$ and suppose that $f \neq 0$. Since f_j is a highest weight vector of the $\mathfrak{g}_{\hat{j}}$-module $\mathfrak{g}$, Lemma 5.1 implies that $\operatorname{ad} e_k(f) = 0$ for $k \neq j$.

Let us prove that $\operatorname{ad} e_j(f) = 0$. Indeed, we have

$$(\operatorname{ad} e_j)(f) = (\operatorname{ad} f_i)^{-a_{ij}+1} \circ \operatorname{ad} e_j(f_j) = (\operatorname{ad} f_i)^{-a_{ij}+1}(h_j) =$$
$$= a_{ji} (\operatorname{ad} f_i)^{-a_{ij}}(f_i).$$

If $a_{ij} = 0$ then $a_{ji} = 0$ by definition of a Cartan matrix (see 1.9); if $a_{ij} \neq 0$ then evidently $(\operatorname{ad} f_i)^{-a_{ij}}(f_i) = 0$. Therefore, in each case $\operatorname{ad} e_j(f) = 0$.

It follows that $U(\mathfrak{n}^+) f = 0$ hence $U(\mathfrak{g}) f$ is a graded ideal in $\mathfrak{g}$ having zero intersection with $\mathfrak{h}$. This contradicts the definition of a contragredient algebra.

5.4. Now we introduce the class of $\mathfrak{g}$-modules, which will play the main role in the sequel.

Definition. A P-graded $\mathfrak{g}$-module V is *locally finite* if for any $v \in V$ we have $f_i^N v = e_i^N v = 0$ for $i = 1, \ldots, n$ and large $N \in \mathbf{N}$. Denote by $\mathcal{O}^f$ the full subcategory of $\mathcal{O}$ consisting of locally finite modules.

The commutation relations 3.1 $(*)$ readily imply that V is locally finite if and only if any $v \in V$ is contained in a finite-dimensional $\mathfrak{g}_i$-submodule for $i = 1, \ldots, n$.

Lemma. For any P-graded $\mathfrak{g}$-module V the subspace $V^f = \{v \in V \mid f_i^N v = e_i^N v = 0$ for $i = 1, \ldots, n$ and large $N\}$ is a graded submodule of V.

Proof. Let v be a weight vector of V^f, i.e., $f_i^m v = e_i^m v = 0$ for $i = 1, \ldots, n$ and some $m \in \mathbf{N}$. We must prove that $f_j^N e_i v = f_j^N f_i v = e_j^N e_i v = e_j^N f_i v = 0$ for $i, j = 1, \ldots, n$ and large N. The verification for e_j^N is the same as for f_j^N.

If $i \neq j$ then $f_j^m e_i v = e_i f_j^m v = 0$; if $i = j$ we have $f_i^{m+1} e_i v = [f_i^{m+1}, e_i] v = -m f_i^m (h_i - m) v = 0$ (see 3.1 $(*)$). It remains to prove that $f_j^N f_i(v) = 0$ for large N and all $j \neq i$. By induction on N, it is easy to prove the identity

$$f_j^N f_i = \sum_{k=0}^{N} \binom{N}{k} (\operatorname{ad} f_j)^k (f_i) f_i^{N-k}.$$

By 5.3, if $N \geqq m - a_{ji}$ then $f_j^N f_i v = 0$ as required.

5.5. By Lemma 5.4, to prove that V is locally finite it suffices to verify that $f_i^N v = e_i^N v = 0$ when v runs over a system of generators of V. In particular, Corollary 5.3. implies that the adjoint representation of $\mathfrak{g}$ is locally finite.

5.6. Definition. A weight $\chi \in P$ is called *integral* if $\bar{\chi}(h_i) \in \mathbf{Z}$ for all $i = 1, \ldots, n$ and *dominant integral* if $\bar{\chi}(h_i) \in \mathbf{Z}^+$ for all $i = 1, \ldots, n$ (see 2.1). Denote the set of dominant integral weights by P_+.

Proposition. If V is a locally finite module and $v \in V_\chi$ is a highest weight vector of V (see 2.3) then $\chi \in P_+$. A simple module $L(\chi)$ is locally finite if and only if $\chi \in P_+$.

Proof. Since v lies in a finite-dimensional $\mathfrak{g}_i$-module for $i = 1, \ldots, n$, Prop. 3.3 implies that $\chi \in P_+$. In particular, if $L(\chi)$ is locally finite then $\chi \in P_+$. The converse statement follows at once from Lemma 5.4 and the next.

