Hence,

$$d_{M_{2,n}}(k)=\sum_{0\leqq s\leqq\left[\frac{k+1}{2}\right]}\left[(k-2s+1)\prod_{1\leqq m\leqq n-2}\frac{s+m}{m}\prod_{2\leqq p\leqq n-1}\frac{k-s+p}{p}\right]^2 .$$

For large k this number is nicely approximated by

$$\frac{k^{4n-5}}{2[(n-1)!\,(n-2)!]^2}\int_0^1 (1-2t)^2\, t^{2n-4}\,(1-t)^{2n-4}\,dt\,,$$

which in turn is easily computed in terms of the Euler B-function. Namely, it equals

$$k^{4n-5}\frac{[B(2n-1,2n-3)-B(2n-2,2n-2)]}{[(n-1)!\,(n-2)!]^2}=$$

$$=\frac{(2n-2)!\,(2n-4)!-[(2n-3)!]^2}{(4n-5)!\,[(n-1)!\,(n-2)!]^2}\,k^{4n-5},$$

and hence,

$$\dim M_{2,n}=4n-4,\qquad \deg M_{2,n}=\frac{(2n-4)!\,(2n-3)!}{[(n-1)!\,(n-2)!]^2}\,. \tag{4}$$

Problem 4. Compute the dimension and the degree of the manifold $A_{2,n}$ of skew symmetric matrices of rank 2 and order n.

Answer: $\dim A_{2,n}=2n-3$, $\deg A_{2,n}=\dfrac{(2n-4)!}{(n-1)!\,(n-2)!}$.

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REPRESENTATIONS OF CONTRAGREDIENT LIE ALGEBRAS AND THE KAC–MACDONALD IDENTITIES

B. L. FEIGIN, A. V. ZELEVINSKY

Institute of Solid State Physics, Chernogolovka

Institute of Earth Physics, Moscow

INTRODUCTION

At Summer School in Budapest 1971 there was given a lecture by *J. N. Bernstein* and *S. I. Gelfand* "Finite-dimensional representations of semisimple Lie algebras". There was developed the approach to finite-dimensional representations based on the study of certain remarkable infinite-dimensional representations (so called Verma modules). Now this approach is well-known and is presented in many excellent text-books (cf., e.g., [11], [22]).

In 1974, V. G. Kac and, independently, R. V. Moody, recognized that the same approach can be applied to the representation theory of some important infinite-dimensional Lie algebras (so-called contragredient Lie algebras) introduced earlier by these authors. It has become apparent in the last few years that contragredient Lie algebras have connections with many other areas of mathematics.

The aim of this paper is to give an introduction to the theory of contragredient Lie algebras and their representations, following lines of the lecture of Bernstein and Gelfand. The text is written for beginners, so we tried to make it as self-contained as possible (we begin with the definition of a Lie algebra!). The reader is assumed to be familiar only with standard facts from linear algebra and some basic terminology from the theory of associative rings and modules over them.

Let us describe in more detail the contents of the paper. In Section 1 we collect together some basic definitions and facts about Lie algebras. Then we introduce the class of contragredient Lie algebras and study their simplest properties.

To any complex $n\times n$-matrix $A=(a_{ij})$ we associate the contragredient Lie algebra $\mathfrak{g}=\mathfrak{g}(A)$ (the precise definition is given in 1.8). The most important contragredient Lie algebras are the so-called Kac–Moody algebras, corresponding to matrices A such that $a_{ii}=2$, a_{ij} are non-positive integers for $i\neq j$, and AD is symmetric for some nonsingular diagonal matrix D. The Kac–Moody algebra $\mathfrak{g}(A)$ can be defined as a Lie algebra generated by $3n$ elements e_i, f_i, h_i $(1\leqq i\leqq n)$ subject to the following relations:

$$[e_i, f_j]=\delta_{ij}h_i,\qquad [h_i, h_j]=0,$$

$$[h_i, e_j] = a_{ij}e_j, \qquad [h_i, f_j] = -a_{ij}f_j,$$

$$(\mathrm{ad}\, e_i)^{1-a_{ij}}e_j = (\mathrm{ad}\, f_i)^{1-a_{ij}}f_j = 0 \qquad (i \neq j).$$

This definition is motivated by the Serre's theorem, which claims that any (finite--dimensional) semisimple Lie algebra is a Kac–Moody algebra ([47]). Working independently, V. G. Kac and R. V. Moody have constructed the remarkable class of infinite-dimensional Kac–Moody algebras, the so-called affine algebras ([23], [41]). Affine algebras correspond to semisimple Lie algebras and their automorphisms of finite order.

The structural theory of semisimple Lie algebras is presented in many text-books (see, e.g., [22], [47]). In order not to repeat it here, we consider in this paper only classical Lie algebras. They are explicitly described in Section 1. For simplicity of exposition, we consider only those affine algebras which correspond to the identity automorphism of a classical Lie algebra.

Section 2 contains basic definitions and general results about representations of contragredient Lie algebras. The main notions introduced here are the category $\mathcal{O}$ and the Verma modules $M(\chi)$.

In Section 3 the representation theory of the simplest contragredient Lie algebra $s\ell_2$ is presented.

The main results of the paper are collected in Sections 5 and 6. In Section 5 the representation theory of Kac–Moody algebras is developed. This theory mirrors precisely (and includes) the classical theory of finite-dimensional representations of semisimple Lie algebras. All results and proofs are due to V. G. Kac [24], [25]. His method is to adapt to Kac–Moody algebras the approach to the classical theory by J. N. Bernstein, I. M. Gelfand and S. I. Gelfand [3].

The main results of Section 5 are the analogies of Weyl's character formula and denominator formula (Theorem 5.10, Corollaries 5.11 and 5.12). The proof of these formulas given in [3] uses the Harish–Chandra theorem concerning the center of the universal enveloping algebra (see [11], [22]), and some symmetry conditions imposed by the action of the Weyl group. The arguments concerning the Weyl group can be generalized directly (see points 5.8, 5.9 and 5.14–5.17 in the text). But an analog of the Harish–Chandra theorem for Kac–Moody algebras is not known; in place of it the Casimir operator is used, which plays the role of a single decisive element of the center of the universal enveloping algebra. The Casimir operator is constructed in Section 4. The construction uses the invariant form on a Kac–Moody algebra, which is also constructed in Section 4.

In Section 6 the case of classical and affine Lie algebras is treated. As it was shown by V. G. Kac [24] and R. V. Moody [42], the denominator formula for affine Lie algebras is just the I. G. Macdonald's identity [39]. We obtain this formula in Theorem 6.16.

Using specializations of the denominator formula, I. G. Macdonald in [39] obtained the remarkable identities for powers of the Dedekind's η-function. The history of these identities and the detailed discussion of them can be found in [13] and [25]. We present some of these identities in their original form and give also the beautiful reformulation of them due to B. Kostant [34] (Theorems 6.17 and 6.22).

The paper is concluded with a brief review of some recent work related to contragredient Lie algebras.

We are grateful to J. N. Bernstein and S. I. Gelfand for permission of using the non-published text of their lecture. We have also used the papers by H. Garland and J. Lepowsky [20], V. G. Kac [25] and I. G. Macdonald [40]. We would like to thank A. Fialowski and D. A. Leites for their interest in our work.

