Exact sequences of tensor categories with respect to a module category

Pavel Etingof (MIT)

joint with Shlomo Gelaki

arXiv:1504.01300

0.1. **Exact sequences of Hopf algebras.** Let B be a (f.d.) Hopf algebra. Recall that a Hopf subalgebra $A \subseteq B$ is normal if it is invariant under the adjoint action of B on itself (this generalizes the notion of a normal subgroup). In this case, A_+B (where $A_+ := \mathrm{Ker}(\varepsilon)|_A$ is the augmentation ideal in A) is a two-sided ideal in B, hence a Hopf ideal, and thus the quotient $C := B/A_+B$ is a Hopf algebra. In this situation, one says that one has a (short) exact sequence of Hopf algebras

$$A \to B \to C. \tag{1}$$

The sequence (1) defines a sequence of tensor functors

$$\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{F} \mathcal{C} \tag{2}$$

between the corresponding tensor categories of finite dimensional comodules, $\mathcal{A} := A - \text{comod}$, $\mathcal{B} := B - \text{comod}$, $\mathcal{C} := C - \text{comod}$.

This sequence has the following categorical properties:

(i) The functor F is surjective (or dominant), i.e., any object Y of $\mathcal{C}$ is a subquotient (equivalently, a subobject, a quotient) of $F(X)$ for some $X \in \mathcal{B}$.

(ii) The functor ι is injective , i.e., is a fully faithful embedding.

(iii) The kernel of F (i.e., the category of objects X such that $F(X)$ is trivial) coincides with the image of ι.

(iv) The functor F is normal, i.e., for any $X \in \mathcal{B}$ there exists a subobject $X_0 \subseteq X$ such that $F(X_0)$ is the largest trivial subobject of $F(X)$.

0.2. **Exact sequences of tensor categories.** Bruguières and Natale called a sequence (2) satisfying conditions (i)-(iv) an exact sequence of tensor categories.

Example 0.1. Let G be a finite group, H its normal subgroup, $K = G/H$. Then we have an exact sequence of tensor categories

$$\mathrm{Rep}K \to \mathrm{Rep}G \to \mathrm{Rep}H.$$

Note that if H is not normal and N is its normal closure, then this sequence with $K := G/N$ satisfies (i)-(iii) but not (iv), which shows the relevance of condition (iv).

The essense of the Bruguières and Natale generalization is that we no longer need the categories $\mathcal{B}$, $\mathcal{C}$ in an exact sequence

$$\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{F} \mathcal{C} \tag{3}$$

to admit a fiber functor, i.e., be representation categories of a Hopf algebra. However, we still need $\mathcal{A}$ to have such a functor, namely the functor $F \circ \iota$. In particular, this means that given finite tensor categories $\mathcal{A}$, $\mathcal{C}$, we don't have an exact sequence

$$\mathcal{A} \to \mathcal{A} \boxtimes \mathcal{C} \to \mathcal{C}$$

unless $\mathcal{A}$ has a fiber functor. The goal of our work is to make a further generalization which would include this example.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be finite tensor categories, and $\mathcal{M}$ be an indecomposable exact $\mathcal{A}$-module category (exact means that $P \otimes X$ is projective for $P \in \mathcal{A}$ projective and $X \in \mathcal{M}$). Let $\mathrm{End}(\mathcal{M})$ be the category of right exact endofunctors of an indecomposable exact $\mathcal{A}$-module category $\mathcal{M}$.

Definition 0.2. An exact sequence with respect to $\mathcal{M}$ is a sequence of tensor functors of the form

$$\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{F} \mathcal{C} \boxtimes \mathrm{End}(\mathcal{M}),$$

such that ι is injective, F is surjective, $\mathcal{A} = \mathrm{Ker}(F)$ (the subcategory of $X \in \mathcal{B}$ such that $F(X) \in \mathrm{End}(\mathcal{M})$), and F is normal (i.e., for any $X \in \mathcal{B}$ there exists a subobject $X_0 \subseteq X$ such that $F(X_0)$ is the largest subobject of $F(X)$ contained in $\mathrm{End}(\mathcal{M})$).

0.3. **Alternative characterizations of exact sequences.** Our first main results are the following theorems, giving alternative characterizations of exact sequences of finite tensor categories.

Let $F : \mathcal{B} \to \mathcal{C} \boxtimes \mathrm{End}(\mathcal{M})$ be an exact monoidal functor which restricts on $\mathcal{A} \subset \mathcal{B}$ to the action of $\mathcal{A}$ on $\mathcal{M}$.

Theorem 0.3. *The following are equivalent:*

(i) F is surjective, $\mathcal{A} = \mathrm{Ker}(F)$ and F is normal, i.e.,

$$\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{F} \mathcal{C} \boxtimes \mathrm{End}(\mathcal{M})$$

is an exact sequence with respect to $\mathcal{M}$.