5.7. Lemma. Let χ be a dominant integral weight and $v_\chi \in L(\chi)$ a canonical generator. Then $f_i^{\bar{\chi}(h_i)+1}(v_\chi)=0$ for $i=1,\ldots,n$.

This follows at once from Cor. 5.2.

5.8. We want to compute the character ch $L(\chi)$ for $\chi \in P_+$. This becomes possible since this character turns out to satisfy some symmetry conditions.

Definition. For any $i=1,\ldots,n$ we define the linear operator r_i: $P \to P$ by

$$r_i(\chi)=\chi-\bar{\chi}(h_i)\cdot\alpha_i$$

(see 2.1). The group generated by operators $r_1,\ldots,r_n$ is called the *Weyl group* (of $\mathfrak{g}$) and denoted by $W=W(\mathfrak{g})$.

Note that all r_i are involutions, i.e. $r_i^2=1$; the subspace of r_i-invariant weights has codimension 1 in P. We have defined W as the group of transformations of the weight space P. Evidently, the subspace $F \subset P$ is W-invariant, and W acts trivially on the quotient P/F. Note that $r_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$; some authors prefer to define W as the group of transformations of F generated by these operators (see, e.g., [20]). One can prove that this definition is equivalent to our one; later we shall prove this fact for classical and affine algebras.

The group W naturally acts on functions on P by $wf(\chi)=f(w^{-1}\chi)$.

5.9. Proposition. (a) The form $\langle\,,\rangle$ on P defined in 4.4, is W-invariant.

(b) Let V be a locally finite $\mathfrak{g}$-module such that all weight subspaces of V are finite-dimensional. Then the set of weights $P(V)$ is W-invariant, and dim $V_{w\lambda}=$ dim V_λ for all $\lambda \in P(V)$, $w \in W$.

(c) The set R of roots of $\mathfrak{g}$ is W-invariant, and dim $\mathfrak{g}_{w\alpha}=$ dim $\mathfrak{g}_\alpha$ for all $\alpha \in R$, $w \in W$.

(d) For all $i=1,\ldots,n$ the transformation r_i permutes the elements of $R_+\backslash\{\alpha_i\}$.

Proof. (a) It suffices to prove that $\langle r_i\chi, r_i\chi\rangle=\langle\chi,\chi\rangle$ for $i=1,\ldots,n$ and $\chi \in P$. We have

$$\langle r_i\chi, r_i\chi\rangle=\langle\chi-\bar{\chi}(h_i)\alpha_i, \chi-\bar{\chi}(h_i)\alpha_i\rangle=$$
$$=\langle\chi,\chi\rangle-2\bar{\chi}(h_i)\langle\chi,\alpha_i\rangle+(\bar{\chi}(h_i))^2\langle\alpha_i,\alpha_i\rangle. \qquad (*)$$

But $\langle\chi,\alpha_i\rangle=\bar{\chi}(h_{\alpha_i})=d_i^{-1}\bar{\chi}(h_i)$ and $\langle\alpha_i,\alpha_i\rangle=\bar{\alpha}_i(h_{\alpha_i})=2d_i^{-1}$ (see 4.4). Substituting these expressions into $(*)$, we see that $\langle r_i\chi, r_i\chi\rangle=\langle\chi,\chi\rangle$, as required.

(b) It suffices to prove that dim $V_{r_i\lambda}=$ dim V_λ for $i=1,\ldots,n$ and $\lambda \in P(V)$. Since V is locally finite, its finite-dimensional subspace $V_\lambda+V_{r_i\lambda}$ generates a finite-dimensional $\mathfrak{g}_i$-module. By Cor. 3.4, $\bar{\lambda}(h_i)=k \in \mathbf{Z}$, and the operators $e_i^{|k|}$ and $f_i^{|k|}$ give isomorphisms between V_λ and $V_{r_i\lambda}$.

(c) Follows at once from (b) and 5.5.

(d) Any element $\alpha \in R_+\backslash\{\alpha_i\}$ has the form $\alpha=\sum_{j=1}^{n}k_j\alpha_j$ with each $k_j \in \mathbf{Z}^+$ and some $k_{j_0}>0$ with $j_0 \neq i$. Since the coefficient of α_{j_0} in α is the same as in $r_i\alpha$, it follows that $r_i\alpha \in R_+\backslash\{\alpha_i\}$, as required.