1. THE SETTING

All vector spaces considered later will be defined over the field $\mathbf{C}$ of complex numbers. As usual, we write $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{Z}^+$ and $\mathbf{N}$ for the sets of real, rational, integer, nonnegative integer, and positive integer numbers.

1.1. Definition. A *Lie algebra* is a vector space $\mathfrak{g}$ with a *bracket* operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ ($(x, y) \mapsto [x, y]$) satisfying the conditions:

(1) $[x, y]$ is a bilinear function of x and y;

(2) $[x, x] = 0$ for any $x \in \mathfrak{g}$;

(3) *(Jacobi identity).* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

By (1) and (2) we have $0 = [x+y, x+y] = [x, y] + [y, x]$, i.e. $[x, y]$ is antisymmetric.

1.2. Examples. (a) Let A be an associative algebra and $\mathfrak{g}$ a vector subspace of A, closed under the bracket operation $[x, y] = xy - yx$. Then $\mathfrak{g}$ is a Lie algebra (the Jacobi identity is verified directly).

(b) An important example is the *general linear algebra* $g\ell(V)$ which is the associative algebra End V of endomorphisms of a vector space V, viewed as Lie algebra. We write $g\ell_n$ instead of $g\ell(\mathbf{C}^n)$. The algebra $g\ell_n$ has the standard basis $\{e_{ij} \mid 1 \leq i, j \leq n\}$, where e_{ij} is the matrix with 1 in the (i, j) position and 0 elsewhere. Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}. \qquad (*)$$

Let $\mathfrak{n}_n^+$, $\mathfrak{n}_n^-$ and $\mathfrak{h}_n$ be subsets of $g\ell_n$ consisting of strictly upper triangular, strictly lower triangular and diagonal matrices, i.e. the subspaces with the bases $\{e_{ij} \mid i<j\}$, $\{e_{ij} \mid i>j\}$ and $\{e_{ii}\}$, respectively. Then $(*)$ implies that each of them is a Lie algebra.

(c) Clearly, the commutator $[x, y]$ of two $n\times n$-matrices has trace 0. Hence $n\times n$-matrices with trace 0 form a Lie algebra; it is called *special linear algebra* and denoted by $\mathfrak{sl}_n$.

(d) Let B be a non-degenerate symmetric bilinear form on a vector space V. Clearly, the subspace $\{x\in\mathfrak{gl}(V)\mid B(xu, v)+B(u, xv)=0$ for all $u, v\in V\}$ is closed under the bracket and so is a Lie algebra. It is called the *orthogonal algebra* and denoted by $\mathfrak{o}(V)$.

(e) Let B be a non-degenerate skew-symmetric bilinear form on V (it is well-known that such a form exists only if V has even dimension). The same definition as in (d) leads to the notion of the *symplectic Lie algebra* $\mathfrak{sp}(V)$.

It is well-known that a non-degenerate symmetric (or skew-symmetric) form on V is essentially unique. So Lie algebras $\mathfrak{o}(V)$ and $\mathfrak{sp}(V)$ in fact depend only on dim V, and we will sometimes denote them by $\mathfrak{o}_n$ and $\mathfrak{sp}_n$ (in the latter case $n=2m$ is even). The algebras $\mathfrak{sl}_n$, $\mathfrak{o}_n$ and $\mathfrak{sp}_{2m}$ are called *classical.*

1.3. In view of Example 1.2 (a), the bracket in a Lie algebra is often called *commutator*. We say that elements x and y *commute* if $[x, y]=0$. A Lie algebra $\mathfrak{g}$ is *commutative* (or *abelian*) if any two elements of $\mathfrak{g}$ commute. The *center* of $\mathfrak{g}$ is the set of elements of $\mathfrak{g}$ commuting with each element of $\mathfrak{g}$.

1.4. The bracket in a Lie algebra will be thought of as a multiplication. This allows one to define the following concepts for Lie algebras just in the same way as for associative algebras: a *subalgebra;* an *ideal* (since the bracket is antisymmetric, any left or right ideal is two-sided); the *quotient algebra* $\mathfrak{g}/\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ by an ideal $\mathfrak{h}$; *homomorphisms, monomorphisms* and *epimorphisms* of Lie algebras; the (direct) *product* of Lie algebras. These notions satisfy all standard properties.

1.5. *A representation of a Lie algebra* $\mathfrak{g}$ *in a vector space* V is a homomorphism of Lie algebras $\rho\colon \mathfrak{g}\to\mathfrak{gl}(V)$ (see 1.2 (b)). The space V is said to be a $\mathfrak{g}$-*module.* We shall often write xv instead of $\rho(x)v$.

Examples. (a) If $\rho(x)=0$ for any $x\in\mathfrak{g}$ then V is a *trivial* $\mathfrak{g}$-module.

(b) The *adjoint representation* ad: $\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ is defined by ad $x(y)=[x, y]$ (the fact that it is a representation follows from the Jacobi identity).

(c) A subalgebra of a general linear algebra $\mathfrak{gl}(V)$ has a natural representation in the space V; it is called the *standard representation.*

(d) The tensor product $V_1\otimes V_2=V_1\otimes_{\mathbf{C}} V_2$ of two $\mathfrak{g}$-modules has a natural structure of a $\mathfrak{g}$-module:

$$x(v_1\otimes v_2)=xv_1\otimes v_2+v_1\otimes xv_2.$$

This allows one to define for any $\mathfrak{g}$-module V its tensor, symmetric and exterior powers denoted by $\otimes^n V$, $S^n(V)$ and $\Lambda^n(V)$.

(e) Similarly, $\mathrm{Hom}(V_1, V_2)=\mathrm{Hom}_{\mathbf{C}}(V_1, V_2)$ becomes a $\mathfrak{g}$-module by

$$(xA)(v)=x(Av)-A(xv), \qquad (x\in\mathfrak{g},\ A\in\mathrm{Hom}(V_1, V_2),\ v\in V_1).$$

The *dual module* V^* is defined by $V^*=\mathrm{Hom}(V, \mathbf{C})$, where $\mathbf{C}$ is the one-dimensional trivial $\mathfrak{g}$-module.

The standard concepts of the module theory such as *submodules, quotient modules, direct sums of modules, homomorphisms, irreducible modules* and so on, are defined for Lie algebras in just the same way as for associative algebras. Moreover, they can be reduced to the case of associative algebras (see the next point).

1.6. Now we shall construct a functor from the category of Lie algebras to the category of associative algebras with unit. Let $\mathfrak{g}$ be a Lie algebra. Consider the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$, i.e.

$$T(\mathfrak{g})=\bigoplus_{n\geq 0} T^n(\mathfrak{g}),$$

where $T^0(\mathfrak{g})=\mathbf{C}$, $T^n(\mathfrak{g})=\otimes^n\mathfrak{g}$. Consider also the two-sided ideal $I\subset T(\mathfrak{g})$ generated by the elements $x\otimes y-y\otimes x-[x, y]$ for $x, y\in\mathfrak{g}$. Put $U(\mathfrak{g})=T(\mathfrak{g})/I$. The correspondence $\mathfrak{g}\mapsto U(\mathfrak{g})$ is the desired functor; $U(\mathfrak{g})$ is called the *universal enveloping algebra of* $\mathfrak{g}$.

