Problem set 5, due October 19

(1) In this problem X is a smooth variety over a field k of characteristic zero, sheaves are considered with respect to the Zarisky topology.
   (a) Let $\omega \in \Omega^1(X)$ be a one form. Show that the map $\mathcal{O} \ni f \mapsto f$, $Vect_X \ni v \mapsto v + \langle \omega, v\rangle$ extends to an automorphism of the sheaf D_X iff ω is closed.
   (b) Let Ω^1_{cl} be the sheaf of closed 1-forms. For $h \in H^1(\Omega^1_{cl})$ define a sheaf of algebras D_h on X, which is locally isomorphic to $D(X)$.
   (c) Recall that isomorphism classes of line bundles on X are in bijection with cohomology classes $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of invertible functions. Consider the morphism of sheaves $\mathcal{O}^* \to \Omega^1_{cl}$, $f \mapsto \frac{df}{f}$. For a line bundle L let $c_1(L) \in H^1(\mathcal{O}^*)$ be the image of the corresponding class in $H^1(X, \mathcal{O}^*)$ under the induced map $H^1(X, \mathcal{O}^*) \to H^1(X, \Omega^1_{cl})$. Identify $D_{c_1(L)}$ with the sheaf of *differential operators acting on the sections of* L.

   We will write D_L instead of $D_{c_1(L)}$.

(2) For which $i \in \mathbb{Z}$ is $\mathbb{P}^n$ "affine with respect to $D_{\mathcal{O}(i)}$"? In other words, for which i does the functor of global sections provide an equivalence of categories between quasicoherent sheaves of $D_{\mathcal{O}(i)}$-modules and modules over global sections of $D_{\mathcal{O}(i)}$?

(3) Assume that X is a D-affine variety, and G is an affine algebraic group acting on X. Let D_X^{glob} be the algebra of global differential operators. Show that the category of G-equivariant D_X-modules is equivalent to the category of D_X^{glob}-modules M endowed with a G-action whose differential coincides with the action of $\mathfrak{g}$ on M coming from the embedding $\mathfrak{g} \to D_X^{glob}$.

(4) Let $X = \mathbb{P}^1$, $G = SL(2)$. Show that D_X^{glob} is equal to the quotient of $U(\mathfrak{g})$ by the ideal generated by the Casimir element C.

(5) Let G be an algebraic group, $K \subset G$ – a subgroup. Recall that a $(\mathfrak{g}, K)$-module is a vector space M endowed with an action of $\mathfrak{g}$ and an algebraic (in particular, locally finite) action of K which are compatible in the obvious sense (i.e. if we let $\mathfrak{k} = Lie(K)$ then we require that the two $\mathfrak{k}$-actions of $\mathfrak{k}$ coming from $\mathfrak{g}$-action and K-action coincide).

(6) Let $G = PGL(2)$ and let $K \subset G$ be the image of the subgroup of diagonal matrices. Use the above results to show that there exist exactly 3 irreducible $(\mathfrak{g}, K)$-modules on which C acts by 0.

   Write the above $(\mathfrak{g}, K)$-modules explicitly.