5.10. Now we shall formulate the main result of this Section.

Theorem. Let $\chi \in P$ be a dominant integral weight and $V \in \mathcal{O}^f$ a module generated by a highest weight vector of weight χ. Then V is naturally isomorphic to $L(\chi)$, and

$$\text{ch } V=\text{ch } L(\chi)=\sum_{w\in W}\det w \text{ ch } M(w(\chi+\rho)-\rho). \qquad (*)$$

Remarks. (a) Recall that the weight ρ was introduced in 4.4. Since $\mathfrak{g}$ is a Kac-Moody algebra, we have $\rho(h_i)=1$ for $i=1,\ldots,n$.

(b) It is easy to see that det $r_i=-1$, so det $w=\pm 1$ for all $w \in W$.

(c) The equality $(*)$ should be understood in the sense of Prop. 2.6.

Before proving this Theorem we shall give some important corollaries.

5.11. Corollary (The Weyl–Kac denominator formula).

$$\prod_{\alpha\in R^+}(1-e^{-\alpha})^{\dim\mathfrak{g}_\alpha}=\sum_{w\in W}\det w\, e^{w\rho-\rho}.$$

This follows from Prop. 2.5 by applying Theorem 5.10 to $\chi=0$.

5.12. Corollary. (The Weyl–Kac character formula). For any dominant integral χ we have

$$\text{ch } L(\chi)=\frac{\sum_{w\in W}\det w\, e^{w(\chi+\rho)-\rho}}{\sum_{w\in W}\det w\, e^{w\rho-\rho}}.$$

5.13. Corollary. Let $\chi \in P_+$. Then the maximal graded submodule $I(\chi)$ of the Verma module $M(\chi)$ equals $\sum_{i=1}^{n}M(\chi_i')$ where $\chi_i'=\chi-(\bar{\chi}(h_i)+1)\,\alpha_i=r_i(\chi+\rho)-\rho$ (see Cor. 5.2).

Proof. Put $V=M(\chi)/\Sigma M(\chi_i')$. Let $v \in V$ be the image in V of the standard generator $v_\chi \in M(\chi)$. Evidently, v generates V and we have $f_i^{\bar{\chi}(h_i)+1}(v)=0$ for $i=1,\ldots,n$. By Lemma 5.4, V is locally finite. Theorem 5.10 implies that $V=L(\chi)$, and our assertion follows.

5.14. The proof of Theorem 5.10 requires some properties of the Weyl group.

Proposition. Let $\chi \in P_+$. Then

(a) $w\chi \in D(\chi)$ for all $w \in W$.

(b) If D is a W-invariant subset of $D(\chi)$ then every element of D is W-conjugate to a dominant integral weight.

Proof. (a) By 5.6 and 5.9 (b), the module $L(\chi)$ is locally finite, and the set $P(L(\chi))$ of its weights is W-invariant. By Prop. 2.7, $\chi \in P(L(\chi)) \subset D(\chi)$, which proves our assertion.

(b) Let $\psi \in D$. Choose $w \in W$ so that in the expression $w\psi = \chi - \sum_i k_i\alpha_i$ $(k_i \in \mathbf{Z}^+)$ the sum Σk_i is minimal. Then $w\psi \in P_+$. Indeed, if $\overline{w\psi}(h_i) = m < 0$ for some i, then $r_i w\psi = w\psi - m\alpha_i$ has an expression $\chi - \sum_i m_i\alpha_i$ with $\Sigma m_i < \Sigma k_i$, which contradicts our choice of w.

5.15. Corollary. If $w \in W$ and $w \neq 1$ then $w\alpha_i \in R_-$ for some i.

Proof. Suppose that $w\alpha_i \in R_+$ for $i = 1, \ldots, n$. Let $\chi \in P_+$. By Prop. 5.14 (a),

$$w^{-1}\chi = \chi - \sum_i k_i\alpha_i, \qquad \text{where} \quad k_i \in \mathbf{Z}^+ .$$

It follows that

$$\chi = w(w^{-1}\chi) = w\chi - \sum_i k_i'\alpha_i, \qquad \text{where} \quad k_i' \in \mathbf{Z}^+ .$$

Again applying Prop. 5.14 (a), we see that $w\chi = \chi$. Evidently, the space P is generated by P_+, so $w = 1$.