We will identify elements of $\mathfrak{g}$ with their images in $U(\mathfrak{g})$. This allows us to identify $\mathfrak{g}$-modules with (left, unitary) $U(\mathfrak{g})$-modules.

If $\mathfrak{g}$ is abelian then $U(\mathfrak{g})$ is by definition the symmetric algebra $S(\mathfrak{g})$. The following theorem provides a generalization for arbitrary Lie algebras.

The Poincaré–Birkhoff–Witt theorem. Let $\{x_i\}_{i\in I}$ be a basis in $\mathfrak{g}$ indexed by a linearly ordered set I. Then the monomials $x_{i_1}^{n_1}x_{i_2}^{n_2}\ldots x_{i_k}^{n_k}$, where $i_1<i_2<\ldots<i_k$ and $n_1,\ldots, n_k$ are non-negative integers, form a basis in $U(\mathfrak{g})$.

The proof can be found in any text-book on Lie algebras (see, e.g., [22]).

Corollary. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be subalgebras of a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ as a vector space. Then $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$ are naturally imbedded in $U(\mathfrak{g})$, and the linear mapping

$$U(\mathfrak{g}_1)\otimes_{\mathbf{C}} U(\mathfrak{g}_2)\to U(\mathfrak{g})$$

sending $x_1\otimes x_2$ to x_1x_2, is an isomorphism of vector spaces.

1.7. A usual way to study algebraic objects is to present them by generators and defining relations. We will not give formal definitions for the case of Lie algebras, but construct the explicit presentation for classical Lie algebras.

First consider the case $\mathfrak{g}=\mathfrak{sl}_n$. Then $\mathfrak{g}$ contains three subalgebras $\mathfrak{n}^+=\mathfrak{n}_n^+$, $\mathfrak{n}^-=\mathfrak{n}_n^-$, $\mathfrak{h}=\mathfrak{b}_n\cap\mathfrak{g}$ (see 1.2) and $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ as a vector space. Let us describe the structure of $\mathfrak{n}^+$ and $\mathfrak{n}^-$. For $i=1, \ldots, n-1$ put $e_i=e_{i,i+1}, f_i=e_{i+1,i}$. Then the formula (*) from 1.2 implies that the elements $e_i(f_i)$ generate the algebra $\mathfrak{n}^+(\mathfrak{n}^-)$. It is easy to verify that $[e_i, e_j]=0$ if $i-j \neq \pm 1$, and $[e_j, [e_j, e_i]]=0$ if $i-j=\pm 1$ (similarly for f_i). In fact, the Lie algebra $\mathfrak{n}^+(\mathfrak{n}^-)$ is defined by these relations, i.e. any other relation between the generators is their consequence (of course, with the axioms of Lie algebras). We will not use this fact (for the proof see [47]).

Now consider the interaction of $\mathfrak{n}^+$ and $\mathfrak{n}^-$. Again by 1.2 (*), we have $[e_i, f_j]=0$ unless $i=j$, and $[e_i, f_i]=h_i$, where $h_i=e_{ii}-e_{i+1,i+1}$. The algebra $\mathfrak{h}$ is abelian, and elements $h_1, \ldots, h_{n-1}$ form its basis. It is easy to verify that $[h_i, e_i]=2e_i$, $[h_i, e_{i\pm 1}]= =-e_{i\pm 1}$, and $[h_i, e_j]=0$ if $|j-i| \geqq 2$; similarly, $[h_i, f_i]=-2f_i$, $[h_i, f_{i\pm 1}]=f_{i\pm 1}$ and $[h_i, f_j]=0$ if $|j-i| \geqq 2$.

Summarizing, we have constructed in $\mathfrak{g}$ the system of $3(n-1)$ generators $f_1, \ldots, f_{n-1}, h_1, \ldots, h_{n-1}$ and $e_1, \ldots, e_{n-1}$. All relations given above can be written as follows. Let $A=(a_{ij})$ be the following integer $(n-1) \times (n-1)$-matrix:

$$A=\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & & & 0 \\ \vdots & & & & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{pmatrix}$$

Then

$$(*)\quad \begin{cases} [h_i, h_j]=0, \quad [e_i, f_j]=\delta_{ij}h_i, \quad [h_i, e_j]=a_{ij}e_j, \\ [h_i, f_j]=-a_{ij}f_j \quad \text{and} \quad (\operatorname{ad} e_i)^{-a_{ij}+1}e_j=0=(\operatorname{ad} f_i)^{-a_{ij}+1}f_j \\ \text{whenever} \quad i \neq j. \end{cases}$$

1.8. It turns out that the representation theory of the Lie algebra $\mathfrak{g}=\mathfrak{sl}_{n+1}$ can be deduced, after all, from the relations 1.7 (*). In fact, these relations form a set of defining relations for $\mathfrak{g}$. All algebras to be considered later (in particular, all other classical algebras) also can be defined by a similar system of relations. Such algebras will be called contragredient; they are the main subject of this paper. Since it is quite difficult to verify for concrete algebras that given relations form a system of defining relations, it is more convenient to give another definition of a contragredient algebra.

We need the notion of a graded Lie algebra. Let Γ be an abelian group. A Lie algebra $\mathfrak{g}$ is called *Γ-graded* if $\mathfrak{g}=\bigoplus_{\alpha\in\Gamma} \mathfrak{g}_\alpha$, where all $\mathfrak{g}_\alpha$ are vector subspaces of $\mathfrak{g}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Basic definition. Let $A=(a_{ij})$ be a $n \times n$-matrix with complex entries and Q a free abelian group with a fixed basis $\alpha_1, \ldots, \alpha_n$. A *contragredient Lie algebra* $\mathfrak{g}$ of rank n corresponding to A is a Q-graded Lie algebra with generators $f_1, \ldots, f_n, h_1, \ldots, h_n$ and $e_1, \ldots, e_n$, satisfying the following properties:

(a) $[h_i, h_j]=0$, $[h_i, e_j]=a_{ij}e_j$, $[h_i, f_j]=-a_{ij}f_j$, $[e_i, f_j]=\delta_{ij}h_i$ for $i, j=1, \ldots, n$.

(b) $h_1, \ldots, h_n$ form a basis of the subspace $\mathfrak{h}=\mathfrak{g}_0$ and $\mathfrak{g}_{\alpha_i}=\mathbf{C}\cdot e_i$, $\mathfrak{g}_{-\alpha_i}=\mathbf{C}\cdot f_i$ for $i=1, \ldots, n$.

(c) Every non-zero Q-graded ideal of $\mathfrak{g}$ has non-zero intersection with $\mathfrak{h}$.

It is easy to verify that such an algebra exists for every matrix A and is determined by A up to an isomorphism; we shall write $\mathfrak{g}=\mathfrak{g}(A)$.

The definition readily implies that there is a (unique) antiautomorphism $\iota\colon \mathfrak{g}\to\mathfrak{g}$ such that $\iota(e_i)=f_i$, $\iota(f_i)=e_i$ and $\iota(h_i)=h_i$ for $i=1, \ldots, n$.

