1. Compute the length of the $\mathbb{C}[x, y]$-module

$$M = \mathbb{C}[x, y]/(y^2 - x^3, x^2 - y^3).$$

Find the maximal ideals associated to M, and for each of them find the length of $M_{\mathfrak{m}}$.

2. Find the Hilbert-Samuel series of $\mathbb{C}[x, y]$ with respect to the ideal (x^2, y^3).

3. Let I be the ideal $(x_1, ..., x_n) \subset \mathbb{C}[x_1, ..., x_n]$. Describe this ideal as a module over $\mathbb{C}[x_1, ..., x_n]$ by generators and relations.

4. Let I be the ideal in $\mathbb{C}[x, y]$ consisting of polynomials which vanish when $y = 0$ and also at the points $(0, 1)$, $(0, -1)$. Find a finite set of generators for I.

5. Let $I \subset R$ be a proper ideal, and $G^\bullet R$ be the associated graded ring with respect to the I-adic filtration. Suppose $\cap_{n \geq 0} I^n = 0$. Show that if $G^\bullet R = R/I \oplus I/I^2 \oplus ...$ is a domain then R is a domain. Is the converse true?

6. Let I be the intersection of finitely many maximal ideals in $\mathbb{C}[x, y]$. Show that I can be generated by two elements.

7. (a) The group μ_n of roots of unity of order $n \geq 2$ acts on $\mathbb{C}[x, y]$ by $a(x, y) = (ax, a^{-1}y)$. Describe the algebra of invariants $\mathbb{C}[x, y]^{\mu_n}$ by generators and relations (you should need three generators).

(b) Let G be the group generated by μ_n and the element g sending (x, y) to $(y, -x)$. Describe the algebra of invariants $\mathbb{C}[x, y]^G$ by generators and relations (again you should need three generators).

8. Which of the following rings are Noetherian:

(i) $\mathbb{Z}[x_1, x_2, ...]/(x_{i+1} - ix_i, i \geq 1)$

(ii) $\mathbb{C}[[x, y]]/(x^2 - y^3)$

(iii) The ring R_∞ of holomorphic functions on $\mathbb{C}$.

(iv) The ring R_r of holomorphic functions on the open disk of radius r.

(v) The ring $R_0 = \cup_{r>0} R_r$ of germs of holomorphic functions near 0 (nested union).

9. Let $R \subset \mathbb{C}[x]$ be the ring of polynomials such that $f(j) = f(-j)$ for $j = 1, ..., m$. Show that R is generated by 2 elements, and find the defining relation between these elements. Is R normal?

10. (a) Let G be a finite subgroup of $GL_n(\mathbb{C})$. Show that the algebra $\mathbb{C}[x_1, ..., x_n]^G$ is normal.

(b) Show that the algebra $R = \mathbb{C}[x, y, z]/(z^2 - xy)$ is normal (you may use problem 7).