5.16. Proposition. Let $\varphi, \psi \in P$ be such that $\bar{\varphi}(h_i) > 0$ and $\bar{\psi}(h_i) \geqq 0$ for $i = 1, \ldots, n$. Then $w\varphi = \psi$ implies that $\varphi = \psi$ and $w = 1$.

Proof. By 4.4 and 4.2,

$$\bar{\varphi}(h_i) = \bar{\varphi}(d_i h_{\alpha_i}) = d_i\langle \varphi, \alpha_i\rangle ,$$

so the conditions $\bar{\varphi}(h_i) > 0$ and $\langle \varphi, \alpha_i\rangle > 0$ are equivalent.

Suppose that $w \neq 1$ and $w\varphi = \psi$. By 5.15, there is an α_i such that

$$w\alpha_i = \Sigma k_j\alpha_j \qquad \text{with} \quad k_j \in \mathbf{Z}^- .$$

Using 5.9 (a), we see that

$$\langle \varphi, \alpha_i\rangle = \langle w^{-1}\psi, \alpha_i\rangle = \langle \psi, w\alpha_i\rangle = \Sigma k_j\langle \psi, \alpha_j\rangle \leqq 0,$$

which is a contradiction.

5.17. Proposition. The element $K = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$ (see 2.1 and 2.5) is W-antiinvariant, that is, $wK = \det w\ K$ for all $w \in W$.

Proof. It suffices to verify that $r_i K = -K$ for $i = 1, \ldots, n$. We have

$$K = e^\rho(1 - e^{-\alpha_i}) \prod_{\alpha \in R^+ \setminus \{\alpha_i\}} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha} .$$

By definition,

$$r_i\{e^\rho(1 - e^{-\alpha_i})\} = e^{\rho - \alpha_i}(1 - e^{\alpha_i}) = -e^\rho(1 - e^{-\alpha_i}) .$$

On the other hand, $\prod_{\alpha \in R^+ \setminus \{\alpha_i\}} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$ is invariant under r_i, by 5.9 (c), (d). It follows that $r_i K = -K$, as required.

5.18. Proof of Theorem 5.10. (1) By Prop. 2.6,

$$\operatorname{ch} V = \sum_{\psi \in P} c_\psi \cdot \operatorname{ch} M(\psi) ,$$

where $c_\psi = 0$ unless $\psi \in D(\chi)$. Moreover, the proof of Prop. 2.6 shows that $c_\chi = 1$.

(2) Let us prove that $\langle \psi + \rho, \psi + \rho\rangle = \langle \chi + \rho, \chi + \rho\rangle$ whenever $c_\psi \neq 0$. By Prop. 4.6 (c), this means that the Casimir operator Δ acts on each $M(\psi)$ occuring in our decomposition, as multiplication by the same constant. Let us remember the proof of Prop. 2.6. We claim that Δ acts on all modules occuring in this proof as multiplication by $\langle \chi + \rho, \chi + \rho\rangle$. Indeed, on each stage of the proof we have proceeded as follows. We have a module, say U, choose some highest weight vectors $v_1, \ldots, v_k$ in U of weights $\mu_1, \ldots, \mu_k$, consider the map $\oplus M(\mu_i) \to U$ and take its kernel M and co-kernel N. Clearly, once it is known that Δ acts on U as a constant, it follows that Δ acts on each $M(\mu_i)$ and hence on M and N, as the same constant. This proves our claim.

(3) Next, we prove that if $\psi + \rho = w(\varphi + \rho)$ for some $w \in W$ then $c_\psi = \det w \cdot c_\varphi$. By Prop. 2.5,

$$\operatorname{ch} V \cdot K = \sum_\psi c_\psi e^{\psi + \rho} \qquad (*)$$

(see 5.17).

We must prove that the right-hand side of $(*)$ is W-antiinvariant. But ch V is W-invariant by 5.9 (b), and K is W-antiinvariant by 5.17, which proves our claim.

(4) Put $D = \{\psi \in P \mid c_{\psi - \rho} \neq 0\}$. Let us show that $D = W(\chi + \rho)$. Let $\psi \in D$. By (1) and (3), D is a W-invariant subset of $D(\chi + \rho)$, so there is a $w \in W$ such that $w\psi \in P_+$ (Prop. 5.14 (b)). Using (2), we see that the weights $\varphi = w\psi$ and $\lambda = \chi + \rho$ satisfy the conditions of Lemma 4.5. Therefore, $w\psi = \chi + \rho$, as required.