Let us introduce some terminology and notation. Let $\mathfrak{g}$ be a contragredient Lie algebra. Put $R=\{\alpha\in Q-\{0\} \mid \mathfrak{g}_\alpha \neq 0\}$; the elements of R are called *roots* of $\mathfrak{g}$, and the decomposition $\mathfrak{g}=\mathfrak{h}\oplus(\bigoplus_{\alpha\in R} \mathfrak{g}_\alpha)$ is called the *root decomposition* of $\mathfrak{g}$. By (b), each α_i is a root for $i=1, \ldots, n$; elements $\alpha_1, \ldots, \alpha_n$ are called *simple roots.* Let $\mathfrak{n}^+(\mathfrak{n}^-)$ be the subalgebra of $\mathfrak{g}$ generated by $e_1, \ldots, e_n$ $(f_1, \ldots, f_n)$. It is easy to verify that $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ (in all our examples this will be evident). A root α is called *positive (negative)* if $\mathfrak{g}_\alpha \subset \mathfrak{n}^+(\mathfrak{n}^-)$; the set of positive (negative) roots of $\mathfrak{g}$ is denoted by $R^+(R^-)$. Put $Q^+=\{\Sigma k_i\alpha_i \mid k_i\in\mathbf{Z}^+\}$ and $Q^-=-Q^+$; then $R^+=R\cap Q^+$, $R^-=R\cap Q^-$ and $R=R^+\cup R^-$. Clearly, if $\alpha=\Sigma k_i\alpha_i\in R^+$ then the subspace $\mathfrak{g}_\alpha\subset\mathfrak{g}$ is spanned by the elements $[e_{i_1}, [e_{i_2}, \ldots, [e_{i_{r-1}}, e_{i_r}] \ldots]]$ where each e_i occurs k_i times (similarly for negative roots). Therefore, $\dim \mathfrak{g}_\alpha<\infty$ for all $\alpha\in R$. Clearly, $\iota(\mathfrak{g}_\alpha)=\mathfrak{g}_{-\alpha}$ for all $\alpha\in Q$. It follows that $-\alpha\in R$ for all $\alpha\in R$ and $\dim \mathfrak{g}_\alpha=\dim \mathfrak{g}_{-\alpha}$; by (b), $\dim \mathfrak{g}_\alpha=1$ if $\pm\alpha$ is a simple root.

Let $F=Q\otimes_{\mathbf{Z}}\mathbf{C}$ be the vector space with the basis $\alpha_1, \ldots, \alpha_n$. Define the linear operator $F\to\mathfrak{h}^*(\alpha\mapsto\bar{\alpha})$ by the formula $\bar{\alpha}_j(h_i)=a_{ij}$ $(i, j=1, \ldots, n)$. The description of the root subspaces $\mathfrak{g}_\alpha$ given above implies easily that

$$[h, x]=\bar{\alpha}(h)x \qquad \text{for} \quad h\in\mathfrak{h}, \quad \alpha\in R \qquad \text{and} \quad x\in\mathfrak{g}_\alpha.$$

Evidently, the mapping $\alpha\mapsto\bar{\alpha}$ is an isomorphism $F \xrightarrow{\sim} \mathfrak{h}^*$ if and only if A is nonsingular.

The assertion that $\mathfrak{sl}_n$ is a contragredient Lie algebra of rank $n-1$ is almost obvious. In this case $R^+=\{\alpha_i+\alpha_{i+1}+\ldots+\alpha_j \mid 1\leqq i\leqq j<n\}$, and for $\alpha=\alpha_i+\ldots+\alpha_j$

we have $\mathfrak{g}_\alpha = \mathbf{C}\cdot e_{i,j+1}$ and $\mathfrak{g}_{-\alpha} = \mathbf{C}e_{j+1,i}$. Properties (a) and (b) are obvious, and (c) follows at once from the equality $[e_{ij}, e_{ji}] = e_{ii} - e_{jj} \in \mathfrak{h}$.

1.9. Our definition of a contragredient algebra does not include the relations $(\operatorname{ad} e_i)^{-a_{ij}+1}(e_j) = (\operatorname{ad} f_i)^{-a_{ij}+1}(f_j) = 0$; they make sense only if a_{ij} is a nonpositive integer for $i \neq j$. Now we introduce an important class of contragredient algebras satisfying these relations.

Definition. The matrix $A = (a_{ij})$ is a *Cartan matrix* if it satisfies the following properties:

(a) $a_{ii} = 2$, a_{ij} is a nonpositive integer for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$.

(b) A is *symmetrizable*, i.e. there is a nonsingular diagonal matrix D such that AD is symmetric. By (a), D can be chosen so that all its diagonal entries are positive rational numbers.

Contragredient algebras corresponding to Cartan matrices are called *Kac–Moody algebras.* We shall prove in Section 5 that relations 1.7 $(*)$ hold in any Kac–Moody algebra (see 5.3 below); recently, O. Gabber and V. Kac proved that they form a set of defining relations ([19]).

1.10. The matrix A is called *decomposable* if it becomes of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ under some simultaneous permutation of rows and columns. Evidently, in this case $\mathfrak{g}(A) = \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$.

Proposition. Let $\mathfrak{g} = \mathfrak{g}(A)$ be a contragredient Lie algebra.

(a) The center Z of $\mathfrak{g}$ consists of elements of $\mathfrak{h}$ commuting with all e_i and f_i $(i = 1, \dots, n)$; $Z = 0$ if and only if A is nonsingular.

(b) If A is a Cartan matrix then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

(c) If A is an indecomposable Cartan matrix then any proper graded ideal of $\mathfrak{g}$ is contained in Z. In particular, if A is nonsingular then $\mathfrak{g}$ has no proper graded ideals.

Proof. (a) Evidently, any graded subspace of Z is a graded ideal of $\mathfrak{g}$. Therefore, by 1.8 (c), $Z \subset \mathfrak{h}$, which proves the first assertion. By 1.8 (a), the element $\sum x_i h_i$ belongs to Z if and only if $\sum_i x_i a_{ij} = 0$ for $j = 1, \dots, n$, which proves the last assertion.

(b) The commutation relations 1.8 (a) show that h_i, e_i and f_i lie in $[\mathfrak{g}, \mathfrak{g}]$ for all i. Since $[\mathfrak{g}, \mathfrak{g}]$ is an ideal and $\mathfrak{g}$ is generated by h_i, e_i and f_i, it follows that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

(c) We must prove that if $h \in \mathfrak{h} - Z$ then the graded ideal I generated by h, coincides with $\mathfrak{g}(A)$. Since $h \notin Z$, it follows that $[h, e_i] \neq 0$ for some i. But $[h, e_i]$ is proportional to e_i hence e_i and $h_i = [e_i, f_i]$ lie in I. Since A is an indecomposable Cartan matrix, there exists $j \neq i$ such that $a_{ij} \neq 0$. It follows that e_j, f_j and h_j also lie in I. Repeating this argument, we obtain that $I = \mathfrak{g}(A)$, as required.

1.11. We shall prove that orthogonal and symplectic Lie algebras also are contragredient. An explicit matrix realization of $\mathfrak{o}_n$ and $\mathfrak{sp}_{2m}$ can be found e.g. in

[22]. We shall give a little more invariant description. This requires some standard facts from linear algebra.

