1. Let $R = \mathbb{C}[x]$.

(a) Describe all saturated multiplicative subsets $S$ in $R$.

(b) For each such $S$, describe the localization $S^{-1}R$.

(c) For which $S$ is $S^{-1}R$ a finitely generated $\mathbb{C}$-algebra? A local algebra?

(d) What are the $S$-saturated ideals of $R$ (for each $S$)? What is the saturation of the non-saturated ones?

(e) For any finitely generated module $M$ over $R$ and any $S$, describe $S^{-1}M$.

**Solution.** (a) Let $S$ be a saturated multiplicative subset. If $0 \in S$ then any $x \in S$, i.e. $S = R$, since $0x = 0$. Assume $0 \notin S$. Every invertible element belongs to $S$, so $\mathbb{C} \subset S$. If

$$p = a_0(x - x_1)^{n_1}...(x - x_k)^{n_k} \in S$$

for $n_i > 0$ then $x - x_i \in S$. So $S = S(Z)$ is determined by the set $Z$ of all $a \in \mathbb{C}$ such that $x - a \in S$; namely, $S$ consists of polynomials $p$ as above such that roots of $p$ are in $Z$. Clearly, any $Z$ can arise.

(b) $0^{-1}R = 0$, $S(Z)^{-1}R = \mathbb{C}[x, \frac{1}{x-a}, a \in Z]$ is the ring of rational finctions which are regular outside of $Z$.

(c) $S(Z)^{-1}R$ is finitely generated as an algebra iff $Z$ is finite. Otherwise, we can create only finitely many poles. To have a local ring, we need $Z$ to be everything or miss just one point.

(d) Each ideal $I$ is principal, generated by $p \in \mathbb{C}[x]$. Consider the nontrivial case $p \neq 0$. Saturated means that $sp \in I$ for some $s \in S = S(Z)$ means $p \in I$. This is equivalent to saying that the roots of $p$ are disjoint from $Z$. The saturation of $I$ in general is $I'$, generated by the polynomial $q$ such that $p = qf$ with $f$ having roots in $Z$ and $q$ having no roots in $Z$.

(e) A finitely generated module is $R^m \oplus E$, where $E$ is finite dimensional. So it suffices to describe $S^{-1}E$. Recall that $E$ is a direct sum of Jordan block modules. Multiplication by $S^{-1}$ just knocks out the Jordan blocks that have eigenvalue in $Z$.

2. Let $R = \mathbb{C}[x, y]/(y^2 - x(x - 1)(x - 2))$.

(a) Describe $X = \mathrm{Spec}R$ (and draw a picture of the real points).

(b) Describe the open sets in $X$ in Zariski topology.

(c) Let $P$ be any point on the curve $y^2 = x(x - 1)(x - 2)$, and $M$ be the ideal consisting of $f \in R$ such that $f(P) = 0$. Show that $M$ is a projective (=locally free) $R$-module of rank 1, but it is not free.

(d) (harder!) Which of the open sets on $X$ are principal? For which $n$ is there a set of distinct points $P_1, ..., P_n$ on the curve $y^2 = x(x-1)(x-2)$ such that $X \setminus \{P_1, ..., P_n\}$ is not principal?

Hint. You may need to use the addition law on the elliptic curve.

**Solution.**

(a) $X = \text{SpecR}$ is the union of the curve $C$ defined by the equation $y^2 = x(x-1)(x-2)$ (in $\mathbb{C}^2$) and the "generic" point $\mathbf{g}$ corresponding to the ideal 0.

(b) Nonempty open sets are complements of finite sets in $C$.

(c) An ideal is a free module iff it is a principal ideal, so we must show that $M$ is not principal, i.e. there is no regular function $f \in R$ such that $M = (f)$. For that $f$ would have to have a simple zero at $P$ and no other zeros on $C$. Let us show that there is no such function. Since $y^2$ expresses in terms of $x$, any $f$ has the form $a(x)y + b(x)$, where $a, b$ are polynomials. If $a = 0$ and $f = b(x)$ is not a constant then $f$ has at least two zeros or one double zero. If $a \neq 0$, then the solutions of $f = 0$ on $C$ are found from the system

$$a(x)y + b(x) = 0,\ y^2 = x(x-1)(x-2).$$

Thus we get

$$a(x)^2 x(x-1)(x-2) = b(x)^2.$$

This equation has degree at least 3 (otherwise the leading term on the left side has to be canceled by something, which can't happen, as RHS is of even degree). So we will have at least three roots, counting multiplicities. This means that we cannot arrange for just one simple zero. So $M$ is not free.

On the other hand, $M$ is locally free of rank 1. To show this, for any maximal ideal $z \in C$, $z \neq P$, pick a regular function $f \in R$ so that $f(z) \neq 0$ and $f(P) = 0$. Then $M_f = R_f$. Also, $M_P$ is the maximal ideal in $R_P$, so it is free over $R_P$ (as it is a principal ideal generated by any $f \in R$ such that $f$ has a simple zero at $P$).

