1) In this problem $X$ is a smooth variety over a field $k$ of characteristic zero, sheaves are considered with respect to the Zariski topology.

(a) Let $\omega \in \Omega^1(X)$ be a one form. Show that the map $\mathcal{O} \ni f \mapsto f$, $\text{Vect}_X \ni v \mapsto v + \langle \omega, v \rangle$ extends to an automorphism of the sheaf $D_X$ iff $\omega$ is closed.

(b) Let $\Omega^1_{cl}$ be the sheaf of closed 1-forms. For $h \in H^1(\Omega^1_{cl})$ define a sheaf of algebras $D_h$ on $X$, which is locally isomorphic to $D(X)$.

(c) Recall that isomorphism classes of line bundles on $X$ are in bijection with cohomology classes $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of invertible functions. Consider the morphism of sheaves $\mathcal{O}^* \to \Omega^1_{cl}$, $f \mapsto df$. For a line bundle $L$ let $c_1(L) \in H^1(\mathcal{O}^*)$ be the image of the corresponding class in $H^1(X, \mathcal{O}^*)$ under the induced map $H^1(X, \mathcal{O}^*) \to H^1(X, \Omega^1_{cl})$.

Identify $D_{c_1(L)}$ with the sheaf of differential operators acting on the sections of $L$.

We will write $D_L$ instead of $D_{c_1(L)}$.

2) For which $i \in \mathbb{Z}$ is $\mathbb{P}^n$ “affine with respect to $D_{\mathcal{O}(i)}$”? In other words, for which $i$ does the functor of global sections provide an equivalence of categories between quasicoherent sheaves of $D_{\mathcal{O}(i)}$-modules and modules over global sections of $D_{\mathcal{O}(i)}$?

3) Assume that $X$ is a $D$-affine variety, and $G$ is an affine algebraic group acting on $X$. Let $D^\text{glob}_X$ be the algebra of global differential operators. Show that the category of $G$-equivariant $D_X$-modules is equivalent to the category of $D^\text{glob}_X$-modules $M$ endowed with a $G$-action whose differential coincides with the action of $g$ on $M$ coming from the embedding $g \to D^\text{glob}_X$.

4) Let $X = \mathbb{P}^1$, $G = SL(2)$. Show that $D^\text{glob}_X$ is equal to the quotient of $U(\mathfrak{g})$ by the ideal generated by the Casimir element $C$.

5) Let $G$ be an algebraic group, $K \subset G$ – a subgroup. Recall that a $(\mathfrak{g}, K)$-module is a vector space $M$ endowed with an action of $\mathfrak{g}$ and an algebraic (in particular, locally finite) action of $K$ which are compatible in the obvious sense (i.e. if we let $\mathfrak{k} = \text{Lie}(K)$ then we require that the two $\mathfrak{k}$-actions of $\mathfrak{k}$ coming from $\mathfrak{g}$-action and $K$-action coincide).

6) Let $G = PGL(2)$ and let $K \subset G$ be the image of the subgroup of diagonal matrices. Use the above results to show that there exist exactly 3 irreducible $(\mathfrak{g}, K)$-modules on which $C$ acts by 0.

Write the above $(\mathfrak{g}, K)$-modules explicitly.