Let V and W be finite-dimensional vector spaces. Then we have the natural isomorphism $\operatorname{Hom}(V, W) = V^* \otimes W$ (if $\xi \in V^*$ and $w \in W$ then the element $\xi \otimes w \in V^* \otimes W$ corresponds to the operator $(v \mapsto \xi(v)w)$). Clearly, if V and W are $\mathfrak{g}$--modules then this isomorphism is compatible with $\mathfrak{g}$-module structures defined in 1.5. In particular, we identify $\otimes^2 V^*$ with $\operatorname{Hom}(V, V^*)$; since $V = (V^*)^*$, we also have $\otimes^2 V = \operatorname{Hom}(V^*, V)$. Therefore, $\Lambda^2 V$ and $S^2 V$ are imbedded into $\operatorname{Hom}(V^*, V)$, while $\Lambda^2 V^*$ and $S^2 V^*$ are imbedded into $\operatorname{Hom}(V, V^*)$.

Proposition. Let V be an n-dimensional vector space and $\mathfrak{g}$ one of the algebras $\mathfrak{sp}_{2n}$, $\mathfrak{o}_{2n}$ and $\mathfrak{o}_{2n+1}$.

(a) $\mathfrak{g}$ admits a **Z**-gradation displayed in the following table:

TABLE 1

$\mathfrak{g}$ \ k	-2	-1	0	1	2
$\mathfrak{sp}_{2n}$	$S^2 V^*$	0	$\mathfrak{gl}(V)$	0	$S^2 V$
$\mathfrak{o}_{2n}$	$\Lambda^2 V^*$	0	$\mathfrak{gl}(V)$	0	$\Lambda^2 V$
$\mathfrak{o}_{2n+1}$	$\Lambda^2 V^*$	V^*	$\mathfrak{gl}(V)$	V	$\Lambda^2 V$

In all cases $\mathfrak{g}_k = 0$ for $|k| > 2$. The bracket acts as follows. The component $\mathfrak{g}_0$ is always $\mathfrak{gl}(V)$ with the standard Lie algebra structure. The structure of a $\mathfrak{g}_0$-module on each $\mathfrak{g}_k$ is standard (see 1.5). It remains to define the bracket in the following 5 cases:

(1) $S^2 V \times S^2 V^* \to \mathfrak{gl}(V)$,

(2) $\Lambda^2 V \times \Lambda^2 V^* \to \mathfrak{gl}(V)$,

(3) $\Lambda^2 V \times V^* \to V$, $V \times \Lambda^2 V^* \to V^*$,

(4) $V \times V \to \Lambda^2 V$, $V^* \times V^* \to \Lambda^2 V^*$,

(5) $V \times V^* \to \mathfrak{gl}(V)$.

(1) and (2). Consider $\Lambda^2 V$ and $S^2 V$ as subspaces of $\operatorname{Hom}(V^*, V)$, and $\Lambda^2 V^*$ and $S^2 V^*$ as subspaces of $\operatorname{Hom}(V, V^*)$. Then the bracket in each case is the restriction of the mapping $\operatorname{Hom}(V^*, V) \times \operatorname{Hom}(V, V^*) \to \mathfrak{gl}(V)$, sending (A, B) to $A \circ B$.

(3) If $A \in \Lambda^2 V \subset \operatorname{Hom}(V^*, V)$ and $\xi \in V^*$, then $[A, \xi] = A(\xi) \in V$ (similarly for the second part).

(4) $[v_1, v_2] = v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1 \in \Lambda^2 V$, $[\xi_1, \xi_2] = -\xi_1 \wedge \xi_2 \in \Lambda^2 V^*$ $(v_i \in V$, $\xi_i \in V^*)$.

(5) $[v, \xi]$ is the image of $v \otimes \xi$ under the natural isomorphism $V \otimes V^* \simeq \mathfrak{gl}(V)$.

(b) $\mathfrak{g}$ is a Kac-Moody algebra of rank n (with the exception of $\mathfrak{g}=\mathfrak{o}_2$). The corresponding matrices A are

$\mathfrak{sp}_{2n}$

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & & \vdots \\ 0 & -1 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & -1 & 0 \\ 0 & & & -1 & 2 & -2 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

$\mathfrak{o}_{2n}$

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & 2 & -1 & -1 \\ 0 & & & -1 & 2 & 0 \\ 0 & 0 & \dots & -1 & 0 & 2 \end{pmatrix}$$

$\mathfrak{o}_{2n+1}$

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & -1 & 0 \\ 0 & & & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

(the matrices for $n=2$ are indicated). The generators and roots of $\mathfrak{g}$ are described as follows. Choose a basis $v_1, \dots, v_n$ in V, hereby identifying $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n$, and let $v_1^*, \dots, v_n^*$ be the dual basis in V^*. By (a), $\mathfrak{g} \supset \mathfrak{gl}(V) = \mathfrak{gl}_n \supset \mathfrak{sl}_n$. We identify $\mathfrak{sl}_n$ with the contragredient subalgebra of $\mathfrak{g}$ of rank $(n-1)$, generated by the elements h_i, e_i and f_i

with $1 \leqq i \leqq n-1$. Thus, the generators $f_1, \dots, f_{n-1}, h_1, \dots, h_{n-1}, e_1, \dots, e_{n-1}$, the roots contained in $\mathbf{Z}\alpha_1 \oplus \dots \oplus \mathbf{Z}\alpha_{n-1}$, and the corresponding root subspaces are those constructed in 1.7, 1.8. The remaining generators, roots and root subspaces are given in Tables 2 and 3.

TABLE 2

$\mathfrak{g}$	f_n	h_n	e_n
$\mathfrak{sp}_{2n}$	$v_n^* \otimes v_n^*$	e_{nn}	$v_n \otimes v_n$
$\mathfrak{o}_{2n}$	$-v_{n-1}^* \wedge v_n^*$	$e_{n-1,n-1} + e_{nn}$	$v_{n-1} \wedge v_n$
$\mathfrak{o}_{2n+1}$	v_n^*	$2e_{nn}$	$2v_n$

TABLE 3

$\mathfrak{g}$	$\alpha \in R^+$	$\mathfrak{g}_\alpha$	$\mathfrak{g}_{-\alpha}$
$\mathfrak{sp}_{2n}$	$\alpha_i + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_{n-1}) + \alpha_n$ $(1 \leqq i \leqq j \leqq n)$	$\mathbf{C}(v_i \otimes v_j + v_j \otimes v_i)$	$\mathbf{C}(v_i^* \otimes v_j^* + v_j^* \otimes v_i^*)$
$\mathfrak{o}_{2n}$	$\alpha_i + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ $(1 \leqq i < j \leqq n)$*	$\mathbf{C}(v_i \wedge v_j)$	$\mathbf{C}(v_i^* \wedge v_j^*)$
$\mathfrak{o}_{2n+1}$	$\alpha_i + \alpha_{i+1} + \dots + \alpha_n \quad (1 \leqq i \leqq n)$	$\mathbf{C}v_i$	$\mathbf{C} \cdot v_i^*$
	$\alpha_i + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_n)$ $(1 \leqq i < j \leqq n)$	$\mathbf{C}(v_i \wedge v_j)$	$\mathbf{C}(v_i^* \wedge v_j^*)$

Here the term $\alpha_j + \dots + \alpha_{n-2}$ for $j > n-2$ is defined as $(\alpha_j + \dots + \alpha_n) - (\alpha_{n-1} + \alpha_n)$, so it equals 0 for $j = n-1$ and $-\alpha_{n-1}$ for $j = n$.