(5) By Prop. 5.16, the mapping $w \mapsto w(\chi + \rho)$ is a bijection between W and D. Since $c_\chi = 1$, we have $c_{w(\chi + \rho) - \rho} = \det w$ for all $w \in W$ (see (3)). Hence

$$\operatorname{ch} V = \sum_{w \in W} \det w \operatorname{ch} M(w(\chi + \rho) - \rho) .$$

(6) It remains to prove that V is naturally isomorphic to $L(\chi)$. Clearly, V is a quotient of $M(\chi)$, so there is a natural epimorphism $p: V \to L(\chi)$. But all arguments above can be applied to $L(\chi)$ instead of V, so ch $L(\chi) =$ ch V. It follows that p is an isomorphism, which completes the proof of Theorem 5.10.

5.19. Now we describe all modules $V \in \mathcal{O}^f$. For simplicity, we assume that V is of finite length, i.e., V has a finite chain of submodules $0 = V_0 \subset V_1 \subset \ldots \subset V_k = V$ such that V_i/V_{i-1} is simple for $i = 1, \ldots, k$ (such a chain is called a *composition series* of V).

Theorem. Any module $V \in \mathcal{O}^f$ of finite length is isomorphic to a direct sum of modules of the type $L(\chi)$ where $\chi \in P_+$.

This Theorem generalizes Prop. 3.3, and the proof will follow lines of that of Prop. 3.3 with necessary modifications.

Let $0=V_0\subset\ldots\subset V_k=V$ be a composition series of V. By Prop. 5.6, each quotient V_i/V_{i-1} is isomorphic to some $L(\chi_i)$, where $\chi_i\in P_+$.

(1) For $s\in\mathbf{C}$ put $V(s)=\{v\in V|(\Delta-s)^N v=0$ for some $N\}$, where Δ is the Casimir operator (see 4.6). We want to show that $V=\bigoplus_{s\in\mathbf{C}} V(s)$. It suffices to prove that each $v\in V$ lies in a finite-dimensional Δ-invariant subspace of V. By Prop. 4.6 (b), (c), if $v\in V_i$ then $\Delta v-\langle\chi_i+2\rho,\chi_i\rangle\cdot v\in V_{i-1}$, so our assertion follows by induction on i. As in Prop. 3.3 (1), we can assume that $V=V(s)$ for some $s\in\mathbf{C}$.

(2) Arguments from 3.3 (2) show that $\langle\chi_i+2\rho,\chi_i\rangle=s$ for all i, so $\langle\chi_i+\rho,\chi_i+\rho\rangle=\langle\chi_j+\rho,\chi_j+\rho\rangle$ for all i,j. By Lemma 4.5, if $\chi_i\neq\chi_j$ then $\chi_i\notin D(\chi_j)$.

(3) We have $\operatorname{ch} V=\sum_{i=1}^{k}\operatorname{ch} L(\chi_i)$, so the sum of weight subspaces V_{χ_i} is k-dimensional. Clearly, $P(V)=\operatorname{supp}\operatorname{ch} V\subset\bigcup D(\chi_i)$. Using step (2), it follows that each $v\in V_{\chi_i}$ is a highest weight vector. By Theorem 5.10, the $\mathfrak{g}$-submodule generated by v is isomorphic to $L(\chi_i)$. The rest of the proof is the same as in 3.3 (3).

5.20. We conclude this Section with a complete description of all finite-dimensional modules over classical Lie algebras. For the remainder of this Section let $\mathfrak{g}$ be a classical Lie algebra.

Lemma. (a) The restriction of the form $\langle\,,\rangle$ on the space $F_{\mathbf{R}}=\mathbf{R}\alpha_1\oplus\ldots\oplus\mathbf{R}\alpha_n$ is positive definite.

(b) W is finite.

Proof. (a) Clearly, $F_{\mathbf{R}}=\{\alpha\in F|h_\alpha\in\mathfrak{h}_{\mathbf{R}}\}$ (see 4.2, 1.13). Therefore, our statement follows at once from definitions (see 4.4) and 1.13.

(b) The proof will be given in the next Section (see 6.4).

5.21. Put $P_+^0=P_+\cap F$ (see 5.6), i.e., P_+^0 is the set of dominant integral weights in F.

Theorem. Let $\mathfrak{g}$ be a classical Lie algebra.