(d) Recall that we have an abelian group law on $C \cup \{\infty\}$ (with $\infty$ playing the role of 0 in the group), defined as follows. If $P, Q$ are points on $C$ then to compute $P + Q$, draw a line through $P, Q$ and take $R$ to be the third intersection point of this line with $C$. Then $P + Q = -R$ is the point mirror symmetric to $R$ with respect to the $x$-axis. (We interpret intersections with multiplicities; e.g., if $P = Q$, we draw the tangent).

The complement $U$ of points $P_1, ..., P_n$ is a principal open set iff there exists a regular function $f$ on $C$ such that $f$ vanishes at $P_1, ..., P_n$ and nowhere else. Let $m_1, ..., m_n$ be the multiplicities of $f$ at $P_1, ..., P_n$. We claim that $f$ with such multiplicities exists iff $\sum_i m_i P_i = 0$ (for the above addition law).

Let us first show that this condition is sufficient. To this end, let $P, Q, R$ be three points such that $P + Q + R = 0$, and let $g_{P,Q,R}$ be the linear function defining the line through $P, Q, R$. Then $g_{P,Q,R}$ has simple zeros at $P, Q, R$ and no other zeros. Also, $x - x_Q$ has simple zeros at $Q$ and $-Q$. Now we can do induction in $\sum m_i$. If $n > 1$, let $P_1 + P_2 = Q$. Also let $f$ be the function vanishing $m_1 - 1$ times at $P_1$, $m_2 - 1$ times at $P_2$, $m_i$ times at $P_i$ for $i \geq 3$, and another time at $Q$. By the induction assumption $f$ exists. Then the function

$$\frac{g_{P_1,P_2,-Q} f}{x - x_Q}$$

is regular and has the required pattern of zeros. If $n = 1$, we set $2P_1 = Q$, and do the same thing.

To prove that the condition is necessary, suppose $\sum m_i P_i = -P \neq 0$, and let $f$ be a function with such pattern of zeros. Then by sufficiency proved above, there is $g$ which has the required pattern of zeros and another zero at $P$. Then $g/f$ is a regular function with a single zero at $P$, which does not exist by (c), a contradiction.

So we see that for $U$ to be principal, there must exist $m_i$ such that $\sum m_i P_i = 0$. This does not hold in general; e.g. if $n = 1$ and $P_1 = P$, this means that $P$ is a finite order element in the group, and it is easy to see that there are countably many elements of finite order, while $C$ is uncountable. In fact, we can see that for any $n$ there are $P_1, ..., P_n$ such that the complement of $P_1, ..., P_n$ is not principal.

(Here I am assuming to be known that a rational function on $C$ is regular iff it has no poles at $C$; this is not hard to show.)

3. Let $R = \mathbb{C}[x_1, ..., x_r]$, and $\phi : R^m \to R^n$ is a module map, given by a matrix $[\phi] = (a_{ij})$, $a_{ij} \in R$. Let $M = \text{Coker}\phi$. Describe the support of $M$ explicitly (as $V(\mathfrak{a})$, where the ideal $\mathfrak{a}$ is defined by polynomial equations). Consider first $m < n$, then $m = n$, then $m > n$.

**Solution.** If $m < n$, the support is the entire $\text{Spec}R$, so let us consider the case $m \geq n$. A linear map $\mathbb{C}^m \to \mathbb{C}^n$ $(m \geq n)$ given by a matrix $A = (a_{ij})$ is not surjective iff all $n$ by $n$ minors of $A$ are zero. So let us show that the ideal $\mathfrak{a}$ is generated by the $n$ by $n$ minors of $A$ (as polynomials of $x_1, ..., x_r$). To this end, assume first that a prime $\mathfrak{p}$ contains $\mathfrak{a}$, i.e., the minors are zero in $R' := R/\mathfrak{p}$. Then the minors are also zero over $\text{Frac}(R')$, so the map is not surjective after localization to the field of fractions. This implies that the map

$$(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})^m \to (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})^n \tag{1}$$

is not surjective, and hence

$$R_\mathfrak{p}^m \to R_\mathfrak{p}^n \tag{2}$$

is not surjective, i.e. $M_\mathfrak{p} \neq 0$, so $\mathfrak{p} \in \text{Supp}M$. On the other hand, assume that $\mathfrak{p}$ does not contain some minor $\Delta$ of $A$. Then $\Delta \neq 0$ in $R'$, so the map is surjective over $\mathrm{Frac}(R')$. Hence the map (1) is surjective. By Nakayama lemma, this implies that the map (2) is surjective, hence $M_{\mathfrak{p}} = 0$.