Proof. (a) Let us consider the most cumbersome case $\mathfrak{g} = \mathfrak{o}_{2n+1} = \mathfrak{o}(W)$. We represent our $(2n+1)$-dimensional vector space W with a form as $V \oplus \mathbf{C} \oplus V^*$, where the form is given by $\langle (v_1, c_1, \xi_1), (v_2, c_2, \xi_2) \rangle = \xi_2(v_1) + \xi_1(v_2) + c_1 c_2$. Each element of $\mathrm{Hom}(W, W)$ is represented by a matrix

$$\begin{array}{c|ccc|} & V & \mathbf{C} & V^* \\ \hline V & A & v & B \\ \mathbf{C} & \xi & c & u \\ V^* & C & \eta & D \\ \hline \end{array},$$

where $A \in \operatorname{Hom}(V, V)$, $B \in \operatorname{Hom}(V^*, V)$, ... It is easy to see that the matrix belongs to $\mathfrak{o}_{2n+1}$ if and only if

$$D=-A^*, \quad B=-B^*, \quad C=-C^*, \quad u=-v^*, \quad \eta=-\xi^*, \quad c=0.$$

Hence, each element of $\mathfrak{g}$ can be identified with a 5-tuple (C, ξ, A, v, B). This implies the decomposition $\mathfrak{g}=\Lambda^2 V^* \oplus V^* \oplus \mathfrak{gl}(V) \oplus V \oplus \Lambda^2 V$ from Table 1. It is easy to verify that the bracket has the form indicated above. The cases $\mathfrak{g}=\mathfrak{sp}_{2n}$ and $\mathfrak{g}=\mathfrak{o}_{2n}$ are treated similarly.

(b) Let the elements $h_1, \ldots, h_n, e_1, \ldots, e_n, f_1, \ldots, f_n$, and the subspaces $\mathfrak{g}_\alpha$ be defined as above. Using the description of commutator given in (a), one can easily verify that $h_1, \ldots, h_n, e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ generate $\mathfrak{g}$, and the decomposition $\mathfrak{g}=\mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\right)$ makes $\mathfrak{g}$ into a Q-graded Lie algebra, satisfying 1.8 (a), (b) with the matrices A described above. The property 1.8 (c) follows at once from the next observation:

(*) If $\mathfrak{g}$ is a classical Lie algebra and α a root of $\mathfrak{g}$ then each of the spaces $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ and $\mathfrak{h}_\alpha=[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional.

1.12. Proposition. Let $\mathfrak{g}$ be a classical Lie algebra and A the corresponding Cartan matrix (the matrices A are given in 1.7 and 1.11 (b)).

(a) A is nonsingular; it is indecomposable if $\mathfrak{g} \neq \mathfrak{o}_4$;

(b) $\mathfrak{g}$ has no proper graded ideals (again if $\mathfrak{g} \neq \mathfrak{o}_4$).

Proof. (b) follows at once from (a) and Prop. 1.9 (b), and the only non-trivial statement in (a) is that A is nonsingular. This means that the linear functionals $\bar{\alpha}_i$ form a basis of $\mathfrak{h}^*$ (see 1.8). Under the notation of 1.11, we have $\mathfrak{h}=\mathfrak{h}_n$ if $\mathfrak{g}=\mathfrak{o}_{2n}, \mathfrak{sp}_{2n}$ or $\mathfrak{o}_{2n+1}$, and $\mathfrak{h}=\mathfrak{h}_n \cap \mathfrak{sl}_n$ if $\mathfrak{g}=\mathfrak{sl}_n$. Let $\{\xi_1, \ldots, \xi_n\}$ be the basis of $\mathfrak{h}_n^*$, which is dual of the basis $\{e_{11}, \ldots, e_{nn}\}$ of $\mathfrak{h}_n$, i.e. $\xi_i(e_{jj})=\delta_{ij}$. The definitions readily imply that if $\mathfrak{g}=\mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ or $\mathfrak{o}_{2n+1}$ then

$$\bar{\alpha}_i=\xi_i-\xi_{i+1} \qquad \text{for} \quad i=1, \ldots, n-1,$$

and for $\mathfrak{g}=\mathfrak{sp}_{2n}$ $(\mathfrak{o}_{2n}, \mathfrak{o}_{2n+1})$

$$\bar{\alpha}_n=2\xi_n\,(\xi_{n-1}+\xi_n, \xi_n).$$

Evidently, in each case the $\bar{\alpha}_i$-s form a basis of $\mathfrak{h}^*$, as required.

1.13. We shall associate to every classical Lie algebra $\mathfrak{g}$ the remarkable infinite-dimensional Kac-Moody algebra $\tilde{\mathfrak{g}}$. This requires a notion of the Killing form on $\mathfrak{g}$.

The *Killing form* B on $\mathfrak{g}$ is defined by

$$B(x, y)=\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y), \qquad x, y \in \mathfrak{g}.$$

Proposition. B is symmetric, non-degenerate and invariant (this means that $B([x, y], z)=B(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$). Moreover, $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)=0$ unless $\alpha+\beta=0$,

and the restriction of B on the subspace $\mathfrak{h}_{\mathbf{R}}=\mathbf{R} \cdot h_1 \oplus \mathbf{R} \cdot h_2 \oplus \ldots \oplus \mathbf{R} \cdot h_n$ is positive definite.

Proof. (1) The facts that B is symmetric and invariant follow directly from the standard properties of the trace and the Jacobi identity (they hold for all finite dimensional Lie algebras).

(2) Let us prove that $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)=0$ unless $\alpha+\beta=0$. Indeed, since $\mathfrak{g}$ is Q-graded, it follows that

$$\operatorname{ad} x \circ \operatorname{ad} y\,(\mathfrak{g}_\gamma) \subset \mathfrak{g}_{\gamma+\alpha+\beta} \qquad \text{for} \quad x \in \mathfrak{g}_\alpha, \quad y \in \mathfrak{g}_\beta,$$

which implies our assertion.

(3) Now we prove that $B(h, h)>0$ for all non-zero $h \in \mathfrak{h}_{\mathbf{R}}$. By definition,

$$B(h, h)=\operatorname{tr}((\operatorname{ad} h)^2)=\sum_{\alpha \in R} (\bar{\alpha}(h))^2 \qquad (\text{see } 1.8).$$

Since $\mathfrak{g}=\mathfrak{g}(A)$ where A is a Cartan matrix, it follows that $\bar{\alpha}(h) \in \mathbf{R}$ for $\alpha \in R$, $h \in \mathfrak{h}_{\mathbf{R}}$. By 1.12, $\bar{\alpha}_i(h) \neq 0$ for some $i=1, \ldots, n$, so $B(h, h)>0$, as required.

(4) It remains to prove that B is non-degenerate. Let $I=\operatorname{Ker} B$. Since B is invariant, it follows that I is an ideal. By (2) and (3), I is Q-graded, and $I \neq \mathfrak{g}$. Therefore, $I=0$ (see 1.12 (b)).