(ii) The natural functor

$$\Phi_* : \mathcal{B} \boxtimes_{\mathcal{A}} \mathcal{M} \to \mathcal{C} \boxtimes \mathcal{M},$$

given by

$$\mathcal{B}\boxtimes_{\mathcal{A}}\mathcal{M} \xrightarrow{F\boxtimes_{\mathcal{A}} id_{\mathcal{M}}} \mathcal{C}\boxtimes \mathrm{End}(\mathcal{M})\boxtimes_{\mathcal{A}}\mathcal{M} =$$

$$= \mathcal{C} \boxtimes \mathcal{M} \boxtimes \mathcal{A}^{*\mathrm{op}}_{\mathcal{M}} \xrightarrow{id_{\mathcal{C}}\boxtimes\rho} \mathcal{C} \boxtimes \mathcal{M},$$

is an equivalence (where $\rho : \mathcal{M} \boxtimes \mathcal{A}^{*\mathrm{op}}_{\mathcal{M}} \to \mathcal{M}$ is the right action of $\mathcal{A}^{*\mathrm{op}}_{\mathcal{M}}$ on $\mathcal{M}$).

(iii) The natural functor

$$\Phi^* : \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{B}) \to \mathrm{Fun}(\mathcal{M}, \mathcal{C}),$$

given by

$$\mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{B}) \xrightarrow{F\circ ?} \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{C}\boxtimes\mathrm{End}(\mathcal{M})) =$$

$$= \mathcal{C}\boxtimes^{\vee}\mathcal{M}\boxtimes_{\mathcal{A}}\mathrm{End}(\mathcal{M}) = \mathcal{C}\boxtimes\mathcal{A}^*_{\mathcal{M}}\boxtimes\mathcal{M}^{\vee} =$$

$$\mathcal{C}\boxtimes\mathcal{M}^{\vee}\boxtimes\mathcal{A}^*_{\mathcal{M}} \xrightarrow{id_{\mathcal{C}}\boxtimes\rho^{\mathrm{op}}} \mathcal{C}\boxtimes\mathcal{M}^{\vee} = \mathrm{Fun}(\mathcal{M}, \mathcal{C}),$$

is an equivalence (where $\mathrm{Fun}(\mathcal{M},\mathcal{C})$ *is the category of right exact functors from* $\mathcal{M}$ *to* $\mathcal{C}$*).*

Recall that for an object X of a finite tensor category $\mathcal{C}$, the Frobenius-Perron dimension $\mathrm{FPdim}(X)$ is the largest real eigenvalue of the operator of multiplication by X in the Grothendieck ring of $\mathcal{C}$, and the Frobenius-Perron dimension of $\mathcal{C}$ is

$$\mathrm{FPdim}(\mathcal{C}) := \sum_i \mathrm{FPdim}(X_i)\mathrm{FPdim}(P_i),$$

where X_i are the simple objects of $\mathcal{C}$ and P_i are their projective covers.

Theorem 0.4. *Assume that* F *is surjective. Then the following are equivalent:*

(i) $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A})\mathrm{FPdim}(\mathcal{C})$.

(ii) $\mathcal{A} = \mathrm{Ker}(F)$ *and* F *is normal (i.e.,* F *defines an exact sequence of tensor categories).*

0.4. **Dualization of exact sequences.** Recall that if

$$A \to B \to C$$

is an exact sequence of finite dimensional Hopf algebras, then we have a dual exact sequence

$$C^* \to B^* \to A^*,$$

and the double dual is the original sequence. This feature is lost in the Bruguières and Natale approach, but is again restored in our generalization. Namely, let

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C} \boxtimes \mathrm{End}\mathcal{M}$$

be an exact sequence of tensor categories with respect to an $\mathcal{A}$-module

$\mathcal{M}$. Also let $\mathcal{N}$ be an exact indecomposable $\mathcal{C}$-module category. Then $\mathcal{N} \boxtimes \mathcal{M}$ is an exact module category over $\mathcal{C} \boxtimes \mathrm{End}\mathcal{M}$, so we can dualize our sequence with respect to this module category and get the dual sequence

$$\mathcal{C}^*_{\mathcal{N}} \to \mathcal{B}^*_{\mathcal{N}\boxtimes\mathcal{M}} \to \mathcal{A}^*_{\mathcal{M}} \boxtimes \mathrm{End}\mathcal{N}.$$

Theorem 0.5. *The dual of an exact sequence of tensor categories is an exact sequence of tensor categories, and the double dual is the original sequence.*

0.5. **Semisimplicity.** Finally, we have the following result.

Theorem 0.6. *If*

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C} \boxtimes \mathrm{End}\mathcal{M}$$

is an exact sequence, and $\mathcal{A}$ *and* $\mathcal{C}$ *are semisimple (i.e. a fusion category), then so is* $\mathcal{B}$*.*