(a) For any $\varphi\in P_+^0$ the simple module $L(\varphi)$ is finite-dimensional. All these modules are mutually non-isomorphic, and any finite-dimensional simple $\mathfrak{g}$-module is isomorphic to one of them.

(b) (The H. Weyl's theorem). Any finite-dimensional $\mathfrak{g}$-module is isomorphic to a direct sum of modules $L(\varphi)$, where $\varphi\in P_+^0$.

Proof. (a) By 2.8, if $\chi\in P$ and $\varphi\in F$ are such that $\bar\chi=\bar\varphi$ then the modules $L(\chi)$ and $L(\varphi)$ are isomorphic as $\mathfrak{g}$-modules (of course, without account of P-gradation). Therefore, by 2.7 and 5.6, any finite-dimensional simple $\mathfrak{g}$-module is isomorphic to some $L(\varphi)$ for $\varphi\in P_+^0$.

By 2.8, the set $P(L(\varphi))$ is determined by the $\mathfrak{g}$-module structure on $L(\varphi)$ (since $L(\varphi)_\lambda=\{v\in L(\varphi)|h\cdot v=\bar\lambda(h)\,v\}$). Since $\varphi\in P(L(\varphi))\subset D(\varphi)$, it follows that $L(\varphi)\neq L(\psi)$ for $\varphi\neq\psi$.

It remains to prove that $L(\varphi)$ is finite-dimensional for $\varphi\in P_+^0$. It suffices to verify that $P(L(\varphi))$ is finite. Using 5.20 (a), (b) and the fact that $P(L(\varphi))\subset D(\varphi)$, we see that this follows from the next assertion:

(∗) If $\psi\in P(L(\varphi))$ then either $\psi\in W\varphi$, or $\langle\psi,\psi\rangle<\langle\varphi,\varphi\rangle$.

Using 5.9 (a), (b), and 5.14 (b), we can assume that $\psi\in P_+^0$. It follows that $\langle\psi,\alpha_i\rangle\geqq 0$ for $i=1,\ldots,n$. We have $\varphi=\psi+\beta$, where $\beta\in Q_+$, so

$$\langle\varphi,\varphi\rangle=\langle\psi,\psi\rangle+2\langle\psi,\beta\rangle+\langle\beta,\beta\rangle\geqq\langle\psi,\psi\rangle+\langle\beta,\beta\rangle\,.$$

By 5.20 (a), if $\beta\neq 0$ then $\langle\psi,\psi\rangle<\langle\varphi,\varphi\rangle$, and (∗) is proved. (This argument is similar to that in 4.5.)

(b) This follows at once from (a), Th. 5.19 and Prop. 3.5.

5.22. Proposition. Let $\varphi\in P_+^0$. Then

$$\dim L(\varphi)=\prod_{\alpha\in R^+}\frac{\langle\alpha,\varphi+\rho\rangle}{\langle\alpha,\rho\rangle}\,.$$

Although the proof is rather simple, it is more convenient for us to give it in the next Section (see 6.8).

6. SPECIALIZATIONS OF THE DENOMINATOR FORMULA AND KAC-MACDONALD IDENTITIES

In this Section we investigate in more detail the Weyl groups of classical and affine Lie algebras. As an application, we shall deduce some beautiful corollaries from the denominator identity 5.11.

6.1. We begin with a general

Definition. Let G be a group and U a G-module. A function $f\colon G\to U$ is a *1-cocycle* (or a *crossed homomorphism*) if

$$f(g_1g_2)=f(g_1)+g_1f(g_2)\qquad\text{for all }\ g_1,g_2\in G\,.$$

The space of 1-cocycles $f\colon G\to U$ is denoted by $C^1(G,U)$.

We need only two properties of 1-cocycles.

(a) Let $S\subset G$ be a set of generators of G. Then every $f\in C^1(G,U)$ is uniquely determined by its values on the elements of S.

(b) Let V be a G-module and $U\subset V$ a G-submodule such that V/U is a trivial G-module. Then for every $v\in V$ the function $f_v\colon G\to U$ defined by $f_v(g)=v-gv$ belongs to $C^1(G,U)$.

The proof is obvious.

6.2. Let $\mathfrak{g}$ be a Kac-Moody algebra. For every weight $\chi\in P=\mathfrak{h}^*\oplus F$ we define the function $f_\chi\colon W\to F$ by $f_\chi(w)=\chi-w\chi$. By 6.1 (b), $f_\chi\in C^1(W,F)$ for any $\chi\in P$.