1.14. Definition. The *affine algebra* $\tilde{\mathfrak{g}}$ corresponding to a classical Lie algebra $\mathfrak{g}$ is defined as follows. As a vector space, $\tilde{\mathfrak{g}}=(\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]) \oplus \mathbf{C}c$, where $\mathbf{C}[t, t^{-1}]$ is the ring of Laurent polynomials in t, and $\mathbf{C}c$ is one-dimensional. The element c lies in the center of $\tilde{\mathfrak{g}}$, and the bracket of two monomials is defined by

$$[x \otimes t^i, y \otimes t^j]=\begin{cases}[x, y] \otimes t^{i+j} & \text{if } i+j \neq 0, \\ [x, y]+iB(x, y)\,c & \text{if } i+j=0\end{cases}$$

(here $x, y \in \mathfrak{g}$). The Lie algebra axioms can be verified directly. We shall write xt^k instead of $x \otimes t^k$ (or simply x if $k=0$).

Remark. The quotient algebra $\tilde{\mathfrak{g}}/\mathbf{C}c=\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$ is called a *loop algebra*. The meaning of the term is that the loop algebra is naturally realized as an algebra of $\mathfrak{g}$-valued functions on the unit circle with finite Fourier series, and the bracket acts pointwise.

1.15. We want to prove that every affine algebra $\tilde{\mathfrak{g}}$ is a Kac-Moody algebra. For this we need some additional information on $\mathfrak{g}$.

Proposition. Let $\mathfrak{g}$ be a classical Lie algebra and R the set of roots of $\mathfrak{g}$. Then $\mathfrak{g}$ has the *maximal root* γ, that is, $\gamma-\alpha \in Q^+$ for all $\alpha \in R$. There is a unique element $h_\gamma^\vee$ of $\mathfrak{h}_\gamma=[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}]$ such that $\bar{\gamma}(h_\gamma^\vee)=2$. If elements $e_\gamma \in \mathfrak{g}_\gamma$ and $f_\gamma \in \mathfrak{g}_{-\gamma}$ are chosen so that $[e_\gamma, f_\gamma]=h_\gamma^\vee$ then $B(e_\gamma, f_\gamma)=\frac{1}{2} B(h_\gamma^\vee, h_\gamma^\vee)>0$.

In the following table we give the explicit values of the root γ, linear form $\bar{\gamma}$ and vectors $h_\gamma^\vee$, e_γ and f_γ for all $\mathfrak{g}$ (we use the notation of 1.11, 1.12).

TABLE 4

$\mathfrak{g}$	γ	$\bar{\gamma}$	$h_\gamma^\vee$	e_γ	f_γ
$\mathfrak{sl}_n$	$\alpha_1+\ldots+\alpha_{n-1}$	$\xi_1-\xi_n$	$e_{11}-e_{nn}$	e_{1n}	e_{n1}
$\mathfrak{sp}_{2n}$	$2(\alpha_1+\ldots+\alpha_{n-1})+\alpha_n$	$2\xi_1$	e_{11}	$v_1\otimes v_1$	$v_1^*\otimes v_1^*$
$\mathfrak{o}_{2n}$	$\alpha_1+2(\alpha_2+\ldots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n$	$\xi_1+\xi_2$	$e_{11}+e_{22}$	$v_1\wedge v_2$	$-v_1^*\wedge v_2^*$
$\mathfrak{o}_{2n+1}$	$\alpha_1+2(\alpha_2+\ldots+\alpha_n)$	$\xi_1+\xi_2$	$e_{11}+e_{22}$	$v_1\wedge v_2$	$-v_1^*\wedge v_2^*$

The proof of all assertions except the equality $B(e_\gamma, f_\gamma)=\frac{1}{2}B(h_\gamma^\vee, h_\gamma^\vee)$ is straightforward, using the explicit description of $\mathfrak{g}$ given in 1.11, 1.12. Since B is invariant and $[h_\gamma^\vee, e_\gamma]=\bar{\gamma}(h_\gamma^\vee)\,e_\gamma=2e_\gamma$, we have

$$B(h_\gamma^\vee, h_\gamma^\vee)=B(h_\gamma^\vee, [e_\gamma, f_\gamma])=B([h_\gamma^\vee, e_\gamma], f_\gamma)=2B(e_\gamma, f_\gamma),$$

as stated.

Later we shall deduce the existence of the maximal root γ from the representation theory of $\mathfrak{g}$ (p. 2.9 below).

1.16. Proposition. If $\mathfrak{g}$ is a classical Lie algebra of rank n then the affine Lie algebra $\tilde{\mathfrak{g}}$ is a Kac-Moody algebra of rank $n+1$. The corresponding Cartan matrices are

$$\mathfrak{g}=\mathfrak{sl}_{n+1}\ (n\geqq 2)$$

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & -1\\ -1 & 2 & -1 & & & 0\\ 0 & -1 & 2 & \ddots & & \vdots\\ \vdots & & -1 & \ddots & \ddots & 0\\ 0 & & & \ddots & \ddots & -1\\ -1 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}$$

$$\mathfrak{g}=\mathfrak{sl}_2$$

$$\begin{pmatrix} 2 & -2\\ -2 & 2 \end{pmatrix}$$

$$\mathfrak{g}=\mathfrak{sp}_{2n}$$

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & & 0\\ -2 & 2 & -1 & 0 & \ldots & 0\\ 0 & -1 & 2 & \ddots & & \vdots\\ & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & -1 & 0\\ & & & -1 & 2 & -2\\ 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}$$

$$\mathfrak{g}=\mathfrak{o}_{2n}$$

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \ldots & & 0\\ 0 & 2 & -1 & 0 & & & 0\\ -1 & -1 & 2 & -1 & & & \vdots\\ 0 & 0 & -1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & & \ddots & \ddots & 0\\ & & & & \ddots & 2 & -1 & -1\\ & & & & & -1 & 2 & 0\\ 0 & 0 & \ldots & & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\mathfrak{g}=\mathfrak{o}_{2n+1}$$

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \ldots & & 0\\ 0 & 2 & -1 & 0 & \ldots & & 0\\ -1 & -1 & 2 & -1 & & & \\ 0 & 0 & -1 & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots\\ & & & & \ddots & -1 & 0\\ & & & & -1 & 2 & -1\\ 0 & 0 & & \ldots & 0 & -2 & 2 \end{pmatrix}$$

The generators and roots of $\tilde{\mathfrak{g}}$ are described as follows. We identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g}\otimes 1$ of $\tilde{\mathfrak{g}}$ and choose the generators $h_1, \ldots, h_n, e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of $\mathfrak{g}$ as in 1.11 (b). Put

$$f_0 = e_\gamma t^{-1}, \quad e_0 = f_\gamma t \quad \text{and} \quad h_0 = \frac{1}{2} B(h_\gamma^\vee, h_\gamma^\vee)\, c - h_\gamma^\vee$$

(see 1.15). Then $\{h_i, e_i, f_i \mid i = 0, 1, \ldots, n\}$ is a standard system of generators of $\tilde{\mathfrak{g}}$. The root decomposition of $\tilde{\mathfrak{g}}$ is

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \Sigma' \mathfrak{g}_\alpha t^k,$$

where $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}c$ and the sum in Σ' is over all pairs (α, k) such that $\alpha \in R \cup \{0\}$, $k \in \mathbf{Z}$ and $(\alpha, k) \neq 0$. The root corresponding to $\mathfrak{g}_\alpha t^k$ is $\alpha + k(\alpha_0 + \gamma)$, so the set of positive roots of $\tilde{\mathfrak{g}}$ is

$$\tilde{R}^+ = R^+ \cup \{\alpha + k(\alpha_0 + \gamma) \mid \alpha \in R \cup \{0\}, k \in \mathbf{N}\}.$$

Proof. Since $B(h_\gamma^\vee, h_\gamma^\vee) > 0$ (see 1.15), it follows that $h_0, h_1, \ldots, h_n$ are linearly independent and span the subspace $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}c$. Clearly, the elements $h_0, h_1, \ldots, h_n$, $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ generate the subalgebra $\mathfrak{g} \oplus \mathbf{C}c$. Using 1.12 (b) and 1.10 (b), we see that $\mathfrak{g}t$ $(\mathfrak{g}t^{-1})$ is generated by e_0 (f_0) as a $\mathfrak{g}$-module and generates the subalgebra $\bigoplus_{k>0} \mathfrak{g}t^k$ $(\bigoplus_{k<0} \mathfrak{g}t^k)$. It follows that $\{h_i, e_i, f_i \mid i = 0, 1, \ldots, n\}$ is a set of generators of $\tilde{\mathfrak{g}}$.

Clearly, $\tilde{\mathfrak{g}}$ becomes $\tilde{Q}$-graded under the root decomposition described above $\left(\tilde{Q} = \bigoplus_{i=0}^{n} \mathbf{Z}\alpha_i = Q \oplus \mathbf{Z}\alpha_0\right)$ and satisfies 1.8 (a), (b) with the appropriate Cartan matrix.

To prove 1.8(c) we must verify that if x is a non-zero element of some root subspace of $\tilde{\mathfrak{g}}$ then the ideal I generated by x has non-zero intersection with $\tilde{\mathfrak{h}}$.

Case 1. $x \in \mathfrak{g}_\alpha t^k$, $\alpha \in R$. Let y be a non-zero element of $\mathfrak{g}_{-\alpha} t^{-k}$. Then $[x, y] \in I \cap \tilde{\mathfrak{h}}$ and $[x, y] \neq 0$ by 1.11 (*).

Case 2. $x = ht^k$, $0 \neq h \in \mathfrak{h}$, $k \neq 0$. Choose some $i = 1, \ldots, n$ such that $\bar{\alpha}_i(h) \neq 0$ (see 1.12). Then $[x, e_i]$ is a non-zero element of $\mathfrak{g}_{\alpha_i} t^k$, and we are reduced to the previous case.

1.17. Remarks. (a) Later we shall prove that every classical Lie algebra (except $\mathscr{o}_2$ and $\mathscr{o}_4$) is simple, i.e. is non-abelian and has no proper ideals (Cor. 3.7 below). Moreover, the class of Kac-Moody algebras corresponding to nonsingular indecomposable Cartan matrices coincides with the class of simple (finite dimensional) Lie algebras ([23]). The classification of these algebras is well-known (see e.g. [22], [47]); besides classical algebras, there are only 5 so called exceptional simple Lie algebras. All our assertions on classical Lie algebras remain true for the exceptional algebras, but for simplicity we shall not consider them in this paper.

(b) Our construction of affine Lie algebras can be generalized, viz., one can construct an infinite dimensional Kac-Moody algebra, corresponding to a classical Lie algebra $\mathfrak{g}$ and certain automorphism σ of $\mathfrak{g}$ of finite order (our construction corresponds to $\sigma = id$) ([40]). For simplicity of exposition, we shall not consider this more general version.

(c) Evidently, if a matrix A' is obtained from A by a simultaneous permutation of rows and columns then $\mathfrak{g}(A) = \mathfrak{g}(A')$. In particular, this gives the so-called exceptional isomorphisms between classical algebras: $\mathscr{sl}_2 \simeq \mathscr{sp}_2 \simeq \mathscr{o}_3$, $\mathscr{o}_4 \simeq \simeq \mathscr{sl}_2 \oplus \mathscr{sl}_2$, $\mathscr{sp}_4 \simeq \mathscr{o}_5$, $\mathscr{o}_6 \simeq \mathscr{sl}_4$.

2. THE CATEGORY $\mathscr{O}$ AND VERMA MODULES

Throughout this section $\mathfrak{g}$ will be a contragredient Lie algebra with the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\right)$. We shall use the notation of 1.8.

2.1. We need some more terminology and notation. Put $P = \mathfrak{h}^* \oplus F$ and define the linear mapping $P \to \mathfrak{h}^*$ $(\chi \mapsto \bar{\chi})$ by $\overline{\chi + \alpha} = \chi + \bar{\alpha}$ $(\chi \in \mathfrak{h}^*, \alpha \in F)$ (see 1.8); trivially, this operator is onto. Elements of P are called *weights*. For $\lambda \in P$ we put $D(\lambda) = \lambda - Q^+ = \{v \in P \mid v = \lambda - \Sigma k_i \alpha_i, k_i \in \mathbf{Z}^+\}$.

Let $\mathscr{E}$ be the space of all functions f on P such that supp f is contained in the union of a finite number of sets of the form $D(\lambda)$. The space $\mathscr{E}$ is a commutative algebra with respect to the convolution operation: $f_1 \cdot f_2(\lambda) = \sum_{\mu \in P} f_1(\mu) f_2(\lambda - \mu)$ (this sum is finite). Denote by $e^\lambda \in \mathscr{E}$ the function $e^\lambda(\mu) = \delta_{\lambda\mu}$. Evidently, $e^\lambda e^\mu = e^{\lambda+\mu}$, and $e^0 = 1$ is the unit of the algebra $\mathscr{E}$. We shall write elements of $\mathscr{E}$ in the form $f = \sum_{\lambda \in P} f(\lambda) e^\lambda$.

2.2. Now we shall define the class of $\mathfrak{g}$-modules, which will be studied in this paper.

Definition. The category $\mathscr{O}$ consists of $\mathfrak{g}$-modules V satisfying the following properties:

(1) V is P-graded, i.e. $V = \bigoplus_{\lambda \in P} V_\lambda$, where $\mathfrak{g}_\alpha V_\lambda \subset V_{\alpha+\lambda}$. If $V_\lambda \neq 0$ then λ is a *weight of* V.

The set of weights of V is denoted by $P(V)$ and the decomposition $V = \bigoplus_{\lambda \in P(V)} V_\lambda$ is called the *weight decomposition*.

(2) For any $h \in \mathfrak{h}$ and $v \in V_\lambda$ we have $hv = \bar{\lambda}(h)v$.

(3) dim $V_\lambda < \infty$ for all $\lambda \in P$, and the function $\lambda \mapsto \dim V_\lambda$ belongs to $\mathscr{E}$. The function ch $V = \sum_\lambda (\dim V_\lambda)\, e^\lambda \in \mathscr{E}$ is called the *character of* V.